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**ON MAXIMUM NORM CONVERGENCE OF
MULTIGRID METHODS FOR ELLIPTIC
BOUNDARY VALUE PROBLEMS**

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On Maximum Norm Convergence of Multigrid Methods for Elliptic Boundary Value Problems

by

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Abstract. We consider multigrid methods applied to standard linear finite element discretizations of linear elliptic boundary value problems in $2D$. In the multigrid method damped Jacobi or damped Gauss-Seidel is used as a smoother. We prove that the two-grid method with ν pre-smoothing iterations has a contraction number with respect to the *maximum norm* that is (asymptotically) bounded by $C\nu^{-\frac{1}{2}} |\ln h_k|^2$, with h_k a suitable mesh size parameter. Moreover, it is shown that this bound is sharp in the sense that a factor $|\ln h_k|$ is necessary.

Key words. Multigrid, convergence analysis, maximum norm, elliptic boundary value problems.

AMS (MOS) subject classification. 65N20

1. Introduction

If one considers elliptic boundary value problems in \mathbb{R}^N ($N = 2, 3$) then multigrid methods can be used to solve efficiently the large sparse linear systems that arise after discretization. In recent years there has been intensive research into the theoretical understanding of the convergence properties of these methods. We refer to Hackbusch [10], McCormick [14] and the references therein. The main feature of multigrid is that for a broad class of problems the contraction number has an upper bound which is smaller than one and independent of the mesh size. In theoretical analyses this has been shown for several variants of multigrid. Usually in these analyses the energy norm is used; sometimes one uses the euclidean norm. Some first results about multigrid convergence in the maximum norm are presented in [19]. In that paper, however, only two-point boundary value problems are treated. In this paper we present convergence results in the *maximum norm* for multigrid applied to a class of elliptic *two dimensional* boundary value problems. We consider a regular linear (nearly) symmetric elliptic boundary value problem on a domain $\Omega \subset \mathbb{R}^2$ and we use linear finite elements on quasi-uniform triangulations.

Two main results of this paper are the following. Firstly, we prove that for a two-grid method with ν damped Jacobi or damped Gauss-Seidel smoothing iterations the contraction number with respect to the maximum norm is (asymptotically) bounded by $C\nu^{-\frac{1}{2}} |\ln h_k|^2$ (with h_k a suitable mesh size parameter). Secondly, it is shown that this bound is sharp in the sense that a factor $|\ln h_k|$ is necessary: For a concrete (very regular) example we prove that the contraction number with respect to the maximum norm of a standard two-grid method with a fixed number of smoothing iterations is bounded from below by $C |\ln h_k|$.

So instead of an “optimal” bound $C\nu^{-1}$ for the contraction number in the energy norm (or the euclidean norm) we obtain a “nearly optimal” bound $C\nu^{-\frac{1}{2}} |\ln h_k|^2$ if we use the maximum norm and this bound is sharp in some way.

We now outline the remainder of this paper. In §2 we introduce a class of elliptic boundary value problems. Some properties of the usual linear finite element discretization on a sequence of quasi-uniform triangulations are derived in §3. An important property, due to Descloux [7], is that the mass matrix has a condition number which is uniformly bounded (for $h_k \downarrow 0$) with respect to the maximum norm. In §4 and §5 we prove the Approximation Property and Smoothing Property (cf. Hackbusch [10]) respectively. In the proof of the Approximation Property we use a regularity result due to Campanato [5] and an L^∞ -finite element error estimate due to Rannacher and Frehse [18,8]. From the latter the factor $|\ln h_k|^2$ originates. The Smoothing Property is proved using a new technique introduced in [20]. In §6 we derive convergence results for the two-grid method and we discuss the multigrid W -cycle. Finally, in §7, we analyze a specific example. We consider the Poisson equation on the unit square and use a linear finite element discretization on a uniform triangulation. We prove a $C|\ln h_k|$ lower bound for the contraction number of a standard two-grid method with a fixed number of smoothing iterations. Our analysis in §7 is based on the approach used by Haverkamp in [11].

2. Continuous problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain with $\partial\Omega$ sufficiently smooth. We consider the following variational boundary value problem:

$$(2.1) \quad \begin{cases} \text{find } \varphi^* \in H_0^1(\Omega) \text{ such that for all } \psi \in H_0^1(\Omega) \\ \sum_{i,j=1}^2 \int_{\Omega} a_{ij} \frac{\partial \varphi^*}{\partial x_i} \frac{\partial \psi}{\partial x_j} dx = \int_{\Omega} f \psi dx . \end{cases}$$

We use the notation

$$(2.2) \quad a(\varphi, \psi) = \sum_{i,j=1}^2 \int_{\Omega} a_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \psi}{\partial x_j} dx \quad (\varphi, \psi \in H_0^1(\Omega)) ,$$

and we make the following assumptions about Ω , f , $a(\cdot, \cdot)$:

$$(2.3) \quad f \in L^2(\Omega)$$

$$(2.4.a) \quad \partial\Omega \in C^{2,\alpha} \quad (\alpha \in]0, 1])$$

$$(2.4.b) \quad a_{ij} = a_{ji} ; \quad a_{ij} \in C^{1,\alpha}(\bar{\Omega}) \quad (\alpha \in]0, 1])$$

$$(2.4.c) \quad \text{there are constants } \lambda_1, \lambda_2 > 0 \text{ such that for all } (\xi_1, \xi_2) \in \mathbb{R}^2 \\ \text{and all } x \in \Omega$$

$$\lambda_2 |\xi|^2 \leq \sum_{i,j=1}^2 a_{ij}(x) \xi_i \xi_j \leq \lambda_1 |\xi|^2 .$$

By $H^{m,p}(\Omega)$, $H_0^{m,p}(\Omega)$ ($1 \leq p \leq \infty$, $m \in \mathbb{N}$) we denote the usual Sobolev spaces with norm $\|\cdot\|_{H^{m,p}}$ and $\|\cdot\|_{H_0^{m,p}}$ respectively. If $p = 2$ we use the notation $H^m(\Omega)$, $H_0^m(\Omega)$.

Remark 2.1. In multigrid convergence theory an important role is played by the regularity of the differential operator. It is well-known that under the assumptions (2.3), (2.4.a-c) we have H^2 -regularity: φ^* is an element of $H^2(\Omega) \cap H_0^1(\Omega)$ and $\|\varphi^*\|_{H^2} \leq c \|f\|_{L^2}$ with c independent of f . If instead of $p = 2$ we take another $p \in]1, \infty[$ then a similar $H^{2,p}$ -regularity result holds (cf. [9]). However, even if $\partial\Omega$ and the coefficients a_{ij} are very smooth, a similar $H^{2,\infty}$ -regularity estimate $\|\varphi^*\|_{H^{2,\infty}} \leq c \|f\|_{L^\infty}$ does not hold, as is shown by the following example (due to [17]). Let Ω be the unit sphere and $a(\varphi, \psi) := \int_{\Omega} \nabla \varphi \cdot \nabla \psi dx$. For $\varepsilon \in [0, 1]$ we define $\varphi_\varepsilon^*(x_1, x_2) = x_1 x_2 \ln(|x| + \varepsilon) - x_1 x_2 \ln(1 + \varepsilon)$ (with $\varphi_0^*(0, 0) := 0$). Then $\varphi_\varepsilon^*|_{\partial\Omega} = 0$ and $\Delta \varphi_\varepsilon^*$ exists on Ω . Now define $f_\varepsilon := -\Delta \varphi_\varepsilon^*$, then $f_\varepsilon \in L^\infty(\bar{\Omega})$ for all $\varepsilon \in [0, 1]$ and even $f_\varepsilon \in C(\bar{\Omega})$ if $\varepsilon \in]0, 1]$. However, if $\varepsilon = 0$ we have $\varphi_0^* \notin H^{2,\infty}(\Omega)$ and if $\varepsilon \in]0, 1]$ we have $\varphi_\varepsilon^* \in H^{2,\infty}$ but $\|\varphi_\varepsilon^*\|_{H^{2,\infty}} \|f_\varepsilon\|_{L^\infty}^{-1}$ is unbounded for $\varepsilon \downarrow 0$.

In the proof of the Approximation Property (§4) w.r.t. the L^∞ -norm another type of regularity result (due to Campanato [5]) is used, in which the usual $L^p(\Omega)$ space is replaced by the space of John-Nirenberg $E^0(\Omega)$ (also called the space of functions of bounded mean oscillation, cf. [12]). This space satisfies $L^\infty(\Omega) \stackrel{\subset}{\neq} E^0(\Omega) \stackrel{\subset}{\neq} L^p(\Omega)$ for all $p \in [1, \infty[$.

3. Discretization and two-grid method

We take a sequence of nested quasi-uniform triangulations $\{\mathcal{T}_k \mid k \in \mathbb{N}_0\}$ as follows. For every k we define $\Omega_k := \cup_{T \in \mathcal{T}_k} T$ and for every \mathcal{T}_k we use a mesh size parameter h_k with $0 < h_{k+1} < h_k < 1$. We make the following assumptions (with constants c_i independent of T and k):

- (3.1.a) $\text{dist}(\partial\Omega_k, \partial\Omega) \leq c_0 h_k^2$;
- (3.1.b) for any two different triangles in \mathcal{T}_k the intersection is empty or consists of a common vertex or of a common side;
- (3.1.c) for every $T \in \mathcal{T}_k$ there is a disc with radius $c_1 h_k$ containing T and a disc with radius $c_2 h_k$ contained in T ;
- (3.1.d) for every $\hat{T} \in \mathcal{T}_k$ there is a subset S of \mathcal{T}_{k+1} such that $\cup_{T \in S} T = \hat{T}$ (this corresponds to mesh refinement);
- (3.1.e) $h_k h_{k+1}^{-1} \leq c_3$.

Continuous piecewise linear functions on such a triangulation \mathcal{T}_k yield a finite dimensional function space

$$(3.2) \quad \Phi_k = \{v \in C(\bar{\Omega}) \mid v \text{ is linear on every } T \in \mathcal{T}_k \text{ and } v \equiv 0 \text{ on } \partial\Omega_k \cup (\bar{\Omega} \setminus \Omega_k)\} .$$

These spaces are nested, i.e.

$$(3.3) \quad \Phi_0 \subset \Phi_1 \subset \dots \subset \Phi_k \subset \dots \subset H_0^1(\Omega) .$$

The collection of interior grid points in \mathcal{T}_k is denoted by $\{x_k^i\}_{i \in J_k}$ for some indexset J_k with $\#J_k = n_k$. We use the notation $U_k = \mathbb{R}^{n_k}$. The standard basis of Φ_k is given by the functions $\varphi_k^i \in \Phi_k$ which satisfy $\varphi_k^i(x_k^j) = \delta_{ij}$ ($i, j \in J_k$). This induces the natural bijection

$$(3.4) \quad P_k : U_k \rightarrow \Phi_k , \quad P_k(u) = \sum_{i \in J_k} u_i \varphi_k^i .$$

On U_k we use a scaled euclidean inner product with corresponding norm

$$(3.5) \quad \langle u, v \rangle_k = h_k^2 \sum_{i \in J_k} u_i v_i , \quad \|u\|_k = \langle u, u \rangle_k^{\frac{1}{2}} .$$

The maximum norm on U_k is denoted by $\|\cdot\|_\infty$. Below, adjoints are always defined with respect to the L^2 -inner product on Φ_k and $\langle \cdot, \cdot \rangle_k$ on U_k . The norms $\|\cdot\|_\infty$ (on U_k) and $\|\cdot\|_{L^\infty}$ (on Φ_k) induce associated operator norms which are denoted by $\|\cdot\|_\infty$.

Remark 3.1. Lemma 3.2 below yields that the sequences $(P_k)_{k \geq 0}$, $(P_k^{-1})_{k \geq 0}$, $(P_k^*)_{k \geq 0}$ and $((P_k^*)^{-1})_{k \geq 0}$ are uniformly bounded in $\|\cdot\|_\infty$. In the analysis of multigrid convergence in the euclidean norm (cf. [10]) such a uniform boundedness result is used too, but then w.r.t. the norms $\|\cdot\|_k$ and $\|\cdot\|_{L^2}$. The latter uniform boundedness is closely related to the well-known fact that the mass matrix $P_k^* P_k$ has a condition number $O(1)$ ($h_k \downarrow 0$) in the euclidean norm. The main argument in the proof of Lemma 3.2 is that the condition number of the mass matrix is $O(1)$ ($h_k \downarrow 0$) w.r.t. the maximum norm, too. This result has been proved by Descloux in [7].

LEMMA 3.2. *Assume (3.1.b,c), then the following holds with constants C_1 and C_2 independent of k :*

- (1) $\|P_k u\|_{L^\infty} = \|u\|_{L^\infty}$ for all $u \in U_k$,
- (2) $C_1 \|\varphi\|_{L^\infty} \leq \|P_k^* \varphi\|_{L^\infty} \leq C_2 \|\varphi\|_{L^\infty}$ for all $\varphi \in \Phi_k$.

Proof. The result in (1) holds because $P_k u$ is piecewise linear and $(P_k u)(x_k^i)$ equals the i -th component of u . Due to (1) the statement in (2) is equivalent with

$$(2') \quad C_1 \|u\|_{L^\infty} \leq \|P_k^* P_k u\|_{L^\infty} \leq C_2 \|u\|_{L^\infty} \text{ for all } u \in U_k .$$

For $i, j \in J_k$ we have (with e_m the m -th unit vector in U_k)

$$(P_k^* P_k)_{i,j} = h_k^{-2} \langle P_k^* P_k e_j, e_i \rangle = h_k^{-2} \int_{\Omega} \varphi_k^i \varphi_k^j dx .$$

So, using (3.1.c), we get

$$\|P_k^* P_k\|_{L^\infty} = \max_i h_k^{-2} \sum_{j \in J_k} \int_{\Omega} \varphi_k^j \varphi_k^i \leq \max_i h_k^{-2} \text{supp}(\varphi_k^i) \leq C_2 ,$$

thus the second inequality in (2') holds. For the first inequality in (2') we note that due to the quasi-uniformity of the triangulations the assumptions in [7] hold. Theorem 2 in [7] then yields the desired result. \square

Galerkin discretization results in a stiffness matrix $L_k : U_k \rightarrow U_k$ defined by

$$(3.6) \quad \langle L_k u, v \rangle_k = a(P_k u, P_k v) \text{ for all } u, v \in U_k .$$

Also we have that

$$(3.7) \quad a(P_k L_k^{-1} g, P_k v) = \langle g, v \rangle_k \text{ for all } g, v \in U_k .$$

For solving a system of the form $L_k u_k = g_k$ we use a standard multigrid method. The iteration matrix of the smoothing method is denoted by S_k . For the prolongation $p = p_k : U_{k-1} \rightarrow U_k$ we use the natural one:

$$(3.8) \quad p = P_k^{-1} P_{k-1} .$$

For the restriction $r = r_k : U_k \rightarrow U_{k-1}$ we take

$$(3.9) \quad r = p^* .$$

The iteration matrix of the two-grid method with ν pre-smoothing iterations is given by

$$(3.10) \quad T_k(\nu) = (I - p L_{k-1}^{-1} r L_k) S_k^\nu = (L_k^{-1} - p L_{k-1}^{-1} r) L_k S_k^\nu .$$

Below, in §4 and §5 we will prove the Approximation Property and Smoothing Property (cf. [10]) with respect to the norm $\|\cdot\|_\infty$.

4. The Approximation Property

We begin this section with a discussion of a rather special finite element L^∞ -error estimate that will be used in the proof of the Approximation Property below.

In most convergence analyses of multigrid the regularity of the underlying boundary value problem is used. If the analysis is based on the energy norm then h -independent convergence can be proved under very weak regularity conditions (cf. eg. [1], [2], [3], [4], [10], [13], [16]). If, however, one wants to prove the Approximation Property in the scaled euclidean norm then an H^2 -regularity estimate

$$(4.1) \quad \|\varphi^*\|_{H^2} \leq c \|f\|_{L^2}$$

is necessary, as is shown in [6]. The Approximation Property then follows from a combination of (4.1) with the following (in which φ_k^* is the Galerkin solution in Φ_k):

$$(4.2.a) \quad \|\varphi_k^* - \varphi^*\|_{L^2} \leq ch_k^2 \|\varphi^*\|_{H^2} \quad (\text{finite element error estimate}),$$

$$(4.2.b) \quad \|\cdot\|_{L^2} \text{ and } \|\cdot\|_k \text{ are uniformly equivalent for } k \rightarrow \infty .$$

In this paper we want to prove the Approximation Property in the maximum norm. Clearly, the analogue of (4.2.b) is given in Lemma 3.2. With respect to (4.2.a) we note that L^∞ -error estimates of the form

$$(4.3) \quad \|\varphi_k^* - \varphi^*\|_{L^\infty} \leq ch_k^2 |\ln h_k| \|\varphi^*\|_{H^{2,\infty}}$$

can be found in the literature (e.g. in [8], [15], [22], [23]). However, it is shown in Remark 2.1 that, even for very regular problems, an $H^{2,\infty}$ -regularity estimate $\|\varphi^*\|_{H^{2,\infty}} \leq c \|f\|_{L^\infty}$ does not hold. Therefore it is not clear how (4.3) can be used to prove the Approximation Property. In [8,18] Rannacher and Frehse prove the following type of (asymptotic) error estimate for $f \in L^\infty$:

$$(4.4) \quad \|\varphi_k^* - \varphi^*\|_{L^\infty} \leq ch_k^2 |\ln h_k|^2 \|f\|_{L^\infty} .$$

This result, which is a substitute for the combination of (4.1) and (4.2.a), and the result of Lemma 3.2 are the main points in the proof of the Approximation Property in Theorem 4.2 below.

Remark 4.1. We briefly comment on the combination of regularity and approximation properties of Φ_k used in the proof of (4.4). Instead of $L^p(\Omega)$ the John-Nirenberg space $E^0(\Omega)$ is used with a suitable norm denoted by $\|\cdot\|_{L^{2,0}}$. For $E^0(\Omega)$ one has $L^\infty(\Omega) \stackrel{\subset}{\neq} E^0(\Omega) \stackrel{\subset}{\neq} L^p(\Omega)$ for every $p \in [1, \infty[$. Instead of $H^2(\Omega)$ the subspace $H^{2,0}(\Omega)$ of functions for which all generalized second derivatives are in $E^0(\Omega)$ is used. In [5] Campanato proves the following regularity result:

$$(4.5) \quad \|\varphi^*\|_{H^{2,0}} \leq c \|f\|_{L^{2,0}} \text{ for all } f \in E^0(\Omega).$$

In [8] Frehse and Rannacher prove the following asymptotic error estimate (for ease $\Omega_k = \Omega$):

$$(4.6) \quad \|\varphi_k^* - \varphi^*\|_{L^\infty} \leq c_1 h_k |\ln h_k| \inf_{\varphi \in \Phi_k} \|\nabla(\varphi - \varphi^*)\|_{L^\infty} + c_2 \inf_{\varphi \in \Phi_k} \|\varphi - \varphi^*\|_{L^\infty}.$$

Furthermore in [18] Rannacher proves the following (less standard) approximation property of the space Φ_k if $\varphi^* \in H^{2,0}(\Omega) \cap H_0^1(\Omega)$:

$$(4.7) \quad \inf_{\varphi \in \Phi_k} \{h_k \|\nabla(\varphi - \varphi^*)\|_{L^\infty} + \|\varphi - \varphi^*\|_{L^\infty}\} \leq c h_k^2 |\ln h_k| \|\varphi^*\|_{H^{2,0}}.$$

Combination of (4.5), (4.6) and (4.7) yields (4.4) (note that $\|f\|_{L^{2,0}} \leq c \|f\|_{L^\infty}$).

THEOREM 4.2. *Assume (2.4.a-c) and (3.1.a-e). Then there are constants k_0 and C_A such that for all $k \geq k_0$ the following holds:*

$$(4.8) \quad \|L_k^{-1} - pL_{k-1}^{-1}r\|_\infty \leq C_A h_k^2 |\ln h_k|^2.$$

Proof. Take $g \in U_k$. In the proof different constants C , all independent of k and g , are used. Let $\varphi \in H_0^1(\Omega)$, $\varphi_k \in \Phi_k$ and $\varphi_{k-1} \in \Phi_{k-1}$ be such that

$$a(\varphi, \psi) = ((P_k^*)^{-1}g, \psi)_{L^2} \text{ for all } \psi \in H_0^1(\Omega)$$

$$a(\varphi_k, \psi) = ((P_k^*)^{-1}g, \psi)_{L^2} \text{ for all } \psi \in \Phi_k$$

$$a(\varphi_{k-1}, \psi) = ((P_k^*)^{-1}g, \psi)_{L^2} \text{ for all } \psi \in \Phi_{k-1}.$$

The asymptotic error estimate (4.4) yields that for k large enough, say $k \geq k_0$, we have

$$\|\varphi_m - \varphi\|_{L^\infty} \leq C h_m^2 |\ln h_m|^2 \|(P_k^*)^{-1}g\|_{L^\infty} \text{ if } m \in \{k, k-1\}.$$

Using (3.1.e) this yields for $k \geq k_0$

$$(4.9) \quad \|\varphi_k - \varphi_{k-1}\|_{L^\infty} \leq C h_k^2 |\ln h_k|^2 \|(P_k^*)^{-1}g\|_{L^\infty}.$$

From (3.7) it follows that $\varphi_k = P_k L_k^{-1}g$ and $\varphi_{k-1} = P_{k-1} L_{k-1}^{-1}rg$. Using Lemma 3.2 and (4.9) we have that for $k \geq k_0$:

$$\begin{aligned} \|(L_k^{-1} - pL_{k-1}^{-1}r)g\|_\infty &= \|P_k L_k^{-1}g - P_{k-1} L_{k-1}^{-1}rg\|_{L^\infty} = \|\varphi_k - \varphi_{k-1}\|_{L^\infty} \leq \\ &C h_k^2 |\ln h_k|^2 \|(P_k^*)^{-1}g\|_{L^\infty} \leq C h_k^2 |\ln h_k|^2 \|g\|_\infty. \end{aligned}$$

□

5. The Smoothing Property

The usual technique for proving the Smoothing Property requires symmetry (or a nearly symmetric situation), and yields results in the euclidean norm or in the energy norm. We refer to Wittum [24], where smoothing and the construction of smoothers are discussed in a general framework. A new approach to the Smoothing Property has been introduced in [20]. The analysis there does not use symmetry and can also be used for the maximum norm. A disadvantage of this new approach is that we need a damping factor less than or equal to 0.5 (whereas the conditions for the damping factor in [24] are less restrictive). The results of this section can be found in a more general setting in [20]. The analysis here is the same as for the 1D case in [19]. For completeness we give proofs here too.

The smoothing iteration we use is based on a splitting

$$(5.1) \quad L_k = W_k - R_k .$$

We make the following assumptions about this splitting:

$$(5.2.a) \quad W_k \text{ is regular and } \|W_k^{-1} R_k\|_\infty \leq 1 \text{ for all } k ,$$

$$(5.2.b) \quad \|W_k\|_\infty \leq ch_k^{-2} \text{ for all } k, \text{ with } c \text{ independent of } k .$$

Remark 5.1. It is well-known that if L_k is weakly diagonally dominant (i.e. $\sum_{j \neq i} |(L_k)_{ij}| \leq |(L_k)_{ii}|$ for all i) then (5.2.a) holds if (5.1) corresponds to the Jacobi or Gauss-Seidel relaxation. It can also be shown (cf. [21]) that the following criterion holds:

if L_k is such that $L_k e \geq 0$ (with $e = (1, 1, \dots, 1)^T$ and “ \geq ” entrywise) and if (5.1) is a (weak) regular splitting then (5.2.a) holds.

This criterion applies to the ILU splitting of an M -matrix L_k .

The proof of the Smoothing Property in Theorem 5.3 below is based on the following elementary lemma.

LEMMA 5.2. *Let A be a square matrix with $\|A\|_\infty \leq 1$. Then the following holds*

$$(5.3) \quad \|(I - A)(I + A)^\nu\|_\infty \leq 2 \binom{\nu}{\lfloor \frac{1}{2}\nu \rfloor} < 2^{\nu+1} \sqrt{\frac{2}{\pi\nu}} \quad (\nu \geq 1) .$$

Proof.

$$(I - A)(I + A)^\nu = (I - A) \sum_{k=0}^{\nu} \binom{\nu}{k} A^k = I - A^{\nu+1} + \sum_{k=1}^{\nu} \left(\binom{\nu}{k} - \binom{\nu}{k-1} \right) A^k .$$

So

$$(5.4) \quad \|(I - A)(I + A)^\nu\|_\infty \leq 2 + \sum_{k=1}^\nu \left| \binom{\nu}{k} - \binom{\nu}{k-1} \right|.$$

Using $\binom{\nu}{k} \geq \binom{\nu}{k-1} \Leftrightarrow k \leq \frac{1}{2}(\nu + 1)$, and $\binom{\nu}{k} = \binom{\nu}{\nu-k}$ we get

$$\begin{aligned} & \sum_{k=1}^\nu \left| \binom{\nu}{k} - \binom{\nu}{k-1} \right| = \\ &= \sum_1^{[\frac{1}{2}(\nu+1)]} \left(\binom{\nu}{k} - \binom{\nu}{k-1} \right) + \sum_{[\frac{1}{2}(\nu+1)]+1}^\nu \left(\binom{\nu}{k-1} - \binom{\nu}{k} \right) \\ &= \sum_1^{[\frac{1}{2}\nu]} \left(\binom{\nu}{k} - \binom{\nu}{k-1} \right) + \sum_{m=1}^{[\frac{1}{2}\nu]} \left(\binom{\nu}{m} - \binom{\nu}{m-1} \right) \\ &= 2 \sum_{k=1}^{[\frac{1}{2}\nu]} \left(\binom{\nu}{k} - \binom{\nu}{k-1} \right) = 2 \left(\binom{\nu}{[\frac{1}{2}\nu]} - \binom{\nu}{0} \right). \end{aligned}$$

Combined with (5.4) this yields the first inequality in (5.3). Elementary analysis yields that

$$\binom{\nu}{[\frac{1}{2}\nu]} < 2^\nu \sqrt{\frac{2}{\pi\nu}} \text{ for all } \nu \geq 1.$$

For details we refer to [20]. □

As a smoother we use a *damped* iteration based on the splitting in (5.1). We consider an iteration with iteration matrix

$$(5.5) \quad S_k = I - \theta W_k^{-1} L_k, \quad \theta \in]0, \frac{1}{2}].$$

THEOREM 5.3. *Assume that (5.2.a,b) holds. Then we have the following inequality with a constant C_S independent of k and ν :*

$$(5.6) \quad \|L_k S_k^\nu\|_\infty \leq C_S \frac{1}{\sqrt{\nu}} h_k^{-2}.$$

Proof. Let $A := I - 2\theta W_k^{-1} L_k = 2\theta W_k^{-1} R_k + (1 - 2\theta)I$. Then due to (5.2.a) and $\theta \in]0, \frac{1}{2}]$ we have $\|A\|_\infty \leq 1$. Using Lemma 5.2 we get

$$\begin{aligned} \|L_k S_k^\nu\|_\infty &= \|L_k (I - \theta W_k^{-1} L_k)^\nu\|_\infty \\ &= \left\| \frac{1}{2\theta} W_k (I - A) \left(\frac{1}{2}\right)^\nu (I + A)^\nu \right\|_\infty \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2\theta} \|W_k\|_\infty \left(\frac{1}{2}\right)^\nu 2^{\nu+1} \sqrt{\frac{2}{\pi\nu}} \\
&\leq \frac{1}{\theta} \sqrt{\frac{2}{\pi}} c \frac{1}{\sqrt{\nu}} h_k^{-2} =: C_S \frac{1}{\sqrt{\nu}} h_k^{-2}. \quad \square
\end{aligned}$$

6. Contraction number of the two-grid method

Theorem 4.2 and Theorem 5.3 combined immediately yield the following result.

THEOREM 6.1. *Assume (2.4.a-c), (3.1.a-e) and (5.2.a,b). Let $T_k(\nu)$ be as in (3.10) with S_k from (5.5). Then there are constants k_0 and C_{TG} independent of k and ν such that for all $k \geq k_0$, we have*

$$(6.1) \quad \|T_k(\nu)\|_\infty \leq C_{TG} \frac{1}{\sqrt{\nu}} |\ln h_k|^2.$$

Clearly Theorem 6.1 shows that if we take $\nu = \nu_k$ then for $\|T_k(\nu_k)\|_\infty \leq c < 1$ to hold, in our upperbound we need $\nu_k = g(h_k)$ with g a logarithmically growing function for $h_k \downarrow 0$. In §7 we show that, with the assumptions as in Theorem 6.1, at least one factor $|\ln h_k|$ is *necessary* in an upper bound for $\|T_k(\nu)\|_\infty$.

With respect to *multigrid* convergence we note the following. Using the technique as given by Hackbusch in [10] it is straightforward to derive a convergence result in the maximum norm for the multigrid W -cycle. A shortcoming of Theorem 6.1 is the requirement $k \geq k_0$ (due to the *asymptotic* error estimates in the proof of the Approximation Property). If we consider a multigrid W -cycle with coarsest triangulation \mathcal{T}_m and if $m < k_0$ then we need bounds for $\|T_k(\nu)\|_\infty$ with $k < k_0$ too. Convergence results in the euclidean norm yield such a bound:

$$\|T_k(\nu)\|_\infty \leq c h_k^{-1} \|T_k(\nu)\|_k \leq \frac{c}{\nu} h_k^{-1}.$$

So we have $\|T_k(\nu)\|_\infty \leq c < 1$ on coarse grids ($k < k_0$) if $\nu = \nu_k \sim h_k^{-1}$. Note that this is not very satisfactory, but also it is of minor importance because it does not affect the order of complexity (for $h_k \downarrow 0$) of the algorithm.

Remark 6.2. In this paper we only consider the *symmetric* variational problem (2.1). From §5 we see that in the proof of the Smoothing Property (Theorem 5.3) this symmetry is not used; the only conditions are (5.2.a,b). The proof of the Approximation Property (Theorem 4.2) remains valid for a more general (e.g. nonsymmetric) problem, provided that the results in Lemma 3.2 and in (4.4) hold. The results in Lemma 3.2 do not depend on $a(\cdot, \cdot)$ at all, but only on the triangulation. So our analysis yields a result as in Theorem 6.1 for a more general second order elliptic boundary value problem too, if the finite element error estimate (4.4) from [8,18] holds. In [8,18] only the symmetric situation as in (2.1) is considered, however, in [17] it is remarked that (4.4) can be carried over to general second order elliptic problems,

provided that $\partial\Omega$ and the coefficients are sufficiently regular, and that the corresponding proofs in [8,18] require only very little technical changes.

7. A lower bound for the contraction number

In this section we show that the estimate in Theorem 6.1 is sharp in the sense that a factor $|\ln h_k|$ is necessary (the power 2 of the $|\ln h_k|$ term, however, may be due to the method of proof).

We consider the Poisson equation on the unit square and use a linear finite element discretization on a uniform triangulation. We analyze the standard two-grid method as in §3 for solving the resulting system of equations. We show (cf. Theorem 7.6 and 7.7 below) that for a fixed number of smoothing iterations the maximum norm of the iteration matrix is bounded from below by $C |\ln h_k|$. Our approach is based on the analysis given in [11]. Also we use an important result from [11] (Lemma 7.3 below).

Let Ω be the unit square and consider the variational boundary value problem as in (2.1), (2.2) with

$$(7.1) \quad a(\varphi, \psi) := \int_{\Omega} \nabla \varphi \cdot \nabla \psi \, dx .$$

We use a uniform triangulation with mesh size parameter $h_k = 2^{-(k+1)}$ ($k \in \mathbb{N}_0$) as indicated in Fig. 1.

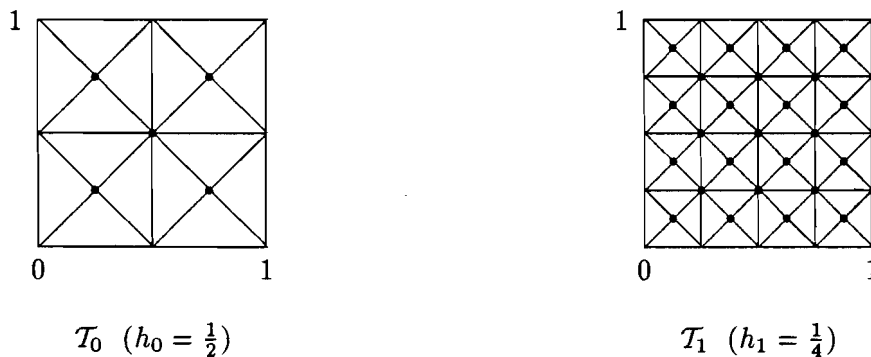


Fig. 1

Now let \mathcal{T}_{k-1} and \mathcal{T}_k be two successive triangulations. We use the definitions of §3 and for ease of notation we introduce $h := h_k$, $\mathcal{T}_h := \mathcal{T}_k$, $J_h := J_k$, $\Phi_h := \Phi_k$, $\varphi_h^i := \varphi_k^i$, $U_h := U_k$, $P_h := P_k$, $\langle u, v \rangle_h := \langle u, v \rangle_k$, $L_h := L_k$. Likewise we use $H := 2h$, \mathcal{T}_H , J_H , Φ_H , φ_H^i , U_H , P_H , $\langle \cdot, \cdot \rangle_H$, L_H . Furthermore the index in J_h (J_H) corresponding to the grid point $Q := (\frac{1}{2}, \frac{1}{2})$ is denoted by i_h^Q (i_H^Q) and the unit vector in U_h (U_H) corresponding to Q is denoted by e_h^Q (e_H^Q).

We now introduce a function $\omega_h \in \Phi_h$ which plays an important role in the analysis below. This function is defined as the piecewise linear interpolant on \mathcal{T}_h of the (h -dependent)

function

$$\omega(x_1, x_2) = \frac{1}{4} \sin\left(\frac{\pi}{h} |x_1 - \frac{1}{2}|\right) \sin\left(\frac{\pi}{h} |x_2 - \frac{1}{2}|\right).$$

We write $w_h := P_h^{-1} \omega_h$. For all grid points x_h^i in \mathcal{T}_h we have $\omega_h(x_h^i) \in \{-\frac{1}{4}, \frac{1}{4}, 0\}$. In Fig. 2 we show the pattern of these function values.

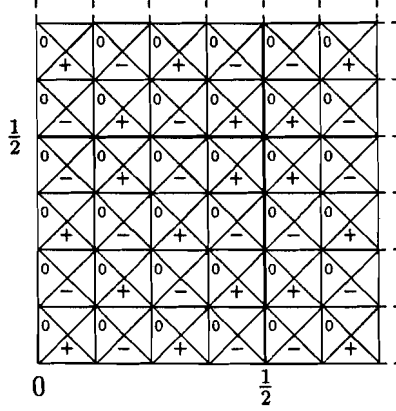


Fig. 2 Values of the function ω_h . +: value $\frac{1}{4}$; -: value $-\frac{1}{4}$; 0: value 0.

It is easy to prove Lemma 7.1 below. A simple proof can be found e.g. in [11].

LEMMA 7.1. For $i, j \in J_h$ the following holds

$$(7.2) \quad a(\varphi_h^i, \varphi_h^j) = \begin{cases} 4 & \text{if } i = j \\ -1 & \text{if } |x_h^i - x_h^j| = \frac{1}{2} \sqrt{2} h \\ 0 & \text{otherwise.} \end{cases}$$

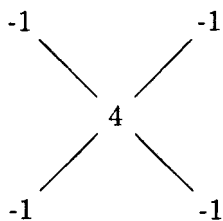
LEMMA 7.2. The following holds:

$$(7.3) \quad L_h w_h = h^{-2}(4w_h - e_h^Q),$$

and for $i \in J_H$:

$$(7.4) \quad a(\omega_h, \varphi_H^i) = \begin{cases} 0 & \text{if } i \neq i_H^Q \\ 1 & \text{if } i = i_H^Q. \end{cases}$$

Proof. We extend the vector $w_h \in U_h$ by taking zero values corresponding to all boundary grid points of \mathcal{T}_h . Then Lemma 7.1 yields that $h^2 L_h$ corresponds to the difference star



in all interior points.

For $i \in J_h$ the value $(h^2 L_h w_h)_i$ can be found by application of this difference star in the grid point x_h^i to the function $\omega_h = P_h w_h$. Fig. 2 then shows that

$$(h^2 L_h w_h)_i = \begin{cases} 4(w_h)_i & \text{if } i \neq i_h^Q \\ -1 & \text{if } i = i_h^Q . \end{cases}$$

So $h^2 L_h w_h = 4w_h - e_h^Q$ holds.

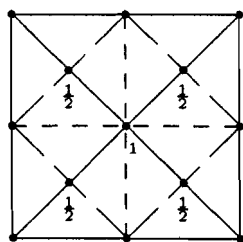
The proof of (7.4) runs as follows. For $i \in J_H$ we have

$$(7.5) \quad a(\omega_h, \varphi_H^i) = \langle L_h w_h, P_h^{-1} \varphi_H^i \rangle_h = 4h^{-2} \langle w_h, P_h^{-1} \varphi_H^i \rangle_h - h^{-2} \langle e_h^Q, P_h^{-1} \varphi_H^i \rangle_h .$$

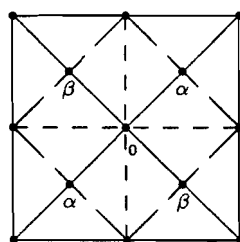
Note that

$$(7.6) \quad h^{-2} \langle e_h^Q, P_h^{-1} \varphi_H^i \rangle_h = (P_h^{-1} \varphi_H^i)_{i_h^Q} = \begin{cases} 0 & \text{if } i \neq i_H^Q \\ 1 & \text{if } i = i_H^Q . \end{cases}$$

With respect to the term $4h^{-2} \langle w_h, P_h^{-1} \varphi_H^i \rangle_h$ in (7.5) we note the following. First we take $i \in J_H$ such that x_H^i is a vertex of 4 triangles in \mathcal{T}_H . Then φ_H^i has nonzero values in 5 grid points of \mathcal{T}_h as indicated in Fig. 3. Using Fig. 2 we see that in these grid points ω_h has values as indicated in Fig. 3.



φ_H^i

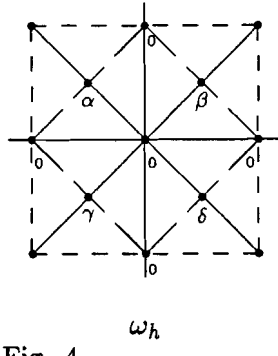
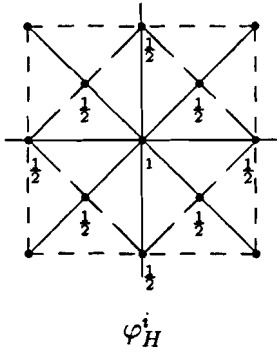


ω_h

Fig. 3

$\alpha, \beta \in \{-\frac{1}{4}, \frac{1}{4}\}$
 $\alpha + \beta = 0$
 $-$: triangles in \mathcal{T}_H

So $4h^{-2} \langle w_h, P_h^{-1} \varphi_H^i \rangle_h = 0$. Secondly, for $i \in J_H \setminus \{i_H^Q\}$ such that x_H^i is a vertex of 8 triangles in \mathcal{T}_H we have values for φ_H^i and ω_h as illustrated in Fig. 4.



$\alpha, \beta, \gamma, \delta \in \{-\frac{1}{4}, \frac{1}{4}\}$
 $\alpha + \beta + \gamma + \delta = 0$
 $\text{---: triangles in } \mathcal{T}_H$

Fig. 4

So again $4h^{-2} < w_h, P_h^{-1} \varphi_H^i >_h = 0$. Finally for $i = i_H^Q$ we have a situation as in Fig. 4 but now with $\alpha = \beta = \gamma = \delta = \frac{1}{4}$, so $4h^{-2} < w_h, P_h^{-1} \varphi_H^i >_h = 2$.

Using these results for $4h^{-2} < w_h, P_h^{-1} \varphi_H^i >_h$ and the result (7.6) in (7.5) proves (7.4). \square

A proof of the following lemma is given by Haverkamp in [11; Lemma 5].

LEMMA 7.3. *If $\psi_H \in \Phi_H$ is such that for $i \in J_H$ we have*

$$a(\psi_H, \varphi_H^i) = \begin{cases} 0 & \text{if } i \neq i_H^Q \\ 1 & \text{if } i = i_H^Q, \end{cases}$$

then $\psi_H(Q) > \pi^{-2} |\ln H|$.

LEMMA 7.4. *The following holds:*

$$\|(I - pL_H^{-1} rL_h) w_h\|_\infty > \pi^{-2} |\ln(2h)| .$$

Proof. We take ψ_H as in Lemma 7.3. Then it follows (cf. Lemma 7.2) that ψ_H is the orthogonal projection of ω_h on Φ_H :

$$a(\psi_H, \varphi_H^i) = a(\omega_h, \varphi_H^i) \text{ for all } i \in J_H .$$

So

$$\langle L_H P_H^{-1} \psi_H, P_H^{-1} \varphi_H^i \rangle_H = \langle rL_h w_h, P_H^{-1} \varphi_H^i \rangle_H \text{ for all } i \in J_H ,$$

and thus

$$\psi_H = P_H L_H^{-1} rL_h w_h .$$

This yields

$$\begin{aligned} \|(I - pL_H^{-1}rL_h)w_h\|_\infty &= \|P_h(I - pL_H^{-1}rL_h)w_h\|_\infty \\ &= \|\omega_h - \psi_H\|_\infty \geq |(\omega_h - \psi_H)(Q)| = |\psi_H(Q)| > \pi^{-2} |\ln(2h)|. \end{aligned} \quad \square$$

We now prove three theorems in which the main results of this section are given. Theorem 7.5 gives an estimate related to the Approximation Property in Theorem 4.2. In Theorem 7.6 and Theorem 7.7 we give lower bounds for $\|T_h(\nu)\|_\infty$. In Theorem 7.6 we take damped Jacobi and damped Gauss-Seidel as a smoother with a damping parameter $\theta \in]0, \frac{1}{2}[$. In Theorem 7.7 we consider damped Jacobi again, but then with $\theta \in]0, 1[$.

THEOREM 7.5. *The following holds:*

$$\|L_h^{-1} - pL_H^{-1}r\|_\infty > \pi^{-2} h^2 |\ln(2h)|.$$

Proof. From Lemma 7.2 we have $\|L_h w_h\|_\infty = h^{-2} \|4w_h - e_h^Q\|_\infty = h^{-2}$. Lemma 7.4 then yields

$$\|L_h^{-1} - pL_H^{-1}r\|_\infty \geq \|(L_h^{-1} - pL_H^{-1}r)L_h w_h\|_\infty / \|L_h w_h\|_\infty > \pi^{-2} h^2 |\ln(2h)|. \quad \square$$

We consider a splitting as in (5.1) corresponding to the Jacobi or Gauss-Seidel method. Note that the conditions (5.2.a,b) are fulfilled. We use a damped version with damping parameter θ ($S_h = I - \theta W_h^{-1}L_h$).

THEOREM 7.6. *For $\theta \in]0, \frac{1}{2}[$ the following holds:*

$$\|T_h(\nu)\|_\infty > 4\pi^{-2}(1 - 2\theta)^\nu |\ln(2h)| \quad (\nu \geq 0).$$

Proof. $\|I - W_h^{-1}L_h\|_\infty \leq 1$ holds and thus $\|W_h^{-1}L_h\|_\infty \leq 2$. So for $\theta \in]0, \frac{1}{2}[$ we have

$$(7.7) \quad \|(I - \theta W_h^{-1}L_h)^{-1}\|_\infty \leq (1 - \theta \|W_h^{-1}L_h\|_\infty)^{-1} \leq (1 - 2\theta)^{-1}.$$

Combining (7.7), Lemma 7.4 and $\|w_h\|_\infty = \frac{1}{4}$ yields

$$\begin{aligned} \|T_h(\nu)\|_\infty &= \|(I - pL_H^{-1}rL_h)S_h^\nu\|_\infty \\ &\geq \|(I - pL_H^{-1}rL_h)w_h\|_\infty / \|S_h^{-\nu}w_h\|_\infty > \pi^{-2} |\ln(2h)| (1 - 2\theta)^\nu 4. \end{aligned} \quad \square$$

THEOREM 7.7. *Let $S_h = I - \theta \frac{1}{4} h^2 L_h$ be the iteration matrix corresponding to the damped Jacobi iteration. Then for $\theta \in]0, 1[$ the following holds with a constant K independent of h :*

$$\|T_h(\nu)\|_\infty > 4\pi^{-2}(1 - \theta)^\nu |\ln(2h)| - K \quad (\nu \geq 0).$$

Proof. The following holds (cf. Lemma 7.2):

$$S_h w_h = (I - \theta \frac{1}{4} h^2 L_h) w_h = (1 - \theta) w_h + \frac{1}{4} \theta e_h^Q .$$

Thus

$$(7.8) \quad S_h^\nu w_h = (1 - \theta)^\nu w_h + \frac{1}{4} \theta \sum_{k=0}^{\nu-1} (1 - \theta)^k S_h^{\nu-1-k} e_h^Q .$$

The analysis of the coarse grid correction w.r.t. the scaled euclidean norm $\|\cdot\|_h = \langle \cdot, \cdot \rangle_h^{\frac{1}{2}}$ yields (see e.g. [10]):

$$(7.9) \quad \|I - pL_H^{-1}rL_h\|_h \leq K , \quad K \text{ independent of } h .$$

Also note that for every $\theta \in]0, 1[$ we have

$$(7.10) \quad \|S_h^m e_h^Q\|_h \leq \|S_h^m\|_h \|e_h^Q\|_h \leq h \quad (m \geq 0) .$$

Combination of Lemma 7.4, (7.8), (7.9), (7.10) and $\|u\|_\infty \leq h^{-1} \|u\|_h$ yields

$$\begin{aligned} \|(I - pL_h^{-1}rL_h) S_h^\nu\|_\infty &\geq 4 \|(I - pL_H^{-1}rL_h) S_h^\nu w_h\|_\infty \\ &\geq 4(1 - \theta)^\nu \|(I - pL_H^{-1}rL_h) w_h\|_\infty - \theta \|(I - pL_H^{-1}rL_h) \sum_{k=0}^{\nu-1} (1 - \theta)^k S_h^{\nu-1-k} e_h^Q\|_\infty \\ &> 4(1 - \theta)^\nu \pi^{-2} |\ln(2h)| - \theta h^{-1} K \sum_{k=0}^{\nu-1} (1 - \theta)^k h \\ &> 4(1 - \theta)^\nu \pi^{-2} |\ln(2h)| - K . \end{aligned}$$

□

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