

## Existence of symmetric metaequilibria and their a-priori probability in metagames

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**Existence of Symmetric Metaequilibria and Their A-Priori  
Probability in Metagames**

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# *Existence of Symmetric Metaequilibria and Their A-Priori Probability in Metagames*

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## **Key Words:**

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## **Abstract:**

This article gives some results on the existence of symmetric metaequilibrium solutions for metagames in the case of two players with  $m$  and  $n$  strategies, respectively. We discuss the a-priori probability of  $m \times n$  metagames having symmetric metaequilibrium solutions and we give a lower bound for this probability. It turns out to be at least  $1-e^{-1}$  for any scale of two-player metagames. For reaching these goals, a metagames algorithm is introduced based on a characteristic function and we consider its features.

## **0. Introduction**

Metagame Theory (Howard, 1971) as a branch of classical game theory is a very useful analysis instrument in the decision making field. Compared with classical game theory, it is often more practical and easier to use because it considers only the order of preferences of the possible outcomes from the point of view of each player. Since concrete payoff values are not used, but only their ordinal significance (D.V. Lindley, 1985), it is very well fit for complex problems in competitive situations. An impressive example based on metagame theory is the conflict analysis methodology of Fraser and Hipel (1984). Metagame theory may give representative models for conflict situations in the real world,

especially in the case of two conflicting parties. In this article some basic properties of such metagames for two players are studied.

The significant outcomes or solutions in metagames analysis or conflict analysis for all players are called symmetric metaequilibria or symmetric metarational outcomes (L.C. Thomas, 1984. N.M. Fraser, K.W. Hipel, 1984). Such solutions are called metaequilibria in the paper for the purpose of simplification. Similar to classical game theory, a metaequilibrium in the case of two players does not always exist. This means that possibly for a given conflict problem we can not get stable outcomes for all players and the problem would have no solution in that situation. This article considers the existence of metaequilibria in the case of two players. We give several theorems for the existence of metaequilibria for metagames in the case of two players with  $m$  and  $n$  strategies, respectively. To prove the theorems, we first survey some main ideas to calculate the metaequilibria in the case of two players. Furthermore, an important characteristic function and its features, such as monotonicity properties are presented. Based on this function the theorems are proved. From these theorems we obtain a lower bound for the a-priori probability that metaequilibria exist. Taking the limit value for  $m, n \rightarrow \infty$  for  $m \times n$  metagames with two players we derive the general lower bound  $1 - e^{-1}$ .

## 1. Metagames: some notations and definitions

For a given metagame with two players, we denote  $I$  as player one,  $II$  as the other player. Player  $I$  has strategy set  $S^I$  with  $m$  strategies, player  $II$  has strategy set  $S^{II}$  with  $n$  strategies. According to the assessment of each strategy combination,  $(i, j) \in S^I \times S^{II}$ , each player forms its preference for any given situation. Suppose player  $I$  has a set of preferences ordered as  $P^I$  for the outcomes  $(i, j) \in S^I \times S^{II}$ , so we have

$$P^I = \{p^1, p^2, \dots, p^{mn}\} \text{ with } p^r \geq p^s, \text{ if } r > s; r, s \in \{1, 2, \dots, mn\}.$$

Suppose player  $II$  has its preferences ordered as  $Q^{II}$ :

$$Q^{II} = \{q^1, q^2, \dots, q^{mn}\} \text{ with } q^u \geq q^v, \text{ if } u > v; u, v \in \{1, 2, \dots, mn\}.$$

If both player  $I$  and player  $II$  have no repeated preferences with  $p^z = p^{z+1}$  and  $q^x = q^{x+1}$  we have

$$P^I = \{p^1, p^2, \dots, p^{mn}\} \text{ with } p^r > p^s, \text{ for any } r > s; r, s \in \{1, 2, \dots, mn\}.$$

$Q^H = \{q^1, q^2, \dots, q^{mn}\}$  with  $q^u > q^v$ , for any  $u > v$ ;  $u, v \in \{1, 2, \dots, mn\}$ .

The normal form (L.C. Thomas, 1984) of metagames consists of a double matrix, as shown in *Table 1*. The rows and columns of the matrix are related to strategies that are available to the players. The elements in the matrix are the pairs of preferences which are the outcomes of the metagame.

$$\begin{array}{c}
 \text{Player I:} \\
 S_1^I \\
 S_2^I \\
 \vdots \\
 S_m^I
 \end{array}
 \begin{array}{c}
 \text{Player II:} \\
 S_1^H \quad S_2^H \quad \dots \quad S_n^H \\
 \left[ \begin{array}{cccc}
 (p, q)_{11} & (p, q)_{12} & \dots & (p, q)_{1n} \\
 (p, q)_{21} & (p, q)_{22} & \dots & (p, q)_{2n} \\
 \vdots & \vdots & \dots & \vdots \\
 (p, q)_{m1} & (p, q)_{m2} & \dots & (p, q)_{mn}
 \end{array} \right]
 \end{array}
 \quad (1-1)$$

**Table 1**

Based on metagame theory, metarationality of an outcome (symmetric metaequilibria or symmetric metarational outcomes) for both players can be determined along the line of the Characterisation Theory (Howard, 1971), (L.C. Thomas, 1984):

For a  $m \times n$  metagame, let  $(P, Q)_{m \times n}$  be the double matrix of pair preferences for both players as introduced above and  $(p)_{ij}, (q)_{ij}$  the preference for situation  $(i, j)$  of each of the players. The symmetric metaequilibria set  $R$  is the intersection of sets  $R_1, R_2, R_3, R_4$ , i.e.,  $R = R_1 \cap R_2 \cap R_3 \cap R_4$ . The sets  $R_1, R_2, R_3, R_4$  are defined below:

$$R_1 = \{ (i, j) \mid (p)_{ij} \geq e_1 \}$$

$$e_1 = \min \{ \max_i (p)_{i1}, \max_i (p)_{i2}, \dots, \max_i (p)_{in} \} \quad (1-2)$$

$$R_2 = \{ (i, j) \mid (q)_{ij} \geq e_2 \}$$

$$e_2 = \max \{ \min_i (q)_{i1}, \min_i (q)_{i2}, \dots, \min_i (q)_{in} \} \quad (1-3)$$

$$R_3 = \{ (i, j) \mid (p)_{ij} \geq e_3 \}$$

$$e_3 = \max \{ \min_j (p)_{1j}, \min_j (p)_{2j}, \dots, \min_j (p)_{mj} \} \quad (1-4)$$

$$R_4 = \{ (i, j) \mid (q)_{ij} \geq e_4 \}$$

$$e_4 = \min\{\max_j(q)_{1j}, \max_j(q)_{2j}, \dots, \max_j(q)_{mj}\} \quad (1-5)$$

$$i=1, 2, \dots, m; j=1, 2, \dots, n.$$

Note that the set  $R$ , the solution set, can also be given as:

$$R=R_p \cap R_q \quad (1-6)$$

$$R_p = \{ (i, j) \mid (p)_{ij} \geq E_m \}$$

$$R_q = \{ (i, j) \mid (q)_{ij} \geq E_n \} \quad (1-7)$$

with

$$\max(e_2, e_4) = E_n$$

$$\max(e_1, e_3) = E_m. \quad (1-8)$$

As known, it is possible that there is not any stable outcome ( $R = \mathbf{O}$ ) for a given metagame in some cases. Here we give a simple example to show  $R = \mathbf{O}$ :

		<i>Player II:</i>		
		$S_1^II$	$S_2^II$	$S_3^II$
<i>Player I:</i>	$S_1^I$	(1, 4) <sub>11</sub>	(5, 8) <sub>12</sub>	(6, 9) <sub>13</sub>
	$S_2^I$	(4, 5) <sub>21</sub>	(2, 7) <sub>22</sub>	(7, 2) <sub>23</sub>
	$S_3^I$	(9, 1) <sub>31</sub>	(8, 3) <sub>32</sub>	(3, 6) <sub>33</sub>

According to the above metagames algorithm, for this concrete metagame we get:

$$E_n = 6, E_m = 7;$$

$$R_p = \{(i, j) \mid (3,1) (3,2) (2,3)\}, R_q = \{(i, j) \mid (1,2) (1,3) (2,2) (3,3)\}.$$

$$\text{Obviously, } R=R_p \cap R_q = \mathbf{O}.$$

Let us now investigate under what conditions metaequilibria do exist.

## 2. Some Characteristic Functions and Its Properties

We define two functions  $N$  and  $M$  from  $\omega \rightarrow \mathbb{N}$ . Here  $\omega$  is a finite subset of  $\mathbb{R}$ , to which we shall refer as a “digital field”. For a given  $\omega$  and definite number  $X \in \mathbb{R}$  we define

$$N(X, \omega) = \mathcal{L}\{x \in \omega, x \geq X\}$$

$$M(X, \omega) = \mathcal{L}\{x \in \omega, x < X\}$$

Where  $\mathcal{L}$  is the cardinality of the given set.

For example:  $\omega = \{1.25, 2, 4.1, 8, 5\}$ , then  $N(2)=4, N(8)=1; M(2)=1, M(8)=4$ .

Several useful properties of  $N(X, \omega)$  and  $M(X, \omega)$  are listed herebelow:

a. strict monotonicity properties:

If  $X^1 > X^2$ , then

$$N(X^2, \omega) > N(X^1, \omega)$$

$$M(X^1, \omega) > M(X^2, \omega)$$

If  $N(X^2, \omega) > N(X^1, \omega)$  or  $M(X^1, \omega) > M(X^2, \omega)$ , then

$$X^1 > X^2$$

b. monotonicity properties:

If  $X^1 \geq X^2$ , then

$$N(X^2, \omega) \geq N(X^1, \omega)$$

$$M(X^1, \omega) \geq M(X^2, \omega)$$

If  $N(X^2, \omega) \geq N(X^1, \omega)$  or  $M(X^1, \omega) \geq M(X^2, \omega)$ , then

$$X^1 \geq X^2.$$

c. complementarity properties

$$N(X, \omega) + M(X, \omega) = \mathcal{L}(\omega).$$

When we combine the characteristic function and its properties with two-player metagames, we get some further interesting properties.

Let us consider a  $m \times n$  metagame with  $e_1, e_2, e_3, e_4, E_m, E_m$  as defined before. We have:

d. If both players have strictly increasing preferences then one can easily derive from (1-

2) – (1-5)

$$n \leq N(e_1, P^I) \leq mn-m+1; m-1 \leq M(e_1, P^I) \leq mn-n.$$

$$n \leq N(e_3, P^I) \leq mn-m+1; m-1 \leq M(e_3, P^I) \leq mn-n.$$

$$m \leq N(e_2, Q^{II}) \leq mn-n+1; n-1 \leq M(e_2, Q^{II}) \leq mn-m.$$

$$m \leq N(e_4, Q^{II}) \leq mn-n+1; n-1 \leq M(e_4, Q^{II}) \leq mn-m.$$

If at least one of the players has equivalent preferences for some situations then

$$n \leq N(e_1, P^I) \leq mn; 0 \leq M(e_1, P^I) \leq mn-n$$

$$n \leq N(e_3, P^I) \leq mn; 0 \leq M(e_3, P^I) \leq mn-n$$

$$m \leq N(e_2, Q^{II}) \leq mn; 0 \leq M(e_2, Q^{II}) \leq mn-m$$

$$m \leq N(e_4, Q^{II}) \leq mn; 0 \leq M(e_4, Q^{II}) \leq mn-m.$$

e. If  $N(E_n, Q^{II}) > M(E_m, P^I)$ , or  $N(E_m, P^I) > M(E_n, Q^{II})$ , then the metagame has at least one metaequilibrium.

Because of property b,  $N(E_n, Q^{II}) > M(E_m, P^I)$  means  $N(E_n, Q^{II}) > mn-N(E_m, P^I)$  or  $N(E_n, Q^{II}) + N(E_m, P^I) > mn$ . Hence, there exists at least one situation  $(s, t)$  with a pair preference  $(p, q)_{st}$  which satisfies  $(q)_{st} \geq E_n$  and  $(p)_{st} \geq E_m$ . In line with (1-6), (1-9), we know that situation  $(s, t)$  is a metaequilibrium. The line of reasoning is the same in case of  $N(E_m, P^I) > M(E_n, Q^{II})$ .

$$f. N(E_m, P^I) = \min\{N(e_1, P^I), N(e_3, P^I)\}$$

$$N(E_n, Q^{II}) = \min\{N(e_2, Q^{II}), N(e_4, Q^{II})\}$$

It is because of (1-8) and property a.

$$g. N(E_m, P^I) \geq n; N(E_n, Q^{II}) \geq m.$$

If both players' preferences are strictly increasing then

$$N(E_m, P^I) \leq mn-m+1; N(E_n, Q^{II}) \leq mn-n+1.$$

If at least one of the players has equivalent preferences for different situations, then

$$N(E_m, P^I) \leq mn; N(E_n, Q^{II}) \leq mn.$$

This is due to the properties d and f.

h. Define  $Q^{row}$  as the “digital field” of all entries  $\subset Q^{II}$  generated by some row of pair preference matrix (1-1) and  $P^{col}, P^{col} \subset P^I$ , the “digital field” generated by the entries of some column, then

$$N(E_n, Q^{row}) \geq 1$$

$$N(E_m, P^{col}) \geq 1.$$

Proof: From (1-3),  $e_2 = \max\{\min(q)_{i1}, \min(q)_{i2}, \dots, \min(q)_{in}\}$ , we have for some  $r$

$e_2 = \min_j (q)_{jr}$ . So for column  $r$ ,  $N(e_2, Q^{colr}) = m$ . This means that for any row we have

$$N(e_2, Q^{row}) \geq 1.$$

From (1-5),  $e_4 = \min_q \{\max_j (q)_{1j}, \max_j (q)_{2j}, \dots, \max_j (q)_{mj}\}$ , we have for some  $s$

$e_4 = \min_j (q)_{sj}$ . Hence for any other rows, there is at least one entry with its  $q$  bigger than

$e_4$ . So  $N(e_4, Q^{rows}) = 1$ ,  $N(e_4, Q^{row\neq s}) \geq 1$ . It leads to  $N(e_4, Q^{row}) \geq 1$  for every row.

Considering the above results and the fact, that  $E_n = \max\{e_2, e_4\}$ , we obtain  $N(E_n, Q^{row}) \geq 1$  for every row.

With the same procedure it can be proved that the  $N(E_m, P^{col}) \geq 1$ .  $\square$

i. From property h, we can easily get another property:

If  $\exists r$  such that  $N(E_m, P^{rowr}) = n$ , then the metagame has a metaequilibrium solution;

or if  $\exists s$  such that  $N(E_n, Q^{cols}) = m$ , then the metagame has a metaequilibrium solution.

### 3. More on Existence Theorems and a-Priori Probability

#### Theorem 1.

*For a  $m \times n$  metagame, if all entries for one column of the preference matrix correspond to the worst preferences  $\{p^i, i=1, 2, \dots, m\}$  for player I, the metagame has a metaequilibrium solution.*

*If all entries of one row of the preference matrix correspond to the worst preferences  $\{q^j, j=1, 2, \dots, n\}$  for player II, the metagame has a metaequilibrium solution as well.*

**Proof:**

Assume that all entries for a column of preference matrix satisfy the condition of Theorem 1 for player I. Obviously, in line with (1-2) and (1-4)  $e_1$  is equal to  $p^m$  and  $e_3$  is also equal to  $p^m$ . So we have  $E_m = p^m$ . Then

$$M(E_m, P^I) = M(p^m, P^I) \leq m-1.$$

According to property g, we have  $N(E_n, Q^II) \geq m$ . So  $N(E_n, Q^II) > m-1 \geq M(E_m, P^I) = M(p^m, P^I)$ . From property e we know that the metagame has a metaequilibrium solution.

For a row, if all its entries satisfy the condition of Theorem 1 for player II, then in line with (1-3) and (1-5),  $e_2$  is equal to  $q^n$  and  $e_4$  is also equal to  $q^n$ . So we have  $E_n = q^n$ . Then

$$M(q^n, Q^II) = M(q^n, Q^II) \leq n-1.$$

According to property g, we have  $N(E_m, P^I) \geq n$ . So  $N(E_m, P^I) > n-1 \geq M(E_n, Q^II) = M(q^n, Q^II)$ . Based on property e we know that the metagame has a metaequilibrium solution.  $\square$

Note, that theorem 1 can also be stated as:

**Corollary 1.** *For a  $m \times n$  metagame of two players, it is always true that*

$$E_m \geq p^m, E_n \geq q^n.$$

*If  $E_m = p^m$ , or  $E_n = q^n$ , the metagame has a metaequilibrium solution.*

*For a metagame without metaequilibrium we must have that  $E_m > p^m$  and  $E_n > q^n$ .*

**Proof:**

According to Eq.(1-2),

$$e_1 = \min\{\max_i(p)_{i1}, \max_i(p)_{i2}, \dots, \max_i(p)_{in}\}, \text{ we have } e_1 = p^r, r \geq m. \text{ It means } p^r \geq p^m.$$

According to Eq.(1-4),

$e_3 = \max\{\min_j(p)_{1j}, \min_j(p)_{2j}, \dots, \min_j(p)_{mj}\}$ , we have  $e_3 = p^t$ ,  $t \geq m$ . It means  $p^t \geq p^m$ .

Since  $E_m = \max\{e_1, e_3\}$ , unequal  $E_m \geq p^m$ .

With the same reasoning we can prove  $E_n \geq q^n$ .  $\square$

**Corollary 2.** *For a  $m \times n$  metagame with strictly increasing preferences for each player, the a-priori probability of the metagame having a metaequilibrium solution, because player I's preferences satisfy the condition of theorem 1, is equal to or greater than*

$$\frac{m!(mn - m)!n}{(mn)!} \quad (3-10)$$

*The a-priori probability of a metagame having a metaequilibrium solution, because player II's preferences satisfy the condition of Theorem 1, is equal to or greater than*

$$\frac{n!(mn - n)!m}{(mn)!} \quad (3-11)$$

**Corollary 3.** *For a  $m \times n$  metagames of two players with strictly increasing preferences satisfying the condition of Theorem 1, the a-priori probability of having a metaequilibrium solution is equal to or greater than*

$$\frac{m!(mn - m)!n}{(mn)!} + \frac{n!(mn - n)!m}{(mn)!} - \frac{m!(mn - m)!n}{(mn)!} \frac{n!(mn - n)!m}{(mn)!} \quad (3-12)$$

From theorem 1 we can get an another interesting consequence: if for one player every strategy he plays when his opponent plays a certain strategy, will lead to his worst preferences, then the metagame must have at least one stable outcome.

**Theorem 2.**

*For a  $m \times n$  metagame of two players, if all entries for one row of the preference matrix correspond to the best preferences  $\{p^i, i = mn - n + 1, mn - n + 2, \dots, mn\}$  for player I, the metagame has a metaequilibrium. The metaequilibrium must exist in this row. If all entries for one column of preference matrix correspond to the best preferences  $\{q^j, j =$*

$mn-m+1, mn-m+2, \dots, mn\}$  for player II, the metagame has a metaequilibrium. The metaequilibrium must exist in this column.

Proof:

Assume all entries of the  $s$ -th row of the preference matrix correspond to the preferences  $\{p^i, i= mn-n+1, mn-n+2, \dots, mn\}$  for player I. In line with (1-2) and (1-4), we have  $e_1=p^{mn-n+1}, e_3=p^{mn-n+1}$ . So  $E_m=p^{mn-n+1}, N(E_m, P^{rows})=n$  for row  $s$ . According to property i, the metagame has a metaequilibrium solution. It is obvious that the in  $s$ -th row there is at least one situation with  $p \geq E_m$  and  $q \geq E_n$ , which means that in the  $s$ -th row there must be a metaequilibrium.

With the same reasoning, we can prove metaequilibrium existence in case all entries for one column are the preferences  $\{q^j, j=mn-m+1, mn-m+2, \dots, mn\}$  for player II.  $\square$

From theorem 2 we can also get another interesting consequence: when one player has a definite strategy to play against any strategy of his opponent, which will lead to his best preferences, then the metagame must have at least one stable outcome.

**Corollary 4.** For a  $m \times n$  metagame of two players, it is true that  $E_m \leq p^{mn-n+1}, E_n \leq q^{mn-m+1}$ .

Proof:

Suppose  $E_m > p^{mn-n+1}$ , we have  $E_m \geq p^{mn-n+1+r} > p^{mn-n+1}$ . Because of  $N(E_m, P^I) \leq N(p^{mn-n+1+r}, P^I)$  (according to property b) and  $N(p^{mn-n+1+r}, P^I) = n-r \leq n-1$ , we have  $N(E_m, P^I) \leq n-1$ . Due to property g,  $N(E_m, P^I) \geq n$ . So  $E_m \leq p^{mn-n+1}$ .

Suppose  $E_n > q^{mn-m+1}$ , we have  $E_n \geq q^{mn-m+1+s} > q^{mn-m+1}$ . Also  $N(E_n, Q^{II}) \leq N(q^{mn-m+1+s}, Q^{II})$  (according to property b) and  $N(q^{mn-m+1+s}, Q^{II}) = n-s \leq m-1$ , so  $N(E_n, Q^{II}) \leq m-1$ . Due to property g,  $N(E_n, Q^{II}) \geq m$ . So  $E_n \leq q^{mn-m+1}$ .  $\square$

Analogous to before we have corollary 5 and corollary 6:

**Corollary 5.** *For a  $m \times n$  metagame of two players with strictly increasing preferences for each player, the a-priori probability of having a metaequilibrium solution because player I's best preferences correspond to one row of the preference matrix is equal to or greater than*

$$\frac{n!(mn-n)!m}{(mn)!} \quad (3-13)$$

*The a-priori probability of a metagame having a metaequilibrium solution because player II's best preferences correspond to one column of the preference matrix is equal to or greater than*

$$\frac{m!(mn-m)!n}{(mn)!} \quad (3-14)$$

**Corollary 6.** *For a  $m \times n$  metagame of two players with strictly increasing preferences satisfying theorem 2 for each player, the a-priori probability of having a metaequilibrium solution is equal to or greater than*

$$\frac{m!(mn-m)!n}{(mn)!} + \frac{n!(mn-n)!m}{(mn)!} - \frac{m!(mn-m)!n}{(mn)!} \frac{n!(mn-n)!m}{(mn)!} \quad (3-15)$$

It is easy to conclude the a-priori probability of having a metaequilibrium solution when two players preferences satisfy both Theorem 1 condition and Theorem 2 condition. The a-priori probability is equal to or greater than

Eq.(3-15)+Eq.(3-15)-Eq.(3-15)×Eq.(3-15).

**Theorem 3.**

*For a  $m \times n$  metagame of two players, if there exists a situation  $(s, t)$  with*

*$(p)_{st} \in \{p^{mn-n+1}, p^{mn-n+2}, \dots, p^{mn}\}$  and  $(q)_{st} \in \{q^{mn-m+1}, q^{mn-m+2}, \dots, q^{mn}\}$ , the metagame has a metaequilibrium solution and the  $(s, t)$  must be a metaequilibrium.*

**Proof:** Because of  $(p)_{st} \geq p^{mn-n+1}$ , according to corollary 4, it is obvious that

$$(p)_{st} \geq E_m .$$

With corollary 4, we also have

$$(q)_{st} \geq E_n .$$

Obviously from (1-6) and (1-9), we know that  $(s, t)$  must be a metaequilibrium.  $\square$

#### Theorem 4.

*For a  $m \times n$  metagame of two players with strictly increasing preferences for each players, the a-priori probability of having a metaequilibrium solution because the players preferences satisfy Theorem 3 is equal to or greater than*

$$1 - \frac{(mn - n)!(mn - m)!}{(mn)!(mn - m - n)!} \quad (3-16)$$

The formulas (3-10) to (3-16) can be obtained counting arrangements and combinations. Here we take Theorem 3 as example to get formula (3-16).

We define

$$A_m^n = n! \binom{m}{n} = \frac{m!}{(m-n)!} .$$

For a  $m \times n$  metagame, the numbers of all cases satisfying Theorem 3 condition are equal to the number of all arrangement cases minus the number of cases unsatisfied Theorem 3, i.e.,

$$A_{mn}^{mn} \times A_{mn}^{mn} - A_{mn}^{mn} \times A_{mn-m}^n \times A_{mn-n}^{mn-n} \quad (3-17)$$

The first  $A_{mn}^{mn}$  accounts for player  $I$ 's ordinal preferences;

The second and the third  $A_{mn}^{mn}$  account for player  $II$ 's ordinal preferences;

$A_{mn-m}^n$  is the number of arrangements of all top preferences for player  $I$  such that there is no intersection with the top preferences of player  $II$ ;

$A_{mn-n}^{mn-n}$  is the number of all arrangements of non-top preferences for player  $I$  in the remaining  $mn-n$  surplus positions;

Formula (3-17) is equivalent to

$$\begin{aligned}
& (mn)!(mn)! - (mn)! \frac{(mn-m)!}{(mn-m-n)!} (mn-n)! \\
& = (mn)! \left[ (mn)! - \frac{(mn-m)!(mn-n)!}{(mn-m-n)!} \right]
\end{aligned} \tag{3-18}$$

When (3-18) is divided by  $A_{mn}^{mn} \times A_{mn}^{mn}$ , the number of all arrangements for two players, we get the probability:

$$1 - \frac{(mn-n)!(mn-m)!}{(mn)!(mn-m-n)!}$$

as mentioned.  $\square$

**Corollary 7.**  $1 - \frac{(mn-n)!(mn-m)!}{(mn)!(mn-m-n)!}$  is a decreasing sequence in  $m$  and  $n$ . It has the limit value  $1 - e^{-1}$ .

Corollary 7 shows that for any metagame the a-priori probability of metaequilibrium existence is more than 0.632.

**Theorem 5.**

*For a  $m \times n$  metagame of two players, if  $E_m = p^r$ ,  $E_n = q^s$ ,  $r+s-2 < mn$ , then the metagame has a metaequilibrium solution.*

Proof:

The proof is simple.

$$N(E_m, P^I) = N(p^r, P^I) \geq mn - r + 1$$

$$M(E_n, Q^{II}) = M(q^s, Q^{II}) \leq s - 1$$

Because of  $r+s-2 < mn$ , so  $N(p^r, P^I) > M(q^s, Q^{II})$  and then  $N(E_m, P^I) > M(E_n, Q^{II})$ .

In line with property e the metagame has a metaequilibrium solution.  $\square$

According to property e and theorem 5 we have corollary 8:

**Corollary 8.** *If a metagame has a metaequilibrium solution, then the number of metaequilibria of the metagame is at least  $N(E_m, P^I) + N(E_n, Q^{II}) - mn$ .*

In order to investigate the accuracy of the bounds given in the above theorems, we have calculated the exact a-priori probability of existing metaequilibrium solution to a metagame:

For  $2 \times 2$  metagame, the a-priori probability of metaequilibrium existence is more than 0.926 due to above theorems. We also calculated the accurate result using simulation based on original metagames formula. The accurate probability is 0.944.

For  $2 \times 3$  or  $3 \times 2$  metagame, the a-priori probability of existing metaequilibrium solution is more than 0.845 based on the above theorems. The accurate result is 0.941.

For  $3 \times 3$  metagame, the a-priori probability of existing metaequilibrium solution is more than 0.779 based on the above theorems. If, for a metagame its range is more than  $3 \times 3$  it is too difficult to get accurate result because of huge quantity of calculation.

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