Functions whose differences belong to a given class

FUNCTIONS WHOSE DIFFERENCES BELONG TO A GIVEN CLASS

BY

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§ 1. *Introduction.* Let the real function $f(x)$, defined for $-\infty < x < \infty$, have the property that the difference $\Delta_h f(x) = f(x+h) - f(x)$ is, for each value of h, a continuous function of x ($-\infty < x < \infty$). Then $f(x)$ need not be continuous itself. For the functional equation $\mathrm{H}(x) + \mathrm{H}(y) = \mathrm{H}(x+y)$ is known to have non-measurable solutions [1]), and for such a function the difference $\Delta_h \mathrm{H}(x)$ is constant, and a fortiori continuous.

The solutions $\mathrm{H}(x)$ of the equation just mentioned are called *additive functions.*

Recently MARY L. BOAS and R. P. BOAS jr. proved [2]) that a function $f(x)$ with continuous difference $\Delta_h f(x)$, for each h, is continuous itself, provided that $f(x)$ is bounded on a set of positive measure. This generalizes the following theorem of OSTROWSKI (see [**6**, **5**]): If an additive function is bounded on a set of positive measure, then it is of the form $\mathrm{H}(x) = c\,x$.

The result of BOAS and BOAS led P. ERDÖS to the following conjecture which will be proved in this paper (§ 2).

Theorem 1.1. *If $f(x)$ is such that, for each h, $\Delta_h f(x)$ is a continuous function of x, then it can be written in the form $g(x) + H(x)$, where $g(x)$ is continuous, and $H(x)$ is additive.*

Clearly the theorem of BOAS and BOAS now immediately follows from OSTROWSKI's theorem.

We shall also consider a number of special cases of the

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1) HAMEL [**3**]: bracketed numbers refer to the bibliography at the end.

2) Unpublished M.S.

following more general problem. Let C be a class of real functions defined for $-\infty < x < \infty$. C will be said to have the *difference property*, if any real function $f(x)$ such that, for each h, $\Delta_h f(x) \in C$, is of the form $f(x) = g(x) + H(x)$, where $g(x) \in C$, and $H(x)$ is additive.

We shall discuss a number of classes, which will be numbered $C_0, C_1, \ldots$. The most important of them which turn out to have the difference property, are: C_0 (continuous functions), $C_2(k)$ (functions with k derivatives), C_3 (analytical functions), C_5 (functions which are absolutely continuous in any finite interval) and C_7 (functions having bounded variation over any finite interval).

The difference property cannot be proved [3]) for the class of measurable functions, not even for the class of bounded measurable functions. For under the assumption that the continuum hypothesis holds, SIERPINSKI (see [9], p. 135, prop. C_{70a}) constructed a non-measurable function $S(x)$ $(-\infty < x < \infty)$ which only attains the values 0 and 1, and such that, for each value of h, $S(x+h) - S(x) = 0$ with exception of at most a countable number of x-values. Clearly, the function $\Delta_h S(x)$ is measurable for each h, but $S(x)$ is not the sum of a measurable function $g(x)$ and an additive function $H(x)$. Otherwise $H(x)$ would be bounded on a set of positive measure. Hence, by OSTROWSKI's theorem, it would be measurable, which is impossible since $g(x)$ is measurable but $S(x)$ is not.

We know that the continuum hypothesis cannot be disproved (see [2]), and so the class of measurable functions cannot be proved to have the difference property.

ERDÖS conjectured furthermore, that if any function $f(x)$ has the property that, for each h, $\Delta_h f(x)$ is measurable, then it can be written in the form $f(x) = g(x) + H(x) + S(x)$, where $g(x)$ is measurable, $H(x)$ is additive and $S(x)$ has, for each h, the property that $S(x+h) = S(x)$ for almost all x. We can prove this (§ 5) for the class of functions integrable (L_2) (over any finite interval) instead of measurable functions.

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[3]) This remark is due to P. ERDÖS.

Another class missing the difference property is the class of all bounded functions (§ 7). The class of RIEMANN integrable functions probably has the difference property, but we could not decide this.

In all cases, the first problem is to locate the function $H(x)$; the second one is to prove that $f(x) - H(x)$ belongs to the given class. The second problem has a different nature in each case, but the first one can be solved quite generally. We shall evolve two methods; the first one is explained in § 2 and generalized in theorem 4.2, the second one uses integration and is given in theorem 4.3. In some cases, e.g. the class of continuous functions, both methods apply.

The key theorem in the first method is

Theorem 1.2 [4]). *If $f(x)$ is defined for $-\infty < x < \infty$, and if, for all x and y we have*

$$| f(x+y) - f(x) - f(y) + f(0) | \leqslant 1,$$

then there exists an additive function $H(x)$, such that $| f(x) - f(0) - H(x) | \leqslant 1$ for all values of x.

The theorem is known for the case that $f(x)$ is defined for $x = 0, \pm 1, \pm 2, \ldots.$ only, satisfying (1.1) for all integers x and y (see [**8**], p. 17 and p. 171). In that case, any additive function is linear.

Theorem 1.2 admits wide generalizations (see theorem 4.1). It remains true, for instance, if x and y run through the elements of an additive abelian group, and $f(x)$ has its values in a BANACH space. We do not give a separate proof of theorem 1.2, since it is contained [5]) in theorem 4.1, and the specialization to theorem 1.2 seems to give no essential simplification in the proof. Moreover, theorem 1.2 is easily deduced from the special case of functions of an integral variable, mentioned above.

Accordingly, theorem 1.1 can be extended, for instance, to functions of many variables: *If $f(x, y)$ is such that, for any h and k, the difference $f(x+h, y+k) - f(x, y)$ is a continuous function of x and y, then we have $f(x, y) = g(x,y) +$*

[4]) Theorem 1.2 expresses that a certain class C_{10} (see § 7) has the difference property.

[5]) Put $\| f(x) \| = | f(0) |$, and consider Remark 2 after theorem 4.1.

$+ \mathrm{H}(x, y)$, *where g is continuous, and H is additive in both variables*. It it is not difficult to make the necessary alterations in the proof.

We return to the ordinary case of functions of one real variable. We can make the following statement which is generally applicable to all classes considered in this paper, except in the cases considered in § 3 [6]). For simplicity we express it for C_0 (continuous functions): In order to prove theorem 1.1 is is sufficient to show

Lemma 1.1. *If $f(x)$ is periodic mod 1 and such that, for each h, $\Delta_h f(x)$ is a continuous function of x, then $f(x) = g(x) + H(x)$, where $g(x)$ is a continuous function of period 1, and $H(x)$ is additive.*

From Lemma 1.1, which will be proved in § 2, we can derive the following extension of theorem 1.1:

Theorem 1.3. *If $f(x)$ is defined in an interval J, and if $f(x)$ is such that, for each h, $\Delta_h f(x)$ is continuous* [7]) *for all values of x which are such that $x \in J$, $x+h \in J$, then we have $f(x) = g(x) + H(x)$, where $g(x)$ is continuous in J, and $H(x)$ is additive.*

Boas and Boas proved their theorem also for this case.

Assuming lemma 1.1 to be true, we shall prove theorem 1.3. Without loss of generality we may assume that the interval $0 \leqslant x \leqslant 2$ is contained in J. Furthermore we may assume that $f(0) = f(1)$, since in the general case it is sufficient to prove the theorem for the function $f_1(x) = f(x) - x\{f(1) - f(0)\}$. We now define the periodic function $f^*(x)$ by $f^*(x) = f(x)$ $(0 \leqslant x \leqslant 1)$, $f^*(x+1) = f^*(x)$ $(-\infty < x < \infty)$. The difference $\chi(x) = f^*(x) - f(x)$ is continuous throughout J. We shall show that for any x_1 we have $f(x) \to f(x_1)$ as x tends to x_1 from the right; the proof for approximation from the left is analogous. Therefore assume $x_1 \in \mathrm{J}$, $x \in \mathrm{J}$, $x \downarrow x_1$.

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[6]) In the cases dealt with in § 3 the analogues of theorem 1.3 are also easily obtained, but it is not convenient (and for the case of analyticity it is impossible) to derive them from the corresponding theorems for the periodic case.

[7]) If J is closed, we have, of course, to consider one-sided continuity in the endpoints.

Let n be the integer satisfying $0 \leqslant x_1 - n < 1$. Then, if $x \downarrow x_1$, we have $f(x) = f^*(x+n)$, $f(x_1) = f^*(x_1+n)$. Therefore $\chi(x) - \chi(x_1) = \Delta_n f(x) - \Delta_n f(x_1) \to 0$ as $x \downarrow x_1$, $\Delta_n f(x)$ being continuous in J.

Since $f^*(x) - f(x)$ is now proved to be continuous, we immediately infer that, for each h $(0 \leqslant h \leqslant 1)$, $\Delta_h f^*(x)$ is continuous for $0 \leqslant x \leqslant 1$. By lemma 1.1, we infer that $f^*(x) = g^*(x) + \mathrm{H}(x)$, $g^*(x)$ continuous everywhere, $\mathrm{H}(x)$ additive. Consequently we have, for $x \in \mathrm{J}$, $f(x) = \{g^*(x) - \chi(x)\} + \mathrm{H}(x)$. Since $\chi(x)$ is continuous in J, this completes the proof of theorem 1.3.

In § 2 we give the proof of lemma 1.1, and a generalization of theorem 1 to functions whose jumps are all $\leqslant 1$. This extension was suggested by Boas and Boas, who proved their theorem for this case also. In § 3 we prove the difference property for some cases where it is a comparatively simple consequence of the difference property for continuous functions. In § 4 we give some fairly general theorems upon which § 5 and § 6 are based. In § 7 we discuss the class of bounded functions.

§ 2. *Continuity; bounded jumps.* We already observed that in order to prove theorems 1.1 and 1.2 it is sufficient to prove lemma 1.1 Assume therefore $f(x)$ to be periodic mod 1, and, for each h, $\Delta_h f(x)$ to be continuous for all x. Consider the function

$$\varphi(x, h) = f(x+h) - f(x) - f(h) + f(0). \qquad (2.1)$$

For any fixed value of h, $\varphi(x, h)$ is a continuous function of x. Since $\varphi(x, h) = \varphi(h, x)$, it is also, for each x, a continuous function of h. Hence, by a theorem of Baire (see [**1**]) there exists a point $x = x_0$, $h = h_0$ where $\varphi(x, h)$ is continuous with respect to the two variables x and h.

Owing to the special form of $\varphi(x, h)$ it is now easily proved that $\varphi(x, h)$ is continuous everywhere. In order to prove the continuity at $x = u$, $h = v$, we write

$$\varphi(u+\lambda, v+\mu) = \varphi(x_0+\lambda, h_0+\mu) + \{f(u+v+\lambda+\mu) - f(x_0+h_0+\lambda+\mu)\} + \{f(x_0+\lambda) - f(u+\lambda)\} + \{f(h_0+\mu) - f(v+\mu)\}.$$

The right-hand-side consists of 4 terms; each one is, for fixed values of u, v, x_0, h_0, a continuous function of the two variables λ and μ at the point $\lambda = \mu = 0$.

Since $f(x)$ is periodic mod 1, the function $\varphi(x, h)$ is a periodic function of both x and h, with period 1. It is a continuous funtion of both variables; consequently it is uniformly bounded:

$$| \varphi(x, h) | \leqslant M \quad (-\infty < x < \infty, \ -\infty < h < \infty) \qquad (2.2)$$

Therefore by theorem 1.2, an additive function $H(x)$ can be found such that

$$f(x)=g(x) + H(x), \ | g(x) - g(0) | \leqslant M. \ (-\infty< x< \infty) \qquad (2.3)$$

At this point we could use the result of BOAS and BOAS which shows that $g(x)$ is continuous. But this can be proved independently in a few lines. For $n = 1, 2, \ldots.,$ we have the identity

$$\sum_{\nu=1}^{n-1} \varphi(\nu h, h) = g(nh) - g(0) - n \{g(h) - g(0)\}. \qquad (2.4)$$

The function $\varphi(x, h)$ is periodic mod 1 and continuous everywhere; it vanishes for $h = 0$. Therefore we can, given $\varepsilon > 0$, determine $\delta > 0$ such that $| \varphi(x, h) | < \varepsilon$ for $| h |< \delta$, $-\infty < x < \infty$. By making $n \to \infty$ in (2.4), using (2.3), we infer that $| g(h) - g(0) | \leq \varepsilon$ $(| h |< \delta)$. So $g(x)$ is continuous at $x = 0$. Since all its differences are continuous we obtain that $g(x)$ is continuous everywhere, and lemma 1.1 is proved.

We next consider the class C_1, consisting of all functions $g(x)$ satisfying

$$\limsup_{\delta \to 0} | g(x + \delta) - g(x) | \leq 1 \qquad (2.5)$$

for each x $(-\infty < x < \infty)$. We shall prove that C_1 has the difference property. First assume $f(x)$ to be periodic mod 1 and to be such that $\Delta_h f(x) \in C_1$ for each h. Then, by a suitable generalisation of BAIRE's argument, which we do not discuss in detail, we can prove that $\varphi(x, h)$ (see (2.1)) is uniformly bounded. It follows, by theorem 1.2, that $f(x) = g(x) + H(x)$, where $H(x)$ is additive and $| g(x) | \leq M$ for some M and all x. Now dropping the assumption of periodicity, we can prove the same for arbitrary functions $f(x)$ whose differences belong to C_1, and for an arbitrary finite x-interval. The argument is the one explained in § 1; we have to bear in mind that any function of C_1 is bounded in any finite interval.

The proof will be now completed by showing

Lemma 2.1. *If $g(x)$ is bounded in any finite interval, and if $\Delta_h g(x) \in C_1$ for each h, then $g(x) \in C_1$.*

Proof. Assume that this is false, that is, for some number u we have either

$$\limsup_{\delta \downarrow 0} \{g(u+\delta) - g(u)\} > A > 1, \tag{2.6}$$

or one of the three other possibilities which violate (2.5). The four cases are completely analogous. We shall assume (2.6) and derive a contradiction.

We know that all the differences $\Delta_h g(x)$ are in C_1; it easily follows that we have, for any x,

$$\limsup_{\delta \downarrow 0} \{g(x+\delta) - g(x)\} > A - 1.$$

We can now construct an infinite sequence $0 = x_0 < x_1 < < x_2 < \ldots. < 1$ such that $g(x_{n+1}) - g(x_n) > A - 1$ $(n = 0, 1, 2, \ldots.)$. This contradicts the fact that $g(x)$ is bounded over $0 \leq x \leq 1$.

§ 3. *Differentiability, analyticity.* In the cases to be dealt with in this § we need not go into the trouble of constructing a suitable additive function. We will be able to depend on theorem 1.1.

Let, for $k = 1, 2, 3, \ldots.$ or $k = \infty$, $C_2(k)$ denote the class of functions $f(x)$ which are, for $-\infty < x < \infty$, differentiable k times. Furthermore, for $k = 1, 2, 3, \ldots.$, $C_2^*(k)$ denotes the class of all functions of $C_2(k)$ whose k-th derivative is continuous. These classes all have the difference property. In order to show this it is sufficient to prove (see theorem 1.1):

Lemma 3.1. *If C is one of the classes $C_2(k)$ or $C_2^*(k)$ for some k, and if the continuous function $f(x)$ is such that $\Delta_h f(x) \in C$ for each h, then we have $f(x) \in C$.*

Proof. First consider $C_2(1)$. Let $f(x)$ satisfy the conditions of the lemma. We can construct a polynomial $P(x)$ satisfying $P(0) = f(0)$, $P(1) > f(1)$, $P(2) = f(2)$, $P(3) < f(3)$,

$P(4) = f(4)$. Clearly $g(x) = f(x) - P(x)$ attains a minimum at a point ξ $(0 < \xi < 2)$, and a maximum at a point η $(2 < \eta < 4)$. We know that $\Delta_{\eta-\xi}\, g(x)$ is differentiable everywhere. Therefore, the function $g(x + \eta) - g(x + \xi)$ has a derivative at $x = 0$, and it follows from the extremal properties of $g(\xi)$ and $g(\eta)$ that $g'(\xi)$ and $g'(\eta)$ both exist (and that $g'(\xi) = g'(\eta) = 0$). As $\{\Delta_h\, g(x)\}'$ exists for each h, it follows that $g'(x)$ exists everywhere.

Secondly, consider $C_2^*(1)$. Assume $f(x)$ continuous, and $\Delta_h f(x) \varepsilon\, C_2^*(1)$ for each h. As $C_2(1)$ has the difference property, we know that $f'(x) = g(x)$ exists. $\Delta_h f(x) \varepsilon\, C_2^*(1)$ now means that $\Delta_h\, g(x)$ is continuous. Therefore, by theorem 1.1, $g(x) = G(x) + H(x)$, $G(x)$ continuous, $H(x)$ additive. But $g(x)$, being a derivative, is measurable; therefore $H(x) = g(x) - G(x)$ is measurable.

It follows that $H(x)$ is continuous (see the theorem of Ostrowski referred to in § 1); hence $f'(x)$ is continuous.

Finally, if we consider $C_2(k)$ or $C_2^*(k)$ with $k > 1$, then we are given $f(x)$ continuous, $\Delta_h f(x) \varepsilon\, C_2(k)$ or $C_2^*(k)$. It follows that $\Delta_h f(x) \varepsilon\, C_2^*(1)$. As $C_2^*(1)$ has the difference property, we obtain that $f(x)$ has a continuous derivative. So if $f'(x) = g(x)$, we have $g(x)$ continuous, $\Delta_h\, g(x) \varepsilon\, C_2(k-1)$ or $C_2^*(k-1)$. The proof now follows by induction.

We shall now prove that C_3 has the difference property, where C_3 is the class of all real functions $f(x)$ which are analytical for $-\infty < x < \infty$.

Let f be such that $f(x + h) - f(x) \varepsilon\, C_3$ for each value of h. We have $C_3 \subset C_2(\infty)$, and $C_2(\infty)$ has the difference property. It follows that $f(x)$ is of the form $g(x) + H(x)$, where $g(x)$ has derivatives of all orders for $-\infty < x < \infty$, and $H(x)$ is additive. Furthermore

$$g(x + h) - g(x) \varepsilon\, C_3 \text{ for each } h. \tag{3.1}$$

We have to show that $g(x)$ is analytical for $-\infty < x < \infty$.

Let, for each h, $N(h)$ denote the smallest positive integer with the property that, uniformly for $n = 1, 2, 3, \ldots .,$

$$| g^{(n)}(h) - g^{(n)}(0) | \leq \{N(h)\}^n \,.\, n! \tag{3.2}$$

The existence of $N(h)$ follows from (3.1).

The interval $(-\infty, \infty)$ is not the union of a countable number of nowhere dense sets. Therefore, if S_K is the set of h's with the property that $N(h) = K$, then there exists an interval $p \leq h \leq q$, and an integer $K > 0$, such that S_K is everywhere dense in $[p, q]$. It follows, since $g^{(n)}(x)$ is continuous, that

$$| g^{(n)}(h) - g^{(n)}(0) | \leq K^n . n! \qquad (n = 1, 2, 3, \ldots ; \ p \leq h \leq q).$$

Consequently we have

$$| g^{(n)}(u) - g^{(n)}(v) | \leq 2K^n . n!$$
$$(n = 1, 2, 3, \ldots ; p \leq u \leq q, \ p \leq v \leq q) \tag{3.3}$$

where K does not depend on the variables n, u, v.

If a function $F(x)$ has n derivatives for $p \leq x \leq q$, then we have the inequality

$$\min_{p \leq x \leq q} | F^{(n)}(x) | \leq \left(\frac{2n}{q-p}\right)^n \max_{p \leq x \leq q} | F(x) | . \tag{3.4}$$

This follows from the well-known formula for the n-th difference:

$$\sum_{\nu=0}^{n} (-1)^{n-\nu} \binom{n}{\nu} F\left\{p + \frac{\nu}{n}(q-p)\right\} = \left(\frac{q-p}{n}\right)^n F^{(n)}(\xi)$$

for some suitable ξ $(p < \xi < q)$.

Applying (3.4) to (3.3), we obtain

$$| g^{(n)}(p) | \leq 2K^n n! + \min_{p \leq x \leq q} | g^{(n)}(x) | \leq 2K^n n! + \left(\frac{2n}{q-p}\right)^n A, \tag{3.5}$$

where A is the maximum of $| g(x) |$ for $p \leq x \leq q$. It follows that $| g^{(n)}(p) | < C^n n!$ $(n = 1, 2, \ldots)$ for some constant C, and so $g(x)$ is analytical at $x = p$. And, since all differences of $g(x)$ are analytical, $g(x)$ is analytical for $-\infty < x < \infty$.

Let, for $a > 0$, $C_3(a)$ denote the class of functions which are analytical for $-\infty < x < \infty$, and which can be continued analytically throughout a strip $| Im\, x | \leq a$. $C_3(\infty)$ denotes the class of all entire functions. These classes also have the difference property. In order to prove this, for

$0 < a < \infty$, we can repeat the proof for C_3, with a few alterations. We replace (3.2) and (3.3) by

$$| g^{(n)}(x + h) - g^{(n)}(x) | \leq N(h)\, a^{-n}\, n! \quad (n = 1, 2, 3, \ldots; \ 0 \leq x \leq 2ea)$$

$$| g^{(n)}(x + u) - g^{(n)}(x + v) | \leq 2K\, a^{-n}\, n! \quad (n = 1, 2, 3, \ldots; \ 0 \leq x \leq 2ea, \ p \leq u \leq q; \ p \leq v \leq q. \qquad (3.6)$$

Now put $q - p = \delta$, and $[2ea\delta^{-1}] + 1 = k$. From (3.6) we infer that in each one of the k intervals $[p, p + \delta]$, $[p + \delta, p + 2\delta]$, $\ldots$, $[p + (k - 1)\delta, \ p + k\delta]$, the fluctuation of $g^{(n)}$ is less than $2K\, a^{-n}\, n!$ Hence we obtain that, for $p \leq u \leq p + 2ea$, $p \leq v \leq p + 2ea$, we have

$$| g^{(n)}(u) - g^{(n)}(v) | \leq 2k\, K\, a^{-n}\, n! = O(a^{-n}\, n!)$$

Now (3.4) being applicable to an interval of length $2ea$, the right-hand-side becomes $O\{(n\, e^{-1}\, a^{-1})^n\} = O(a^{-n}\, n!)$. The end of the proof is the same as for C_3.

$C_3(\infty)$ also has the difference property. This follows from the fact that it is $\lim C_3(a)$, in the set-theoretical sense.

The class C_4 consisting of all polynomials has the difference property. For, assume $\Delta_h f \in C_4$ for each h, then it follows by theorem 1.1 that $f(x) = g(x) + H(x)$, where $H(x)$ is additive, and $g(x)$ is continuous. The differences of $g(x)$ are polynomials. We can determine a polynomial $P(x)$ such that $P(x + 1) - P(x) = g(x + 1) - g(x)$, the right-hand-side being a polynomial. Now $g(x) - P(x) = g_1(x)$ is a continuous function of period 1. Therefore $| g_1(x) | \leq M$ for all x. So for each h, $\Delta_h\, g_1(x)$ is a bounded polynomial, and hence it is of the form $c(h)$, not depending on x. Clearly we have $c(nh) = n\, c(h)$ for all integers n, and $| c(h) | \leq 2M$ for all h. It follows that $c(h)$ vanishes identically, and so $g_1(x)$ is a constant. This proves that $g(x) = g_1(x) + P(x)$ is a polynomial.

§ 4. *Some general theorems.* We consider a Banach space Ω of real functions $f(x)$ $(-\infty < x < \infty)$, that is a space with the following properties:

(I) Ω is linear, that is, $g_1 \in \Omega$, $g_2 \in \Omega$ imply $ag_1 + bg_2 \in \Omega$ for each pair of real numbers a, b.

(II) For any $g \in \Omega$ a non-negative norm $\| g \|$ is defined; we have $\| a g \| = | a | . \| g \|$ for any real number a and any $g \in \Omega$; and $\| g_1 + g_2 \| \leq \| g_1 \| + \| g_2 \|$ for any pair $g_1 \in \Omega$, $g_2 \in \Omega$.

(III) $\| g \| = 0$ if and only if g vanishes identically.

(IV) Ω is complete with respect to this norm, that is to say, if the sequence $\{g_n\}$ ($g_n \in \Omega$, $n = 1, 2, 3, \ldots$) is such that $\| g_n - g_m \| \to 0$ as $n \to \infty$, $m \to \infty$, then there exists a function $g \in \Omega$ such that $\| g_n - g \| \to 0$ as $n \to \infty$.

We shall first prove the following extension of theorem 1.2:

Theorem 4.1. *Let* $f(x)$ [8]) *be such that* $\Delta_h f(x) \in \Omega$ *for all values of* h, *and* $\Delta_h \Delta_k f(x) \in \Omega$, $\| \Delta_h \Delta_k f(x) \| \leq 1$ *for all values of* h *and* k. *Then there exists a real-valued additive function* $H(x)$, *such that* $\Delta_h \{f(x) - H(x)\} \in \Omega$, $\| \Delta_h \{f(x) - H(x)\} \| \leq 1$ *for all values of* h.

Remarks 1. The theorem remains true, without essential alterations in the proof below, if we replace the condition (III) by the weaker condition (III*):

(III*) If $g \in \Omega$, and $\| g(x + h) - g(x) \| = 0$ for all h, then there exists a constant [9]) $C \in \Omega$, such that $\| g(x) - C \| = 0$.

2. The theorem remains true, and the proof needs no alterations, if the domain of x, which was the set of real numbers, is replaced by an arbitrary additive abelian group G, and if the range of values of $f(x)$, which also was the set of real numbers, is replaced by an arbitrary linear space S. Then Ω has to be a sub-set of the set of all functions G to S, and Ω is assumed to satisfy (I), (II) and (III*). H(x) is an additive function, defined on G, with values in S. As to (I) and (II) we may remark that it is sufficient to require these conditions for rational values of a and b only.

An attractive example is, that S is a Banach space itself, and Ω is the set of all functions G into S. Taking for $f(x) \in \Omega$ the norm $\| f \|_\Omega$ to be equal to the norm $\| f(0) \|_S$ of $f(0)$

[8]) $f(x)$ need not be an element of Ω.

[9]) A constant is a function having the same value for all values of x.

in S, then (I), (II) and (III*) are satisfied. We then obtain from Theorem 4.1: *If* $f(x)$ *is a function of* G *into* S, *and* $\| f(x+y) - f(x) - f(y) + f(0) \|_S \leq 1$ *for all* $x \in G$, $y \in G$, *then there exists an additive function* $H(x)$ *of* G *into* S, *such that* $\| f(x) - f(0) - H(x) \|_S \leq 1$ *for all* x.

Taking for both H and S the set of real numbers, we obtain theorem 1.2 as a special case.

P r o o f o f t h e o r e m 4.1. We specialize by taking for h and k integral multiples of a fixed number ξ. In the first place we have ($n = 1, 2, \ldots.$),

$$\Delta_{n\xi} f(x) = \sum_{\nu=1}^{n-1} \Delta_{\xi} \Delta_{\nu\xi} f(x) + n \Delta_{\xi} f(x), \tag{4.1}$$

whence $\| \Delta_{n\xi} f(x) \| \leq n + n\, O(1) = O(n)$.

Now let N be a large positive integer, and put $N = n + lm$, where $l \geqslant 1$, $0 \leq n < m$; l, m, n are integers. We have

$$\Delta_{N\xi} f(x) = \sum_{\nu=1}^{l-1} \Delta_{m\xi} \Delta_{\nu m\xi} f(x) + \\ + l \Delta_{m\xi} f(x) + \Delta_{n\xi} \Delta_{lm\xi} f(x) + \Delta_{n\xi} f(x). \tag{4.2}$$

whence

$$\| \Delta_{N\xi} f(x) - l \Delta_{m\xi} f(x) \| \leq l + 1 + O(n). \tag{4.3}$$

Furthermore we have $\| (lN^{-1} - m^{-1}) \Delta_{m\xi} f(x) \| = \\ = O(N^{-1}).O(m)$, and so

$$\| N^{-1} \Delta_{N\xi} f(x) - m^{-1} \Delta_{m\xi} f(x) \| = O(m^{-1}) + O(mN^{-1}).$$

Applying this a second time to M instead of N, with the same value of m, we easily obtain, if $g_N = N^{-1} \Delta_{N\xi} f(x)$, that $\| g_N(x) - g_M(x) \| \to 0$ as $N \to \infty$, $M \to \infty$. Hence it follows from (IV), that there exists a function $p_\xi(x) \in \Omega$ such that $\| g_N(x) - p_\xi(x) \| \to 0$ as $N \to \infty$.

Again consider (4.3), and take $n = 0$, $N = lm$. Dividing by l and making $l \to \infty$ we obtain $\| \Delta_{m\xi} f(x) - m\, p_\xi(x) \| \leq 1$ ($m = 1, 2, 3, \ldots.$).

We have $\| \Delta_{m\xi} f(x+a) - \Delta_{m\xi} f(x) \| = \| \Delta_a \Delta_{m\xi} f(x) \| \leq 1$, whence it follows, by making $m \to \infty$, that $\| p_\xi(x+a) - \\ - p_\xi(x) \| = 0$. Therefore, by (III*), there exists a constant $H(\xi)$ such that $\| p_\xi(x) - H(\xi) \| = 0$, and so

$$\| \Delta_{m\xi} f(x) - m\, H(\xi) \| \leq 1. \tag{4.4}$$

We can show that $H(\xi)$ is an additive function of ξ. We have

$$\| \Delta_{m(\xi+\eta)} f(x) - \Delta_{m\xi} f(x) - \Delta_{m\eta} f(x) \| = \| \Delta_{m\xi} \Delta_{m\eta} f(x) \| \leq 1,$$

and it follows, by (4.4) that $\| H(\xi+\eta) - H(\xi) - H(\eta) \| \leq m^{-1}$. Making $m \to \infty$ we infer, for all ξ and η,

$$\| H(\xi + \eta) - H(\xi) - H(\eta) \| = 0.$$

If we assume (III), it follows that $H(\xi)$ is additive, and substitution of $m = 1$ in (4.4) gives the final result

$$\| \Delta_h f(x) - H(h) \| \leq 1. \qquad (h \text{ arbitrary})$$

If we assume (III*) instead of (III) then we have to argue differently. In most cases, the fact that one constant $\neq 0$ has the norm zero implies that any constant has the norm zero, and then we obtain $\| \Delta_h f(x) \| \leq 1$ for all h. If, however, the generalizations mentioned in Remark 2 are considered, the proof can be completed as follows. Let Ω_C denote the set consisting of all constant functions in Ω, and let Ω'_C denote the subset of Ω_C consisting of all constants with the norm zero. Clearly Ω_C and Ω'_C are linear spaces. Ω_C can be written as a direct sum [10] $\Omega_C = \Omega'_C + \Omega^*_C$. If $A \in \Omega_C$, its component in Ω^*_C be denoted by A^*.

We now define $H^*(\xi)$ by $H^*(\xi) = \{H(\xi)\}^*$. Clearly $\| H^*(\xi) - H(\xi) \| = 0$ for all ξ, and so $\| H^*(\xi + \eta) - H^*(\xi) - H^*\eta) \| = 0$ for all ξ and η. If we put $A = H^*(\xi + \eta) - H^*(\xi) - H^*(\eta)$, we have $A \in \Omega^*_C$, $\| A \| = 0$, and so $A = 0$. Therefore $H^*(\xi)$ is additive, and $\| \Delta_h f(x) - H^*(h) \| \leq 1$ for all h.

Let Ω_1 be a space of real functions defined for $-\infty < x < \infty$, and assume that (I), (II), (III) [11] and (IV) hold. We furthermore require

(V) If $g(x) \in \Omega_1$, and ir h is a real number, then $g(x+h) \in \Omega_1$, and $\| g(x+h) \| = \| g(x) \|$.

(VI) If $g \in \Omega_1$, then we have $\| \Delta_h g(x) \| \to 0$ as $h \to 0$.

[10]) If Ω_C has infinite dimension, the axiom of choice is needed here.

[11]) Instead of (III) the condition (III*) is also sufficient for the present purpose.

A consequence is, that $\| \Delta_h g(x) \|$ is a continuous function of h for all values of h[12]).

(VII) Every function of Ω_1 is periodic mod 1.

Theorem 4.2. *Let f be periodic mod 1, and let f be such that $\Delta_h f \in \Omega_1$, for each value of h. Then there exists an additive function $H(x)$ such that $\Delta_h \{f(x) - H(x)\} \in \Omega_1$,*

$$\| \Delta_h \{f(x) - \mathrm{H}(x)\} \| \leq \mathrm{M}, \tag{4.5}$$

M not depending on h, and

$$\lim_{h \to 0} \| \Delta_h \{f(x) - \mathrm{H}(x)\} \| = 0. \tag{4.6}$$

Proof. A proof can be given completely analogous to the one in § 2. First, by BAIRE's theorem, the function

$$\varphi(h, k) = \| \Delta_h \Delta_k f(x) \|$$

is continuous in both variables, in at least one point $h = a$, $k = b$. We shall show that it is continuous everywhere.

Consider a point (c, d) and put $h = c + \lambda$, $k = d + \mu$, and $c = a + p$, $d = b + q$. We have

$$\Delta_{c+\lambda} \Delta_{d+\mu} f(x) - \Delta_c \Delta_d f(x) = \{\Delta_{a+\lambda} \Delta_{b+\mu} f(x) - \Delta_a \Delta_b f(x)\} +$$
$$+ \{\Delta_{\lambda+\mu} \Delta_{p+q} f(a+b+x)\} - \{\Delta_\lambda \Delta_p f(a+x)\} - \{\Delta_\mu \Delta_q f(b+x)\}.$$

The norm of each one of the four terms tends to zero as $\lambda \to 0$, $\mu \to 0$. Consequently $\varphi(c + \lambda, d + \mu) \to \varphi(c, d)$, and the continuity has been established.

Furthermore, $\varphi(h, k)$ is periodic in both variables, and so we have $|\varphi(h, k)| \leq \mathrm{M}$ for all values of h and k[13]). Now theorem 4.1. shows the truth of (4.5).

In order to prove (4.6) we use (4.1) for small values of ξ and large values of n. Given ε, we can, owing to the continuity of $\varphi(h, k)$, determine $\delta > 0$ such that we have $\| \Delta_\xi \Delta_\eta f \| < \varepsilon$ for $|\xi| < \delta$ and arbitrary values of η. Hence we have, by (4.1)

$$\| \Delta_\xi f(x) - n^{-1} \Delta_{n\xi} f(x) \| < \varepsilon. \quad (|\xi| < \delta, \; n = 1, 2, \ldots .).$$

Making $n \to \infty$ and using $\lim n^{-1} \Delta_{n\xi} f(x) = \mathrm{H}(\xi)$ we obtain (4.6).

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12) See [4], p. 141.

13) A slightly different argument for this part of the proof can be found in the beginning of the proof of theorem 4.3. Applying (VI) to (4.7) and using $\Delta_h \Delta_k = \Delta_k \Delta_h$, we see that (4.7) holds for all h and k.

Theorem 4.2 can be extended by admitting $f(x)$ to be a function defined for all elements x of a compact additive abelian topological group satisfying the second countability axiom. We do not go into details.

We shall now consider another general theorem of the same type as theorem 4.1. We shall drop, among others, the very restrictive condition (VI), but, on the other hand, some assumption has to be made on the integrability of the functions of the class.

Let Λ be a normed linear space of periodic functions of one real variable, satisfying the conditions (I), (II), (V) and (VII). Moreover, assume that it has the following properties:

(VIII) Every function of Λ is RIEMANN integrable [14]).

(IX) If the function $\varphi \varepsilon \Lambda$ satisfies $\| \varphi(x+h) - \varphi(x) \| \leq 1$ for all values of h belonging to some everywhere dense set, and if $\psi(x) = \int_0^1 \{\varphi(x+h) - \varphi(x)\}\, dh$ belongs to Λ, then we have $\| \psi(x) \| \leq 1$.

(X) If $g(x)$ is constant for all x, then $g \varepsilon \Lambda$.

Theorem 4.3. *If* $f(x)$ [15]) *is periodic mod* 1, *and if, for each value of* h, *we have* $\Delta_h f(x) \varepsilon \Lambda$, *then there exists an additive function* $H(x)$, *such that* $\| \Delta_h \{f(x) - H(x)\} \| \leq M$, M *not depending on* h.

Proof. As the interval $0 \leq h \leq 1$ is not the union of a countable number of nowhere dense sets, there exists a natural number N such that the set S_0 for which $\| \Delta_h f \| \leq N$ is everywhere dense in some interval $p \leq h \leq q$ $(0 \leq p < q \leq 1)$. Considering the fact that $\omega(h) = \| \Delta_h f(x) \|$ is sub-additive [16]), that is $\omega(h_1 + h_2) \leq \omega(h_1) + \omega(h_2)$, we easily infer that there exists a set S, everywhere dense in $0 \leq h \leq 1$, such that $\omega(h)$ is bounded over S:

$$\| \Delta_h f(x) \| \leq M', \qquad (h \varepsilon S)$$

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[14]) Integrability (L_1) would be sufficient here, but condition (IX) practically excludes this generalisation.

[15]) $f(x)$ need not be integrable itself.

[16]) See [4], p. 141.

and consequently, by (V),

$$\| \Delta_k \Delta_h f(x) \| \leq 2M' = M. \quad (h \in S,\ 0 \leq k \leq 1) \tag{4.7}$$

The function $\Delta_k f(x)$ is in Λ, hence it is integrable (R) with respect to x. Furthermore, it is periodic mod 1. Therefore, putting

$$\int_0^1 \Delta_k f(x + y)\, dx = H(k), \tag{4.8}$$

the function $H(k)$ does not depend on y. $H(k)$ is an additive function of k. For,

$$H(k+l) = \int_0^1 \Delta_{k+l} f(x)\, dk = \int_0^1 \{\Delta_k f(x+l) + \Delta_l f(x)\}\, dx = H(k) + H(l).$$

Putting $f(x) - H(x) = g(x)$, we infer from (X) that $\Delta_h g(x)$ is in Λ for every h. Now we have, for each k,

$$\int_0^1 \Delta_k g(x+y) dx = \int_0^1 \Delta_k f(x+y) dx - \int_0^1 H(k)\, dx = H(k) - H(k) = 0.$$

and so

$$\int_0^1 \Delta_k \{g(x + h) - g(x)\}\, dh = -\Delta_k g(x).$$

Now apply (IX), taking $\varphi(x) = \Delta_k g(x)$. It follows from (4.7) that $\| \Delta_h \varphi(x) \| \leq M$ for $h \in S$. It results that $\| \psi(x) \| \leq M$, that is to say $\| \Delta_k g(x) \| \leq M$. This holds for arbitrary k, and so the proof is completed.

A simple example of a Λ-space is given by the continuous functions of period 1, where the norm $\| f \|$ denotes the maximum of $| f |$. This means that Theorem 4.3 gives a new proof for a part of lemma 1.1 (§ 2). Another example is the class of periodic functions having bounded variation on $0 \leq x \leq 1$, if $\| f \|$ denotes the total variation of f over a period. The latter is an example of a more general type of spaces defined as follows.

Let U be a finite or infinite set of linear functionals L, each L being of the form

$$L\{f\} = \sum_{k=1}^{m} a_k f(b_k).$$

Here the a's and b's are real numbers, and the number m need not be uniformly bounded for all L's of U. Let Λ consist of all RIEMANN integrable functions $f(x)$ which satisfy

$$\sup_{0 \leq h \leq 1,\, L \in U} | L\{f(x+h)\} | < \infty. \tag{4.9}$$

For $f \in \Lambda$, the left-hand-side of (4.9) be denoted by $\|f\|$. Now it is not difficult to prove that Λ satisfies the conditions required for theorem 4.3.

As a further example we mention the class of functions f with the property that the difference quotients $(f(x) - f(y))/ / (x - y)$ $(x \neq y)$ all have absolute value $\leq$ C, C depending on f only. In that case we have to take for the L's the functionals $L_s \{f\} = s^{-1} \{f(s) - f(0)\}$, where s runs through the numbers $0 < s \leqslant 1$.

In theorem 4.3 we may, of course, simply take for Λ the class of all functions integrable (R), with period 1, if we take $\| f \| = \int_0^1 | f(x) | \, dx$.

In Theorem 4.3 RIEMANN integrable functions occur. We can get a similar theorem by replacing these by functions integrable (L_1). If we do this in condition (VIII), and at the same time replace in condition (IX) the words „some everywhere dense set" by „some set of outer measure 1", then the theorem remains valid. Corresponding alterations have to be made in the proof. We do not go into details.

§ 5. *Special* BANACH *spaces.* This § is based upon theorem 4.2.

In the first place we can show that lemma 1.1 is contained in Theorem 4.2. If $\| f \|$ is defined as the maximum of $| f |$ for $0 \leq x \leq 1$, the set of all continuous functions of period 1 is easily seen to satisfy the conditions of Theorem 4.2. Applying (4.6) we find that $f(x) = g(x) + H(x)$, where $\| \Delta_h g(x) \| \to 0$ as $h \to 0$. This implies that $g(x)$ is continuous.

Let C_5 denote the class of functions on $-\infty < x < \infty$, which are absolutely continuous on any finite sub-interval. C_5 has the difference property. Again, it is sufficient to consider the case of periodicity mod 1 (see § 1). The set of these functions is a BANACH space satisfying the conditions of theorem 4.2, it we define $\| f \|$ as the sum of the maximum of $| f |$ and the total variation of f on $0 \leqslant x \leqslant 1$ (see [**7**, **10**]). It follows, by (4.5) and (4.6), that $f(x) = g(x) + H(x)$ where $H(x)$ is additive, and $g(x)$ is a continuous function with the following properties: If $\omega(h)$ denotes the total variation of $g(x+h) - g(x)$, then $\omega(h) \leq M$ (M not depending on h), and $\omega(h) \to 0$ as $h \to 0$.

The following lemma, essentially due to A. PLESSNER [**7**], now shows that $g(x)$ is absolutely continuous:

Lemma 5.1. *If $F(x)$ is integrable and periodic mod 1, and if is such that the total variation $\omega(h)$ of $F(x+h) - F(x)$ on $0 \leqslant x \leqslant 1$ satisfies, fore some constant M,*

$$\omega(h) \leqslant M \quad (0 \leqslant h \leqslant 1), \qquad \lim_{h \to 0} \omega(h) = 0,$$

then $F(x)$ is absolutely continuous.

Instead of $\omega(h) \leqslant M$, PLESSNER assumed that $F(x)$ has bounded variation itself. But, his proof can be used for the present lemma as well, without any alterations [17]).

Let, for $p \geqslant 1$, $C_6(p)$ denote the class of functions $f(x)$, defined for $-\infty < x < \infty$, which belong to $L_p(a, b)$ for each pair (a, b), that is to say, $f(x)$ is measurable, and the LEBESGUE integral $\int_a^b | f(x) |^p \, dx$ is finite.

The periodic functions of $C_6(p)$ form a BANACH space, if we take as the norm $\| f \| = \int_0^1 | f |^p \, dx$. The conditions of theorem 4.2 are satisfied [18]), and hence we obtain (4.5) and (4.6). But it cannot be deduced that $C_6(p)$ has the difference property (see § 1).

We can prove, however, for $p = 2$.

Theorem 5.1. *If $f(x)$ is such that, for each value of h, $\Delta_h f(x) \in C_6(2)$, then we have a decomposition.*

$$f(x) = g(x) + H(x) + S(x),$$

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[17]) cf. (6.2), where it is shown that PLESSNER's condition follows from ours.

[18]) With (III*) instead of (III).

where $g(x) \in C_6(2)$, $H(x)$ *is additive, and* $S(x)$ *satisfies, for each value of* h,

$$S(x+h) = S(x) \text{ for almost all values of } x.$$

Proof. As in § 1, we can write $f(x) = g_1(x) + f_1(x)$, where $g_1(x) \in C_6(2)$, and $f_1(x)$ is periodic mod 1, and such that $\Delta_h f_1(x) \in C_6(2)$ for each h. Now applying (4.5) we find the decomposition $f_1(x) = f_2(x) + H(x)$. $H(x)$ is additive; $f_2(x)$ such that $\Delta_h f_2 \in C_6(2)$, for each h, and

$$\int_0^1 | f_2(x+h) - f_2(x) |^2 \, dx \leq M, \tag{5.1}$$

M not depending on h.

Let $a_n(h)$ denote the complex Fourier coefficients of $\Delta_h f_2$:

$$a_n(h) = \int_0^1 \{f_2(x+h) - f_2(x)\} \, e^{-2\pi n i x} \, dx.$$

We have, for arbitrary real values of h and k,

$$\int_0^1 \Delta_h f_2(x+k) \, e^{-2\pi n i x} \, dx + \int_0^1 \Delta_k f_2(x) \, e^{-2\pi n i x} \, dx = \\ = \int_0^1 \Delta_{h+k} f_2(x) \, e^{-2\pi n i x} \, dx,$$

whence it follows

$$e^{2\pi n i k} \, a_n(h) + a_n(k) = a_n(h+k). \tag{5.2}$$

The right-hand-side being symmetrical with respect to h and k, we obtain

$$a_n(h) \, . \, (e^{2\pi n i k} - 1) = a_n(k) \, . \, (e^{2\pi n i h} - 1).$$

First take $n \neq 0$. For k we take a fixed irrational number. Putting $a_n(k)/(e^{2\pi n i k} - 1) = a_n$, we obtain:

$$a_n(h) = (e^{2\pi n i h} - 1) \, a_n. \tag{5.3}$$

By (5.1) we have

$$\sum_{-\infty}^{\infty} | a_n(h) |^2 \leq M \tag{5.4}$$

for each value of h. In the first place it follows that $a_0(h) = 0$ for each h. For, the function $a_0(h)$ is additive, by (5.2), and bounded. Furthermore, we deduce from (5.3) and (5.4),

$$\sum_{-N}^{N} | a_n |^2 \int_0^1 | e^{2\pi n i h} - 1 |^2 \, dh \leq M$$

for each positive integer N. It follows that $\Sigma_{-\infty}^{+\infty} | a_n |^2 \leq \frac{1}{2}M$, and it results from the theorem of RIESZ-FISCHER that there exists a function $g_2 \in L_2$, whose FOURIER coefficients are a_n. Hence, by (5.3), $\Delta_h g_2(x)$ has the same FOURIER coefficients as $\Delta_h f_2(x)$. Putting $f_2(x) = g_2(x) + S(x)$ we infer that all FOURIER coefficients of $\Delta_h S(x)$ are zero. Consequently $\int_0^1 | \Delta_h S(x) |^2 dx = 0$, and the theorem is proved.

§ 6. *Bounded variation.* The class C_7 of all functions which have bounded total variation over any finite interval $a \leq x \leq b$ has the difference property. In order to prove this, it is sufficient again to do it for the periodic functions mod 1:

Theorem 6.1. *Let $f(x)$ have the period* 1, *and assume that, for any value of h, the difference $\Delta_h f(x)$ has bounded variation over* $0 \leq x \leq 1$. *Then $f(x)$ can be written in the form $g(x) + H(x)$, where $g(x)$ has bounded variation, and $H(x)$ is additive.*

Proof. Consider the space C_7^* of functions of period 1 with bounded variation over a period. This variation be denoted by var $g(x)$. As the norm of g ($g \in G_7^*$) we take

$$\| g \| = \text{var } g + \max_{0 \leq x \leq 1} | g(x) |.$$

With this norm we may apply theorem 4.3 (see the end of section 4), and so $f(x) = g(x) + H(x)$, where $g(x)$ satisfies $\| \Delta_h g(x) \| \leq M$, M not depending on h; $H(x)$ is additive. Therefore it remains to be proved that any periodic function $g(x)$ satisfying, for each h,

$$\text{var } \{\Delta_h g(x)\} \leq M, \quad \max_{0 \leq x \leq 1} | \Delta_h g(x) | \leq M, \tag{6.1}$$

has bounded total variation itself.

This is very easy to prove if we assume $g(x)$ to be integrable. Using an idea occurring in PLESSNER's proof of lemma 5.1, we write

$$\int_0^1 \{g(x + h) - g(x)\} \, dh = - g(x) + A, \tag{6.2}$$

where A does not depend on x. Taking the total variation of both sides, we obtain, in virtue of (6.1), that var $g(x) \leq$ M [19]).

If we drop the assumption of integrability, the proof becomes more intricate. We shall first consider $g(x)$ for rational values of x only. Let R denote the set of rational numbers in $0 \leq x \leq 1$. If S is an arbitrary sub-set of $0 \leq x \leq 1$, then we define $\mathrm{var}_S\, f$ as the least upper bound of all finite sums $\Sigma_{i=1}^{n} \mid f(x_i) - f(x_{i-1}) \mid$, where the x's are chosen from S, and $0 \leq x_0 < x_1 < \ldots . < x_n \leq 1$.

We can show, by an identity analogous to (6.2), that (6.1) implies $\mathrm{var}_R\, g(x) \leq M$.

Let N be a natural number. Then we have

$$\sum_{j=1}^{N} \left\{ g\left(x + \frac{j}{N}\right) - g(x) \right\} = -N\, g(x) + A(x), \qquad (6.3)$$

where $A(x)$ only depends on the residue class of x mod N^{-1}. Now let S denote the set of numbers kN^{-1} ($k=0, 1, \ldots, N$). $A(x)$ is constant on S. It follows from (6.3) and (6.1) that

$$N\, \mathrm{var}_S\, g(x) \leq \sum_{j=1}^{N} \mathrm{var}_S \{\Delta_{j/N}\, g(x)\} \leq N.M.$$

If T is an arbitrary finite set of rational numbers, then we take N as their common denominator, and we infer $\mathrm{var}_T\, g(x) \leq \mathrm{var}_S\, g(x) \leq M$. It follows that $g(x)$ has bounded variation over the rationals: $\mathrm{var}_R\, g(x) \leq M$.

Any function $g(x)$, defined on R, with the property $\mathrm{var}_R\, g(x) \leq M$, can be extended to the irrationals such that the total variation over $0 \leq x \leq 1$ is still $\leq M$. This is easily proved by writing $g(x)$ as the difference of two monotonic functions, and solving the extension problem for both. If the entension is denoted by $g^*(x)$, then we have $g(x) = g^*(x)$ for $x \in R$. Henceforth we shall consider the difference $g(x) - g^*(x)$, and so it suffices to prove the following

L e m m a 6.1. *If $g(x)$ is periodic mod 1, $g(x) = 0$ for x rational, and if $g(x)$ satisfies (6.1) for each h, then we have* var $g(x) < \infty$.

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[19]) This would finish the proof if it would have been proved first that the class of all functions integrable (R) has the difference property. For then, in theorem 6.1 we would know that $g(x)$ is integrable (R).

P r o o f. Let $\zeta_1, \ldots, \zeta_n$ be a set of real numbers, $0 \leq \zeta_1 < \ldots < \zeta_n < 1$. We choose rational numbers r_j such that

$$0 \leqslant \zeta_1 < r_1 < \zeta_2 < r_2 < \ldots < \zeta_n < r_n < 1, \quad (6.3)$$

and S denotes the set of all ζ's and all r's. Then we have, for arbitrary values of h,

$$\operatorname{var}_S \Delta_h g(x) \geq \sum_{j=1}^{n} | \Delta_h g(\zeta_j) - \Delta_h g(r_j) |,$$

and so (6.1) shows that we have

$$\sum_{j=1}^{n} | g(\zeta_j + h) - g(\zeta_j) - g(r_j + h) | \leq M, \quad (6.4)$$

whatever h may be.

In the first place we apply (6.4) for a rational number h, so that $g(r_j + h)$ vanishes. We can take as many ζ's as we please; therefore

$$\sum_{\zeta \in J} | g(\zeta + h) - g(\zeta) | \leq M, \quad (h \in R) \quad (6.5)$$

where the sum is extended over all ζ's in the interval $J = [0, 1]$. In this sum there are, of course, at most a countable number of terms $\neq 0$.

For a second application of (6.4) we take a number ζ, and rational numbers $s_1, \ldots, s_n$ such that the sums $\zeta_i = \zeta + s_i$ satisfy $0 \leqslant \zeta_1 < \zeta_2 < \ldots < \zeta_n < 1$. Furthermore we take a rational number t such that the numbers $r_i = t + s_i$ satisfy (6.3). To this end it is sufficient to take t in the interval $\zeta < t < \zeta + \delta$, where δ is the smallest of the numbers $s_2 - s_1, s_3 - s_2, \ldots, s_n - s_{n-1}, 1 - \zeta_n$. Now (6.4) becomes, if we write ξ instead of h,

$$\sum_{j=1}^{n} | g(\zeta + s_j + \xi) - g(\zeta + s_j) - g(s_j + t + \xi) | \leqslant M.$$

Using (6.5), with $h = t$, we obtain

$$\sum_{s \in S} | g(\zeta + \xi + s) - g(\zeta + s) - g(\xi + s) | \leqslant 2M, \quad (6.6)$$

where the sum is extended over an arbitrary finite set S ($S \subset R$). Writing down (6.6) for $\zeta = \xi, 2\xi, 3\xi, \ldots, (k-1)\xi$, respectively, we obtain by addition

$$\sum_{s \in S} | g(k\xi+s) - k\, g(\xi+s) | \leqslant 2(k-1)\, M. \quad (k=2, 3, \ldots) \quad (6.7)$$

Let S be finite and fixed for a moment. We know from (6.1) that $g(x)$ is bounded. Therefore, we have $\Sigma \mid g(k\xi + s) \mid = O(1)$ as $k \to \infty$. We now infer from (6.7) that $\Sigma \mid g(\xi + s) \mid \leqslant 2M$, where the sum is extended over an arbitrary finite sub-set S of R. It follows that

$$\sum_{s \in R} \mid g(\xi + s) \mid \leqslant 2M. \tag{6.8}$$

We return to (6.4), where the ζ's are arbitrary, but for $0 \leqslant \zeta_1 < \ldots . < \zeta_n < 1$, and we write ξ instead of h. By (6.8) we have

$$\sum_{j=1}^{n} \mid g(r_j + \xi) \mid \leqslant 2M,$$

and so, by (6.4),

$$\sum_{j=1}^{n} \mid g(\zeta_j + \xi) - g(\zeta_j) \mid \leqslant 3M.$$

This being true for any finite set of ζ's taken from $J = [0, 1]$, we find,

$$\sum_{\zeta \in J} \mid g(\zeta + \xi) - g(\zeta) \mid \leqslant 3M, \tag{6.9}$$

which extends (6.5) to arbitrary values of ξ.

It follows from (6.9) in a few lines that $g(x)$ has bounded variation. Take a natural number N, and choose N different numbers $\xi_1, \ldots, \xi_N$ from J, and N different rationals $r_1, \ldots, r_N$ from R. We have, by (6.5) and (6.9)

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \mid g(\xi_j + r_i) - g(\xi_j) \mid \leqslant MN$$

and

$$\sum_{j=1}^{N} \sum_{i=1}^{N} \mid g(\xi_j + r_i) - g(r_i) \mid \leqslant 3MN.$$

Here we have $g(r_i) = 0$, and so

$$N \sum_{j=1}^{N} \mid g(\xi_j) \mid = \sum_i \sum_j \mid g(\xi_j) \mid \leqslant$$

$$\leqslant \sum_i \sum_j \mid \{g(\xi_j + r_i) - g(r_i)\} + \{g(\xi_j) - g(\xi_j + r_i)\} \mid \leqslant 4MN.$$

The ξ's being arbitrary, we finally obtain

$$\sum_{\xi \in J} \mid g(\xi) \mid \leqslant 4M,$$

and hence var $g(x) \leqslant 8M$. This completes the proof of lemma 6.1.

§ 7. *Bounded functions.* In this § we consider three different classes. C_8: the class of all functions $g(x)$ bounded over $-\infty < x < \infty$; C_9: the class of all functions $g(x)$ bounded over any finite interval; C_{10}: the class of all functions $g(x)$ satisfying $| g(x) - g(0) | \leqslant 1$ for all values of x.

The class C_{10} has the difference property. Actually, this statement is equivalent to theorem 1.2. For, $\Delta_h f(x) \in C_{10}$ is equivalent to (1.1), and, if $H(x)$ is additive, then $f(x) - H(x) \in C_{10}$ is equivalent to $| f(x) - f(0) - H(x) | \leqslant 1$.

C_8 does not have the difference property; a simple counterexample is $f(x) = \log\ (x^2 + 1)$, whose differences are all bounded in $-\infty < x < \infty$.

C_9 neither has the difference property. Here we have no measurable counterexamples [20]), but we can construct a non-measurable one, depending on the axiom of choice.

Let the set of numbers $1 = b_1, b_2, b_3, \ldots$ be a HAMEL base (see [3]) for the set of real numbers, that is to say, any real x can be written uniquely in the form $x = \Sigma\, k_i b_i$, where the k_i are rational, and all but a finite number of the k's are zero. Now define $T(x)$ by $T(x) = k_2$, where k_2 is the coefficient of b_2. Clearly $T(x)$ is an additive function, and $T(x) = T(x + r)$ for all rational values of r. Now put $f(x) = \log \{(T(x))^2 + 1\}$.

If h is a fixed number, then we have

$$f(x+h) - f(x) = \log \{(T(x) + T(h))^2 + 1\} - \log \{(T(x))^2 + 1\}$$

and hence

$$| f(x+h) - f(x) | < C(h) \quad (-\infty < x < \infty)$$

We shall show that $f(x)$ is not of the form $g(x) + H(x)$, where $g(x)$ is bounded on any finite interval, and $H(x)$ is additive. Assume that $f = g(x) + H(x)$, where g is bounded on $0 \leq x \leq 1$, and H is additive. We may assume that

[20]) In order to show this, apply theorem 4.3, taking Λ to be the class of all measurable periodic bounded functions, and taking $\| g \| = \max\ | g |$ as the norm.

$H(1) = 0$, for this can be obtained by adding a linear function to g and subtracting it from H.

The functions $H(x)$ and $f(x)$ are periodic mod 1. It follows that g is also periodic; therefore it is bounded for $-\infty < x < \infty$. Consequently

$$f(nb_2) - H(nb_2) = O(1) \text{ for } n \to \infty.$$

But $T(nb_2) = n$, $f(nb_2) = \log(n^2+1)$, whereas $H(nb_2) = n\,H(b_2)$, and, as $n \to \infty$, $|\log(n^2+1) - na| \to \infty$ whatever a may be.

This proves that C_9 fails to have the difference property.

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