

Upper bounds for the length of normal forms and for the length of reduction sequence in lambda-typed lambda calculus

Citation for published version (APA):

Bruijn, de, N. G. (1985). *Upper bounds for the length of normal forms and for the length of reduction sequence in lambda-typed lambda calculus*. Technische Universiteit Eindhoven.

Document status and date:

Published: 01/01/1985

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
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August 12, 1985

M23

Revised form, September 20, 1985

Upper bounds for the length of normal forms
and for the length of reduction sequences
in lambda-typed lambda calculus

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1. For notations and definitions, as far as not explained in this note itself, we refer to [1]. Nevertheless it should be noted that for the material of the present note there is no substantial difference between the system of [1] and the system Λ of [4]. Almost all ideas involved in this note are already in [4]; the only new idea is to use Nederpelt's normalization proof for estimating the length of the normal forms. The estimate we give here for the length of the normal form is presented as a very fast growing function of the length of the given non-normal expression. It is of the order of the diagonal of the Ackermann function.
2. We consider the minireductions of [1] section 4, slightly generalized since we shall admit these operations with the

use of AT-couples ([1] section 4.3) instead of just AT-pairs ([1] section 4.2).

A lambda tree will be called almost-normal if each one of its AT-couples admits AT-removal ([1] section 4.5). The idea will be that we start from a lambda tree, apply a sequence of local beta-reductions and hope to end up with an almost-normal lambda tree. From that point onwards it will be a simple action to get to a normal lambda tree, just omitting AT-couples until there are no more left.

The term "almost-normal" will also be used for a subset of the set of points of a lambda tree. It will mean that this subset contains no end-points that refer to the T-part of an AT-couple.

3. The number of end-points in a lambda tree will be called its length. If a lambda tree (V,lab) is semicorrect, it has a norm, denoted by $norm(V,lab)$ ([1] section 5.9). This is again a lambda tree; its length is $length(norm(V,lab))$, and will be abbreviated to $ln(V,lab)$.

The notion of norm was also defined for subtrees, and that gives rise to the similar extension for ln .

If a tree (V,lab) is represented as a formula P , we just write $ln(P)$ instead of $ln(V,lab)$.

4. Let us develop some terminology around the structure of the main line (see [1] section 5.2) of a lambda tree. The rightmost label on the main line is either a dummy or τ , all the others are either T or A. In the formula these T's and A's correspond to abstractors $[...:...]$ and applicators $\langle...\rangle$. We can draw a graph of the sequence if we represent T's and A's by line segments with positive and negative slope, respectively. An example is shown in figure 1.

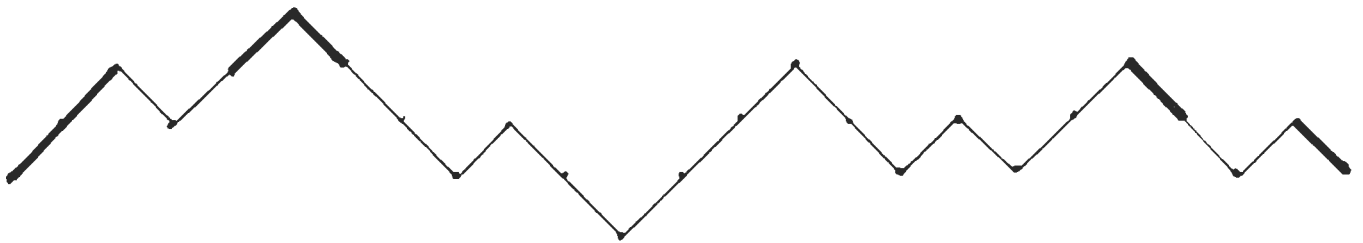


Figure 1. Graph of TTATTAATAATTTAATATTAATA

In figure 1 we have drawn the T's as heavy lines if they are "visible" from the left, and the A's as heavy lines if they are visible from the right. These visible T's can be said to form the western facade, and the visible A's the eastern facade. All the other A's and T's can be combined pairwise to AT-couples. This coupling can be visualised as follows: we consider the graphs as a longitudinal section of a folded paper strip. Next we apply glue to the upper side of the strip and press the strip together (pressing from the left and from the right). Taking the pressure away, we see that the AT-couples are pasted together, and that western and eastern facade are uninterrupted lines (figure 2).



Figure 2. Showing western facade, eastern facade, and AT-couples.

To every T and to every A in the main line sequence there can be attached a positive number: the length of the norm of the left-hand subtree. Or, in terms of expressions instead of trees, to the abstractor $[x:P]$ we attach the number $\ln(P)$, and to the applicator $\langle Q \rangle$ we attach $\ln(Q)$. Note that these norms are defined only by virtue of the lower part of the tree; that part gives a meaning to the types of the dummies involved in P and Q .

Similarly we can attach a value of \ln to the rightmost end-point of the lambda tree, either $\ln(x)$, where x is a dummy, or $\ln(\tau)$ ($=1$).

Let $[x_1:P_1], \dots, [x_k:P_k]$ be the abstractors of the western facade, and $\langle Q_1 \rangle, \dots, \langle Q_m \rangle$ the applicators of the eastern facade.

Let us call

$$\ln(P_1) + \dots + \ln(P_k)$$

the opening value of our lambda tree,

$$\ln(Q_1) + \dots + \ln(Q_m)$$

the deficit of the sequence of abstractors and applicators. The length of the norm of the rightmost end-point will be called the closing value.

We note that all terminology developed thus far for lambda trees, can be used for its subtrees too.

5. If a lambda tree is norm-correct ([1] section 5.9) then in every AT-couple $\langle Q \rangle [x:P]$ we have $\ln(Q) = \ln(P)$. Moreover we have $\ln([x:P]R) = \ln(P) + \ln(R)$, $\ln(\langle Q \rangle R) = \ln(R) - \ln(Q)$. By the way, this shows that the purely accidental notational similarity between "length of norm" and "natural logarithm" is not so unfortunate after all. The formation of $[x:P]R$ can be felt as a kind of product, and $\langle Q \rangle R$ as a kind of quotient (with the denominator in front).

6. The following theorem for norm-correct lambda trees can be proved by routine methods:

Theorem 1. Let E be a subtree in a norm-correct lambda tree, let p be its opening value, d its deficit, q its closing value.

Then we have

$$\ln(E) = p - d + q$$

and

$$\ln(E) > 0, \quad d < q.$$

7. We introduce a slight modification of the notion of almost-normality, to be called fractured almost-normality. Let P be a lambda tree, and let it be written in the form $WU\omega$, where W and U are sequences of abstractors and applicators, and ω is either τ or a dummy. Let the eastern facade of W consist of the applicators $\langle Q_1 \rangle, \dots, \langle Q_m \rangle$.

We say that P is fractured almost normal with respect to W if all its AT-couples admit AT-removals, possibly except for AT-couples of which the applicator is one of $\langle Q_1 \rangle, \dots, \langle Q_m \rangle$.

So in $WU\omega$ there may occur bound instances of dummies which are bound by abstractors in U which form AT-couples with one of $\langle Q_1 \rangle, \dots, \langle Q_m \rangle$. If such bound instances do not occur in U itself (leaving the final x as the only possibility), we can express this by saying that WU is almost-normal.

If $WU\omega$ is fractured almost-normal with respect to W , and if WU is almost-normal, then we have two possibilities:

(i) The final ω does not admit local beta-reduction. In other words, ω is either τ or it is a dummy bound by an abstractor that does not form an AT-couple with an applicator in the eastern facade of W .

In this case $WU\omega$ is obviously almost-normal.

(ii) The final ω is a dummy x , bound by an abstractor forming an AT-couple with some $\langle Q_j \rangle$ from the eastern facade of W . Let this Q_j have the form $V\sigma$. (σ is τ or a dummy). We apply local beta-reduction, which transforms WUx into $WUV\sigma$.

In this situation it is easy to see that $WUV\sigma$ is fractured almost-normal with respect to WU . The dummies in $V\sigma$ bound by

abstractors outside $V\sigma$ cannot give rise to beta-reduction, since $V\sigma$ is a transplantation of a piece of W , and W is almost-normal. But the dummies in $V\sigma$ bound by abstractors in $V\sigma$ itself, can give rise to local beta-reductions; those abstractors form AT-couples with applicators on the eastern facade of WU .

We shall show that

$$\text{deficit}(WU) < \text{deficit}(W) \quad (1)$$

under the assumption that our lambda tree WUx is norm-correct.

According to section 6 we have $\text{deficit}(U) < \ln(x)$. But the eastern facade of WU can contain more than the one of U : it also contains that part of the eastern facade of W that does not form couples with pieces of the western facade of Ux . One such piece of the eastern facade of W is $\langle Qj \rangle$, mentioned above. Therefore

$$\text{deficit}(WU) \leq \text{deficit}(W) - \ln(Qj) + \text{deficit}(U).$$

Since WU is norm-correct we have $\ln(Qj) = \ln(x)$. And by Theorem 1 we have $\text{deficit}(U) < \ln(x)$. So $\text{deficit}(U) < \ln(Qj)$, and (1) follows.

8. It was shown in [2] that a lambda tree P of length n satisfies $\ln(P) \leq 2^{n-1}$. It follows that if a lambda tree has length n then its deficit is at most $n \cdot 2^{n-1}$. This estimate is rough; let us write $H(n)$ for the best possible estimate.

9. Assume that it has been proved for a particular value of n

that any norm-correct lambda tree of length $<n$ can be reduced, by repeated local beta-reduction (no AT-removals), to an almost-normal form of length at most q . Then it is not hard to show that if $R\omega$ (where R is a sequence of abstractors and applicators and ω is either τ or a dummy) is a norm-correct lambda tree of length n then we can reduce the sequence R to an almost-normal form of length at most $(n-1)q$.

10. We define the functions G and V by recursion.

The function G is a function of three integer variables k, h, n , all having non-negative values only. The values of G are non-negative integers. We define them by recursion, according to the lexicographic order of the pairs (k,n) :

(i) if $k = 0$ or $n = 0$:

$$G(k,h,n) = h + n + 1,$$

(ii) if $k > 0, n > 0$:

$$G(k,h,n) = G(k-1, h + (n-1)G(k,h,n-1), h).$$

This definition is very similar to the one of the Ackermann function (see [3]).

It is easy to show that G is a monotonically increasing function of each one of its variables.

By means of this function G we define the function V : $V(n)$ will be defined for all positive integers n , and its values will be positive integers. We define

$$V(1) = 1,$$

$$V(n+1) = \max(G(H(n), nV(n), nV(n)) , nV(n) + 1),$$

where H is the function mentioned in section 8.

11. We now come back to the situation sketched in section 7.

We shall prove the following

Theorem 2. Let $WR\omega$ be a norm-correct lambda tree, fractured almost-correct with respect to W . Let h be the length of W , n the length of R , k the deficit of W . Then $WR\omega$ can be reduced (by local beta-reductions, without AT-removal) to an almost-normal form of length $\leq G(k,h,n)$.

Proof. We apply induction with respect to k , next for every value of k induction with respect to n , and once k and n have been fixed we prove the statement for all h simultaneously. If $k = 0$ or $n=0$ then $WR\omega$ is almost normal itself; its length is $h + n + 1$.

Let $k > 0$, and assume that the statement is true for all smaller values of k (with arbitrary h and n). We apply induction with respect to n . The case $n=0$ has been settled, so we take $n > 0$, and we assume that the statement is true for all smaller values of n .

We now apply the idea exposed in section 9, with $q=G(k,h,n-1)$. The idea has to be slightly modified because of the W in front and the fractured almost-normality, but we shall leave this to the reader. Because of our induction assumption we conclude that our WR can be reduced to some almost-normal WU (the reduction cannot affect W), where the length of U is $\leq nG(k,h,n-1)$.

We can now apply section 7. In case (i) $WU\omega$ is almost-normal, in case (ii) $WU\omega$ reduces to $WUV\sigma$, the length of V is at most h (V being a copy of a part of W), $WUV\sigma$ is fractured almost-normal with respect to WU , and the

deficit of WU is less than k . So by induction $WUV\sigma$ reduces to something of length at most $G(k-1, h+(n-1)G(k, h, n-1), h)$. By the definition of G this is at most $G(k, h, n)$.

12. We can now prove our main result.

Theorem 3. A norm-correct lambda tree of length m can be reduced (local beta-reductions, no AT-removals) to an almost-correct lambda tree with length $\leq V(m)$.

Proof. We apply induction with respect to m . The case $m=1$ is trivial. Having proved the result for $m \leq n$, we shall treat the case $m = n + 1$.

Let the tree be presented as $Q\omega$ (where ω is either τ or a dummy). So Q has length n . According to section 9 we can reduce Q to an almost-normal form W of length $\leq nV(n)$.

If ω equals τ , or a dummy that does not admit beta-reduction, then $W\omega$ is almost-normal, so it suffices to remark that $nV(n) + 1 \leq V(n + 1)$.

If ω is a dummy that admits beta-reduction, then this reduction leads to $WU\sigma$ (where again σ is either τ or a dummy), and this is fractured almost-normal (see section 7), so we can apply section 11. The length of W is at most $nV(n)$. Furthermore, U is a copy of a subexpression of W , so its length is $\leq nV(n)$ too (but it is not hard to show that it is $\leq V(n)$ itself). The deficit of W can be estimated by $H(n)$ according to section 8 (remark that the deficit did not change by passing from Q to W ; without that remark we would have to use the estimate $H(nV(n))$ instead of $H(n)$). Applying Theorem 2 we get an almost-normal form of length at most $G(H(n), nV(n), nV(n))$, and this equals $V(n+1)$.

13. We shall indicate how strong normalization for norm-correct lambda trees follows from Theorem 3.

First we mention that for our local beta-reduction the Church-Rosser property holds. This is essentially contained in Nederpelt [4], where it is proved for β_i -reduction (β_i -reduction is the effect of local beta-reduction applied to all dummies bound by one and the same abstractor).

The normal forms corresponding to local beta-reduction are what we have called almost-normal. From Theorem 3 we have the reducibility to such an almost-normal form, and the Church-Rosser property expresses the uniqueness.

If local beta-reduction transforms the lambda tree (V, lab) into a lambda tree (V', lab') then (V, lab) is almost entirely embedded in (V', lab') : we have $V \subset V'$, and lab' coincides with lab on V with the exception of the end-points to which the beta-reduction was applied. It follows that (V, lab) , and all its reducts (by repeated local beta-reduction), are embedded in the almost-normal form, if we take exception for the labels of end-points.

By a local beta-reduction the length of a lambda tree will increase, apart from a trivial case where it remains constant. The trivial case is that the AT-couple in question has its applicator in the form $\langle x \rangle$, where x is either τ or a dummy.

This increase of length enables us to prove strong normalization.

Theorem 4 (strong normalization for local beta-reduction). Let P be a norm-correct lambda tree of length n , and let the sequence

$$P = P_0, P_1, P_2, \dots, P_m$$

be such that every P_{i+1} is obtained from P_i by some local beta-reduction. Then we have $m \leq (V(n))^3 + V(n)$.

Proof. Let Q be the almost-normal form of P . We have

$$\text{length}(P_i) \leq \text{length}(P_{i+1}) \leq \text{length}(Q) \leq V(n).$$

We show that a sub-sequence $P_j, P_{j+1}, \dots, P_{j+h}$ can have constant length only if $h \leq (V(n))^2$. In such a sub-sequence the tree is constant, only the labels at end-points can change. At such an end-point a dummy is sometimes replaced by another one whose abstractor lies lower in the tree, or by τ . The number of candidates for the change is at most equal to the length of the tree, so at most $V(n)$. A candidate can never turn up a second time for the same position. Irrespective of the order in which these replacements are carried out, we conclude to $h \leq (V(n))^2$.

In the sequence P_1, \dots, P_n we have at most $V(n) - 1$ steps where the length increases, and, at most $V(n)$ subintervals of length $\leq (V(n))^2$ where the length is constant. This proves the theorem.

14. If we admit both local beta-reduction and AT-removal, we get to what is usually called the normal form. It is easy to understand that the longest reduction sequences are obtained by taking first all the local beta-reduction steps and then the AT-removals. A tree with length $\leq V(n)$ can not have more than $V(n)$ AT-couples, so a reduction sequence (starting from a norm-correct tree) can not have more than $(V(n))^3 + 2V(n)$ steps.

Ordinary beta-reduction consists of a sequence of local beta-reductions followed by AT-removal. Therefore we also have

the bound $(V(n))^3 + 2V(n)$ for the number of ordinary beta-reduction steps.

15. We have presented bounds for the length of the normal form expressed in terms of the length of the initial tree. A simple example shows that it can not be done in terms of the length of the norm of the tree.

We can present an infinite sequence of norm-correct (and even correct) lambda trees, all with the same norm. It will be clear how the sequence is defined if we just present its sixth entry:

$$[v: \tau][x:v][y:[u:v]v] \lllll \langle x \rangle y \rangle y \rangle y \rangle y \rangle y.$$

Actually these can occur in an Automath book on natural numbers: there v is the type of naturals, y is the successor function and x is the number 1.

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