Smoke dispersion from a high chimney

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SMOKE DISPERSION FROM A HIGH CHIMNEY

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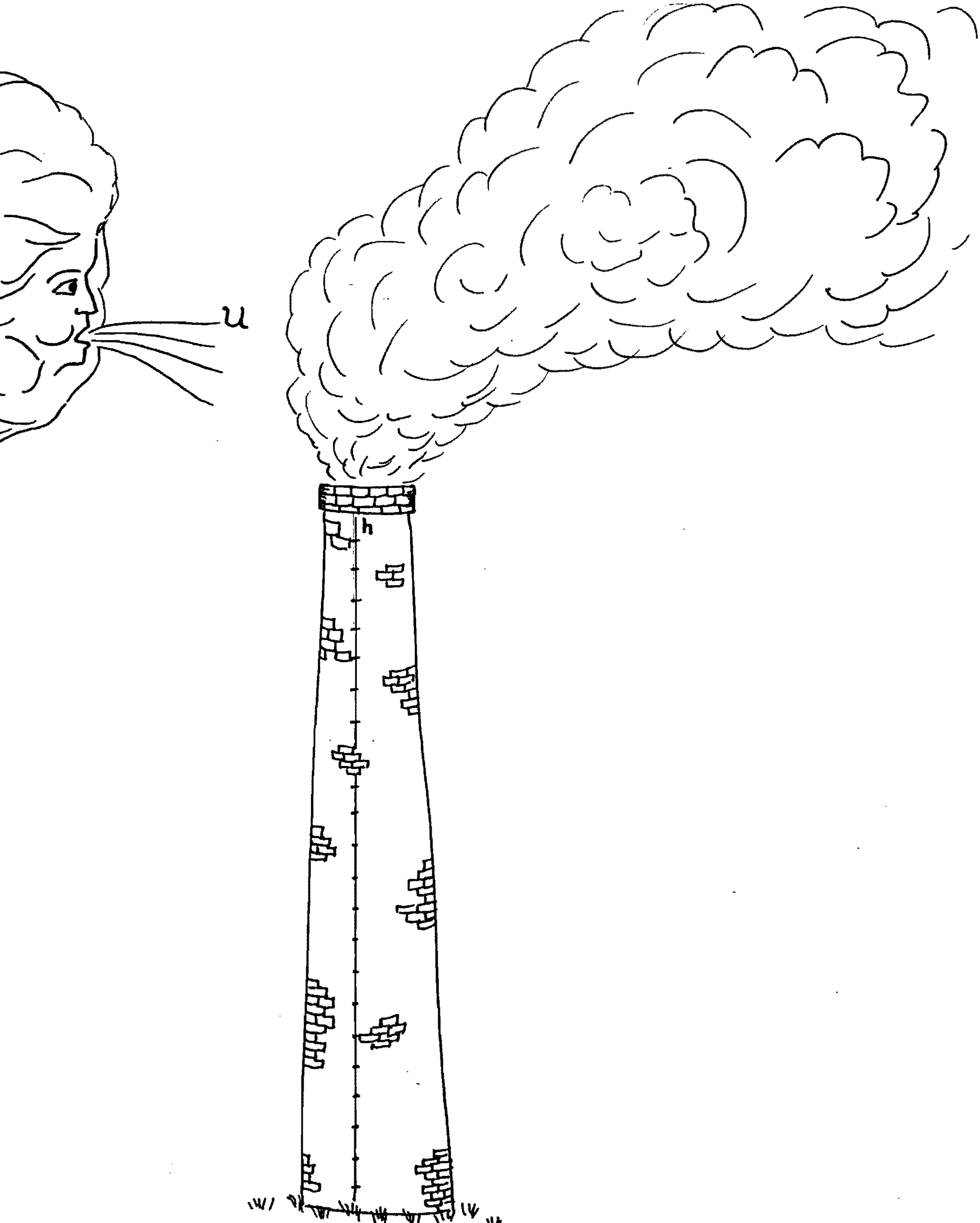
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1 Preface

The aim of this paper is to describe the distribution of pollutant on the ground emitted by a high chimney. Of special interest is the position and the value of the maximum in the steady state case. We have to model the following situation:
Smoke, which is a mixture of gases and particles, is moving under the influence of quite a lot of different phenomena. Among these are diffusion, wind, gravitation, thermal convection or the velocity of the smoke on leaving the chimney. Of course, it is almost impossible to take into account all these influences. Therefore we consider a number of different models which take into account only diffusion and wind effects. In the last chapter the results are compared.
This project is based on the lecture 'heat and mass-transfer' by Prof. McKee held at Eindhoven in March 1991. The model described in chapter 1 was presented in this lecture.

2 The General Diffusion - Wind - Model

In our models we will only consider the influence of diffusion, wind and absorption on particle motion. The effects of gravity can be neglected because the particles are generally small ($1\mu m$). Furthermore we assume that the smoke has been cooled down to the temperature of the surrounding air when it leaves the chimney. That is for example true for modern power stations. At least under this assumption thermal convection can also be neglected.
At the ground the smoke will be deposited at a rate λ which depends on the terrain and the local concentration of pollutant.
This leads to the following description of the process :
The air is treated as a moving medium which carries the smoke density as a quantity comparable to the temperature in the heat equation. This means that the exchange of smoke between air volumes is only described by diffusion. The influence by the other forces, e.g. thermal convection, is implied by the air velocity we assume to be given. The describing differential equation then results from the conservation of mass:
For an arbitrary volume element Ω holds:

$$\underbrace{\frac{d}{dt}\int_{\Omega} c\, dV}_{Increase\ of\ mass\ in\ \Omega\ per\ unit\ time} = \underbrace{-\int_{\partial\Omega} c\vec{u}\vec{n}\, dS}_{mass\ pushed\ into\ \Omega\ by\ the\ wind} + \underbrace{\int_{\partial\Omega} D\nabla c\vec{n} dS}_{mass\ diffusing\ into\ \Omega}$$

Taking into account that we consider the stationary case and applying Gauss we find:

$$\nabla(c\vec{u}) = \nabla(D\nabla c) \qquad (1)$$

Absorption on the ground is modelled by the following boundary condition:

$$\nabla c\vec{n} = \lambda c$$

where $\vec{n}$ is the normal vector on the ground.

3 2-Dimensional No-Ground-Flux - Model

To get concise mathematical results, we will first consider a very simple version of the general model:

Simplifications

- Constant diffusion coefficient
- Horizontal winds of constant velocity U
- Constant emission Q per unit time
- The environment of the chimney is a plane.
- The problem is considered in the plane of the prevailing wind.
- No absorption on the ground
- In the direction of the prevailing wind diffusion is dominated by the wind effects.

We use the following non-dimensional variables :

$$x' = \frac{x}{\frac{Uh^2}{D}} \;,\; z' = \frac{z}{h} \;,\; c' = \frac{D}{Qh^2}$$

Skipping the primes and assuming that $\frac{D}{Uh} \ll 1$, equation (1) reduces to

$$\frac{\partial}{\partial x}c(x,z) = \frac{\partial^2}{\partial z^2}c(x,z) \text{ on } \mathbb{R}^+ \times \mathbb{R}^+ \tag{2}$$

($\frac{D}{Uh} \ll 1$ means that the mass transport due to wind dominates that due to diffusion.)

The following boundary conditions seem to be reasonable:
No flux through the ground :

$$\frac{\partial}{\partial z}c(x,z=0) = 0$$

No sinks in the environment :

$$\lim_{x \to \infty} \frac{\partial}{\partial x} c(x, z) = 0$$

$$\lim_{z \to \infty} \frac{\partial}{\partial z} c(x, z) = 0$$

Point source on top of the chimney (at the height h)

$$c(x = 0, z) = \delta(z - 1)$$

The problem can be solved by applying Laplace transforms:

$$c(x, 0) = \frac{1}{\sqrt{\pi x}} e^{-\frac{1}{4x}} \tag{3}$$

which has a maximum value of $\sqrt{\frac{2}{\pi e}}$ at $x = \frac{1}{2}$.

For details cf [1] .

4 2-Dimensional Ground-Flux - Model

4.1 Model

In this and the subsequent chapters we give up some of the most unrealistic simplifications which we made in chapter 3. We start by allowing absorption on the ground :

Generalizations w.r.t. the model of chapter 3

- The ground is absorbing the pollutant at a rate proportional to its concentration.

Switching to the non-dimensional variables

$$x' = \frac{x}{\frac{Uh^2}{D}} \ , \ z' = \frac{z}{h} \ , \ c' = \frac{D}{Qh^2}c \ , \ \lambda' = h\lambda$$

the modified problem reads as follows:

$$\frac{\partial}{\partial x}c(x,z) = \frac{\partial^2}{\partial z^2}c(x,z) \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^+$$

$$\lim_{x\to\infty} \frac{\partial}{\partial x}c(x,z) = 0$$

$$\lim_{z\to\infty} \frac{\partial}{\partial z}c(x,z) = 0$$

$$c(x=0,z) = \delta(z-1)$$

$$\frac{\partial}{\partial z}c(x,z=0) = \lambda c(x,z=0)$$

λ is a positive constant.

In order to find a solution we take the Laplace transforms ($\mathcal{L}$) with respect to x and z:

$$
\begin{array}{rlcl}
 & \frac{\partial}{\partial x}c(x,z) & = & \frac{\partial^2}{\partial z^2}c(x,z) \\
\overset{\mathcal{L}\text{w.r.t.x}}{\Rightarrow} & c(x,z)e^{-px}|_0^\infty + p\overline{c}(p,z) & = & \frac{\partial^2}{\partial z^2}\overline{c}(p,z) \\
\overset{c(x=0,z)=\delta(z-1)}{\Rightarrow} & \frac{\partial^2}{\partial z^2}\overline{c}(p,z) - p\overline{c}(p,z) & = & -\delta(z-1) \\
\overset{\mathcal{L}\text{w.r.t.z}}{\Rightarrow} & \frac{\partial}{\partial z}\overline{c}(p,z)e^{-qz}\Big|_0^\infty + q\overline{c}(p,z)e^{-qz}|_0^\infty & & \\
 & +q^2\overline{\overline{c}}(p,q) - p\overline{\overline{c}}(p,q) & = & -e^{-q} \\
\overset{\frac{\partial}{\partial z}\overline{c}(p,z=0)=\lambda\overline{c}(p,z=0)}{\Rightarrow} & \overline{\overline{c}}(p,q) & = & \frac{(\lambda+q)\overline{c}(p,z=0)-e^{-q}}{q^2-p}
\end{array}
$$

The inverse Laplace transform of the latter expression can be found using tables and applying the convolution theorem:

$$
\begin{aligned}
\Rightarrow \quad \overline{c}(p,z) &= \left(\lambda\frac{\sinh\sqrt{p}z}{\sqrt{p}} + \cosh\sqrt{p}z\right)\overline{c}(p,z=0) \\
&\quad -\int_0^z \frac{\sinh\sqrt{p}(z-t)}{\sqrt{p}}\delta(t-1)dt \\
&= \left(\lambda\frac{\sinh\sqrt{p}z}{\sqrt{p}} + \cosh\sqrt{p}z\right)\overline{c}(p,z=0) - \frac{\sinh\sqrt{p}(z-t)}{\sqrt{p}}\ \chi_{[1,\infty)} \\
\Rightarrow \quad \frac{\partial}{\partial z}\overline{c}(p,z) &= (\lambda\cosh\sqrt{p}z + \sqrt{p}\sinh\sqrt{p}z)\overline{c}(p,z=0) - \cosh\sqrt{p}z\ \chi_{[1,\infty)} \\
\overset{\overline{c}(p,\infty)=0}{\Rightarrow} \quad \overline{c}(p,z=0) &= \lim_{z\to\infty}\frac{\cosh\sqrt{p}(z-1)}{\lambda\cosh\sqrt{p}z + \sqrt{p}\sinh\sqrt{p}z} \\
&= \lim_{z\to\infty}\frac{e^{\sqrt{p}z}\ e^{-\sqrt{p}}}{\lambda e^{\sqrt{p}z} + \sqrt{p}e^{\sqrt{p}z}}
\end{aligned}
$$

So we get :

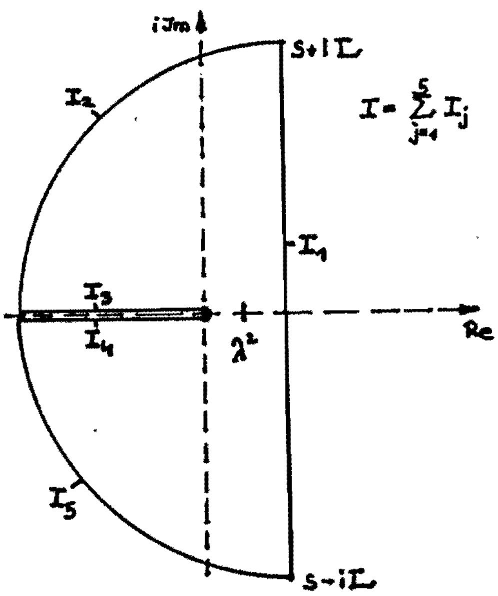
$$
\mathbf{\overline{c}(p,z=0)} = \frac{\mathbf{e^{-\sqrt{p}}}}{\lambda+\sqrt{\mathbf{p}}} \tag{4}
$$

The last step will be to find the inverse Laplace transform. At least to our knowledge, however, it is impossible to give it explicitly. Therefore we will revert to Mellin's integral formulation of the inverse Laplace transform.

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{e^{-\sqrt{p}}}{\sqrt{p}+\lambda}\}(x) &= \lim_{L\to\infty}\frac{1}{2\pi i}\int\limits_{s-iL}^{s+iL}\frac{e^{-\sqrt{p}}e^{px}}{\sqrt{p}+\lambda}dp \\ &= \lim_{L\to\infty}\frac{1}{2\pi i}\int\limits_{s-iL}^{s+iL}\frac{e^{-\sqrt{p}}e^{px}(\sqrt{p}-\lambda)}{p-\lambda^2}dp \\ &=: I_1\end{aligned}$$

for an arbitrary s $> \lambda^2$

Choosing the usual branch of $\sqrt{x}$ we find that the integrand is analytic in $\mathbb{C}\backslash(\mathbb{R}^- \cup \{\lambda^2\})$ and has a first order singularity in λ^2 .



We can calculate the contour integral shown above by applying the residual theorem:

- I
  For L $> s > \lambda^2$ we get
  $\mathrm{I} = res_{\lambda^2} = e^{-\sqrt{p}}e^{px}(\sqrt{p}-\lambda)\Big|_{p=\lambda^2} = 0 \qquad \forall\, \mathrm{L} > s > \lambda^2$

- $\underline{I_2 + I_5}$
  For $g : p \mapsto \frac{e^{-\sqrt{p}}}{\sqrt{p}+\lambda}$ holds

  1. The integral of the modulus of g is bounded over each semi-circle of radius L $> s > \lambda^2$:
     $C_{L,s} := \{p \in \mathbb{C} | p = s + Le^{i\phi}; \phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\}$
  2. $\sup_{p \in C_{L,s}} |g(p)| \to 0 \quad for \mathrm{L} \to +\infty$

  proof: w.l.o.g. $\phi \in \left[\frac{\pi}{2}, \pi\right]$

$$\begin{aligned}
\lim_{L\to\infty} \left| \frac{-e^{\sqrt{s+Le^{i\phi}}}}{\sqrt{s+Le^{i\phi}}+\lambda} \right| &= \lim_{L\to\infty} \left| \frac{-e^{\sqrt{Le^{i\phi}}}}{\sqrt{Le^{i\phi}}} \right| \\
&= \lim_{L\to\infty} \frac{\left|e^{-\sqrt{L}\cos\frac{\phi}{2}}\right| \left|e^{-i\sqrt{L}\sin\frac{\phi}{2}}\right|}{\sqrt{L}} \\
&\leq \lim_{L\to\infty} \sup_{\phi\in\left[\frac{\pi}{2},\pi\right]} \frac{e^{-\sqrt{L}\cos\frac{\phi}{2}}}{\sqrt{L}} \\
&\leq \lim_{L\to\infty} \frac{1}{\sqrt{L}} \\
&= 0
\end{aligned}$$

  by Jordan's theorem follows : $I_2 + I_5 \to 0$ for L $\to +\infty$

- $\underline{I_3 + I_4}$

$$I_3 + I_4 = \frac{1}{2\pi i} \int_{\substack{s-L \\ arg(p)=\pi}}^{0} g(p)e^{px}dp + \frac{1}{2\pi i} \int_{\substack{0 \\ arg(p)=-\pi}}^{s-L} g(p)e^{px}dp$$

  Putting $p = -u^2$ we find :

$$\mathrm{dp} \; = \; -2\mathrm{u} \; \mathrm{du} \quad \text{and} \quad \sqrt{\mathrm{p}} = \begin{cases} iu & for \;\; arg(p) = \pi \\ -iu & for \;\; arg(p) = -\pi \end{cases}$$

$\Rightarrow$

$$
\begin{aligned}
I_3 + I_4 &= \frac{1}{2\pi i} \int_0^{\sqrt{L-s}} \frac{e^{-iu}(\lambda - iu) - e^{iu}(\lambda + iu)}{u^2 + \lambda^2} e^{u^2 x} 2u \, du \\
&= -\frac{2}{\pi} \int_0^{\sqrt{L-s}} \frac{u \cos u + \lambda \sin u}{\lambda^2 + u^2} e^{-u^2 x} u \, du
\end{aligned}
$$

$$
\Rightarrow I_3 + I_4 \to -\frac{2}{\pi} \int_0^{\infty} \frac{u \cos u + \lambda \sin u}{\lambda^2 + u^2} e^{-u^2 x} u \, du \ for\ L \to +\infty
$$

This integral converges faster than the original integral I_1. Therefore it is much more suitible for numerical analysis.

- $\underline{I_1}$

  Since

$$
I_1 = I - I_2 - I_3 - I_4 - I_5
$$

we get the following concentration on the ground:

$$
\mathbf{c_\lambda(x,0) = I_1 = \frac{2}{\pi} \int_0^{\infty} \frac{u \cos u + \lambda \sin u}{\lambda^2 + u^2} e^{-u^2 x} u \, du} \tag{5}
$$

In the special case $\lambda = 0$ we get :

$$
c_0(x,0) = \frac{2}{\pi} \int_0^{\infty} \cos u e^{-u^2 x} du = re \left[\frac{1}{\pi} \int_{-\infty}^{\infty} e^{iu - u^2 x} du \right]
$$

$$
= re \left[\frac{1}{\pi} \int_{-\infty}^{\infty} e^{-\left(\sqrt{x}u - \frac{i}{2\sqrt{x}}\right)^2} e^{-\frac{1}{4x}} du \right] \overset{v = \sqrt{x}u - \frac{i}{2\sqrt{x}}}{=} \frac{1}{\sqrt{x}\pi} e^{-\frac{1}{4x}} \int_{-\infty}^{\infty} e^{-v^2} dv
$$

$$
= \frac{e^{-\frac{1}{4x}}}{\sqrt{\pi x}}
$$

This result coincides with that of chapter 3.

4.2 Numerical Analysis

First we want to calculate the concentration as a function of x for several absorption parameters λ, i.e. we have to find a good aproximation for :

$$c_\lambda(x,0) = \frac{2}{\pi}\int_0^\infty \frac{u\cos u + \lambda \sin u}{\lambda^2+u^2} e^{-u^2 x} u \, du$$

Since we cannot integrate numerically over an infinite integral, we have to look for an upper bound $T < \infty$, such that

$$\int_T^\infty \cdots du \; < \; \varepsilon$$

where ε is a bound for the admissible error.
For all u ≥ 0 we get :

$$\left| \frac{u\cos u + \lambda \sin u}{\lambda^2+u^2} u \right| \; \leq \; 2$$

$$\Rightarrow \forall T \geq 0 \; : \; \left| \frac{2}{\pi}\int_T^\infty \frac{u\cos u + \lambda \sin u}{\lambda^2+u^2} e^{-u^2 x} u \, du \right| \leq \frac{4}{\pi}\int_T^\infty e^{-u^2 x} du$$

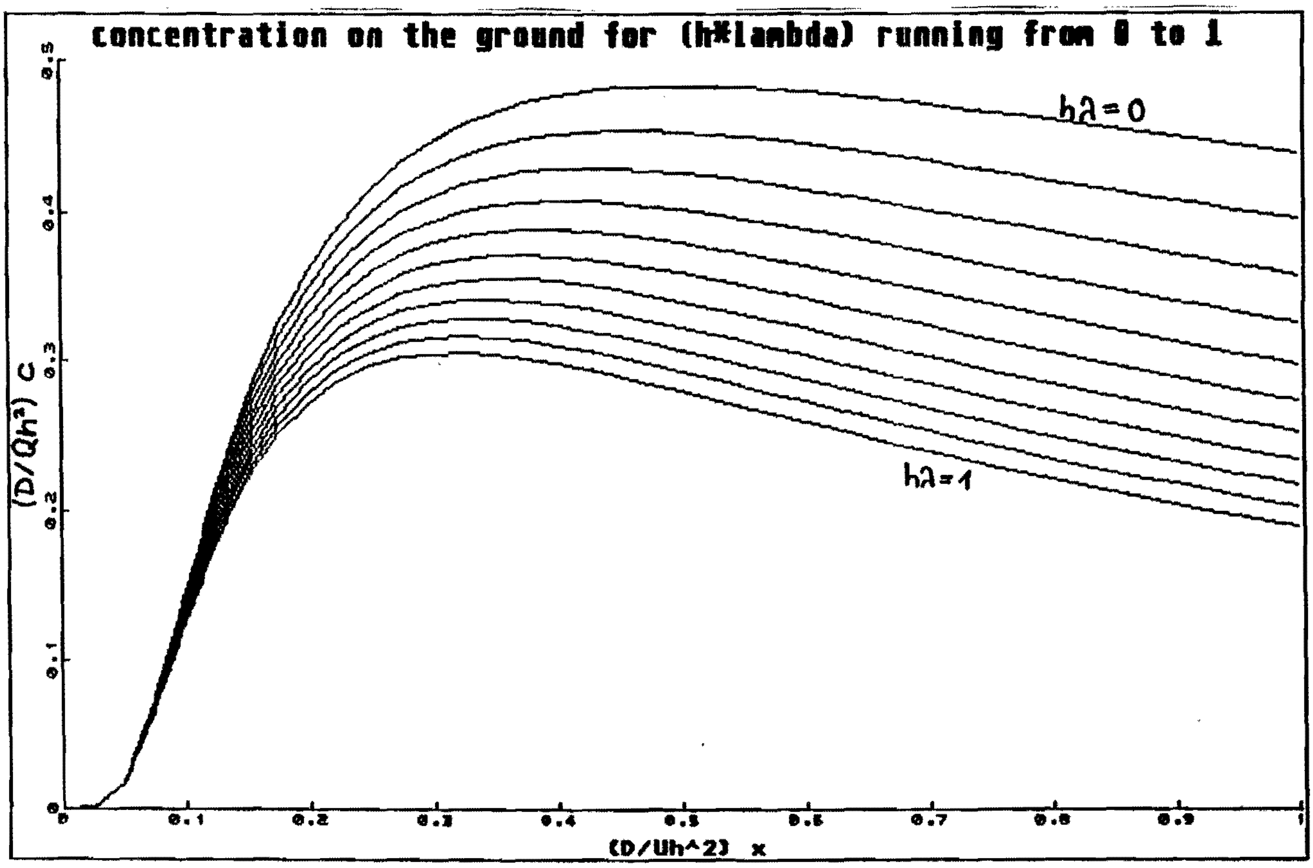
$$\overset{v=\sqrt{2x}u}{=} \frac{4}{\sqrt{\pi x}} \int_{\sqrt{2x}T}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv = \frac{4}{\sqrt{\pi x}}(1-\phi(\sqrt{2x}T))$$

Thus the absolute error is sufficiently small, if

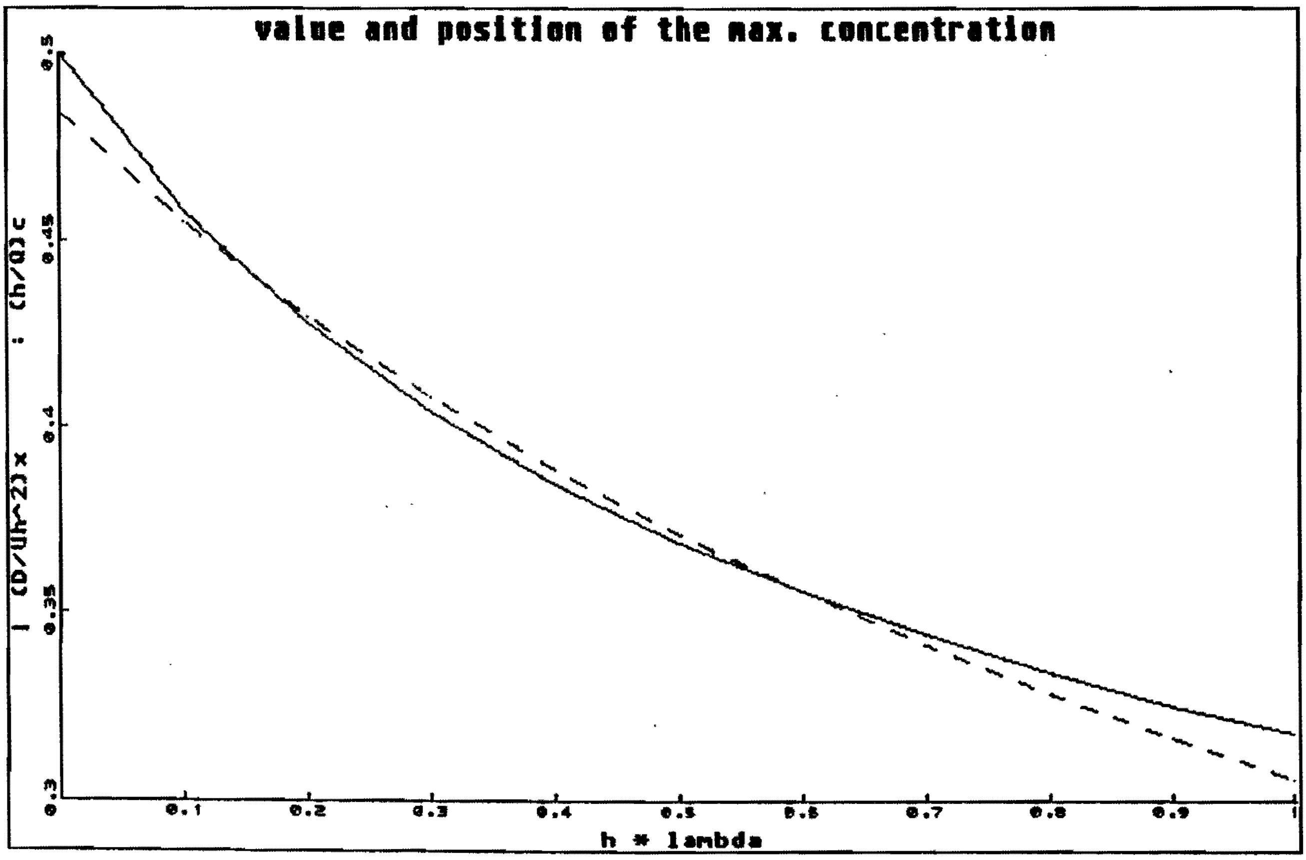
$$\frac{4}{\sqrt{\pi x}}(1-\phi(\sqrt{2x}T)) \leq \varepsilon \Longleftrightarrow T(x,\lambda) \geq \frac{\phi^{-1}\left(1-\frac{\sqrt{\pi x}\varepsilon}{4}\right)}{\sqrt{2x}}$$

remark :

- For numerical reasons, one should integrate from T back to 0 such that the small values are summed up first.
- For the integration we used Romberg's extrapolation method, because it enables us to control the relative approximation error.
- Note that the scaling in the next and all the following pictures refers to the original variables and not to the non-dimensional ones !!



Now we will illustrate how the value and the position of the maximal concentration depend on λ. Therefore we have to find the zero of the derivative of (5). The derivatives are calculated analytically and the roots by applying Newton's method.
The dotted line refers to the value of the maximal concentration and the full line to its position.



5 3-Dimensional No-Ground-Flux - Model

5.1 Model

Up to now our models have had one important disadvantage :
Diffusion has been restricted to the z-axis whereas in reality it is a 3-dimensional process. This disadvantage is coped by the approach due to Carslaw and Jaeger [2] .
The main idea is to model the emission of pollutant as the superposition of delta impulses which extend independently while the medium in which diffusion takes place is transported uniformly by the wind. This is valid because the corresponding differential equations are linear.
The results of Carslaw and Jaeger, however, are only applicable if we accept the no-ground-flux condition. As for the preceding models we give a short overview over the main features of this model:

Generalizations w.r.t. the model of chapter 3

- low winds are allowed (no restriction to $\frac{D}{hU}$)
- the problem is solved for the 3-dim. upper halfspace $H^+ = \{(x,y,z) \in \mathbb{R}^3 \big| z \geq 0)\}$

Once more we introduce non-dimensional variables:

$$x' = \frac{U}{D}x, \; y' = \frac{U}{D}y, \; z' = \frac{U}{D}z,$$
$$c' = \frac{U^2}{DQ}c, \; h' = \frac{U}{D}h$$

Skipping the primes our problem can be formulated as follows:

$$\begin{aligned} \frac{\partial}{\partial x}c(x,y,z) &= \Delta c(x,y,z) + \delta(x)\delta(y)\delta(z-h) \text{ on } \mathrm{H}^+ \qquad (6) \\ \lim_{x\to\pm\infty} c(x,y,z) &= 0 \\ \lim_{z\to\infty} c(x,y,z) &= 0 \\ \frac{\partial}{\partial z}c(x,y,z=0) &= 0 \end{aligned}$$

We will construct the solution in five steps:

1. We remember the solution of the time dependent heat equation on an unbounded domain for the initial distribution $c(\vec{x}, t = t_0) = \delta(\vec{x})$ at time t_0 .
2. The solution is extended to the case of a moving medium (wind) .
3. We integrate the solutions referring to different initial times t_0.
4. We consider the behaviour of the solution as time tends to infinity in order to get a steady state solution.
5. Another source is introduced at (x=0,y=0,z=-h). The resulting steady state solution of the free heat equation fulfils the desired boundary condition:
   $\frac{\partial}{\partial z} c(x, y, z = 0) = 0$

The free heat flow problem reads as follows :

$$\begin{aligned} \frac{\partial c}{\partial t} &= \Delta c \text{ on } \mathbb{R}^3 \\ c(x, y, z, t = t_0) &= \delta(x)\delta(y)\delta(z - h) \end{aligned}$$

and has the solution [3, page 329]

$$c(x, y, z, t) = \begin{cases} \frac{1}{8(\pi(t-t_0))^{\frac{3}{2}}} e^{-\frac{x^2+y^2+(z-h)^2}{4(t-t_0)}} & t > t_0 \\ 0 & t < t_0 \end{cases}$$

From this it is not hard to find that

$$\begin{aligned} \frac{\partial c}{\partial t} &= -\frac{\partial c}{\partial x} + \Delta c \\ c(x, y, z, t = t_0) &= \delta(x)\delta(y)\delta(z - h) \end{aligned}$$

is solved by

$$c(x,y,z,t) = \begin{cases} \frac{1}{8(\pi(t-t_0))^{\frac{3}{2}}} e^{-\frac{(x-(t-t_0))^2+y^2+(z-h)^2}{4(t-t_0)}} & t > t_0 \\ 0 & t < t_0 \end{cases}$$

This means physically that the effects of wind and diffusion are independent and thus can be superposed.

Summing up the contributions referring to different initial times t_0 we find:

$$\begin{aligned} c(x,y,z,t) &= \int_0^t \frac{e^{-\frac{(x-(t-\tau))^2+y^2+(z-h)^2}{4(t-\tau)}}}{8(\pi(t-\tau))^{\frac{3}{2}}} d\tau \\ &= \frac{1}{8\pi^{\frac{3}{2}}} \int_0^t \frac{e^{-\left(\frac{R^2}{4(t-\tau)}-\frac{x}{2}+\frac{t-\tau}{4}\right)}}{(t-\tau)^{\frac{3}{2}}} d\tau \end{aligned}$$

where $R^2 := x^2 + y^2 + (z-h)^2$

Setting $\xi := \frac{R}{\sqrt{2}\sqrt{t-\tau}}$ we get for $t \to \infty$:

$$\begin{aligned} \lim_{t\to\infty} c(R,t) &= \lim_{t\to\infty} \frac{1}{8\pi^{\frac{3}{2}}} e^{\frac{x}{2}} \frac{\sqrt{8}}{R} \int_{\frac{R}{\sqrt{2t}}}^{\infty} e^{-\frac{\xi^2-\left(\frac{R}{2\xi}\right)^2}{2}} d\xi \\ &= \frac{e^{\frac{x}{2}}}{\sqrt{2\pi}^3 R} \int_0^\infty e^{-\frac{\left(\xi-\frac{R}{2\xi}\right)^2}{2}} d\xi \end{aligned}$$

It holds :

$$\forall R \geq 0 \quad I(R) := \int_0^\infty e^{-\frac{\left(\xi-\frac{R}{2\xi}\right)^2}{2}} d\xi = \int_0^\infty e^{-\frac{\xi^2}{2}} d\xi = \sqrt{\frac{\pi}{2}}$$

This can be seen as follows :

1. The value of I at R = 0 can be found in the literature :
   $I(0) = \sqrt{\frac{\pi}{2}}$ [4, page 66 , Integral 3]

2. To show that I is independent of R we differentiate I w.r.t. R. It turns out that I fulfils the following differential equation :
   $\frac{dI}{dR} = 2(I(R) - I_0)$
   For the initial value $I(0) = I_0$ this equation has the unique solution I(R) = I(0).

Hence we get :

$$c(x, y, z) = \frac{1}{4\pi R} e^{-\frac{R-x}{2}}$$

This is the steady state solution for the free heat equation with a point source in (0,0,h). From symmetry arguments follows that adding a point source in (0,0,-h) results in a concentration which solves our original problem :

$$c(x, y, z) = \frac{1}{4\pi} \left(\frac{e^{-\frac{\sqrt{x^2+y^2+(z-h)^2}-x}{2}}}{\sqrt{x^2 + y^2 + (z-h)^2}} + \frac{e^{-\frac{\sqrt{x^2+y^2+(z+h)^2}-x}{2}}}{\sqrt{x^2 + y^2 + (z+h)^2}} \right) \tag{7}$$

To prove this formally put (7) into the differential equation (6) and check the corresponding boundary conditions.
On the ground (7) takes the special form:

$$\mathbf{c(x, y, 0)} = \frac{\mathbf{1}}{\mathbf{2\pi\sqrt{x^2 + y^2 + h^2}}} \mathbf{e}^{-\frac{\sqrt{\mathbf{x^2+y^2+h^2}}-\mathbf{x}}{\mathbf{2}}} \tag{8}$$

Now we will try and find the point where c takes its maximum value. By symmetry arguments follows that $y_{max} = 0$. Differentiating c w.r.t. x leads to the necessary condition:

$$x_{max}^3 - (1 + \frac{h^2}{4})x_{max}^2 + h^2 x_{max} - \frac{h^4}{4} = 0 \tag{9}$$

The largest root of this polynomial will give us x_{max}. In general the roots of a third order polynomial can be expressed explicitly by Cardano's formula.

From a numerical point of view, however, it is easier to calculate them by Newton's method.
For large values of h we find:

$$\lim_{h\to\infty} \frac{x_{max}(h)}{\frac{h^2}{4}} = 1 \qquad (10)$$

$$\lim_{h\to\infty} \frac{c_{max}(h)}{\frac{2}{\pi e h^2}} = 1 \qquad (11)$$

This means

$$x_{max} \approx \frac{h^2}{4}$$

$$c_{max} \approx \frac{2}{\pi e h^2}$$

(10) becomes obvious, if we define $\alpha_h := \frac{x_{max}}{h^2}$ and plug $\alpha_h h^2$ into equation (9). We get :

$$\forall h > 0 \quad \alpha_h(\alpha_h - \frac{1}{4}) + O(\frac{1}{h^2}) = 0$$

$$\Rightarrow \lim_{h\to\infty} \alpha_h^2(\alpha_h - \frac{1}{4}) = 0$$

$$\Rightarrow \alpha_\infty \in \left\{0, \frac{1}{4}\right\}$$

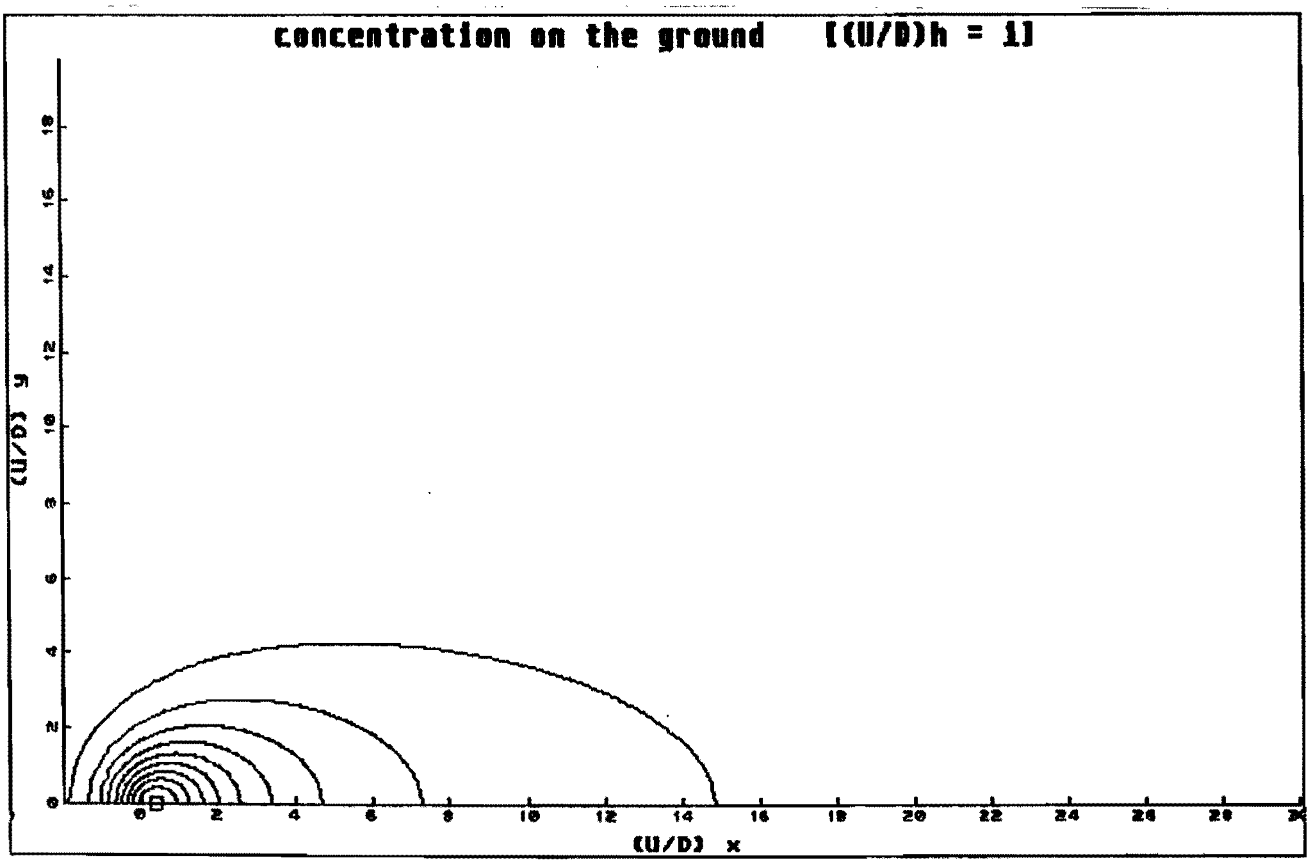
Since for all h > 0 there is an x_{max} (h) > 0 we finally get :

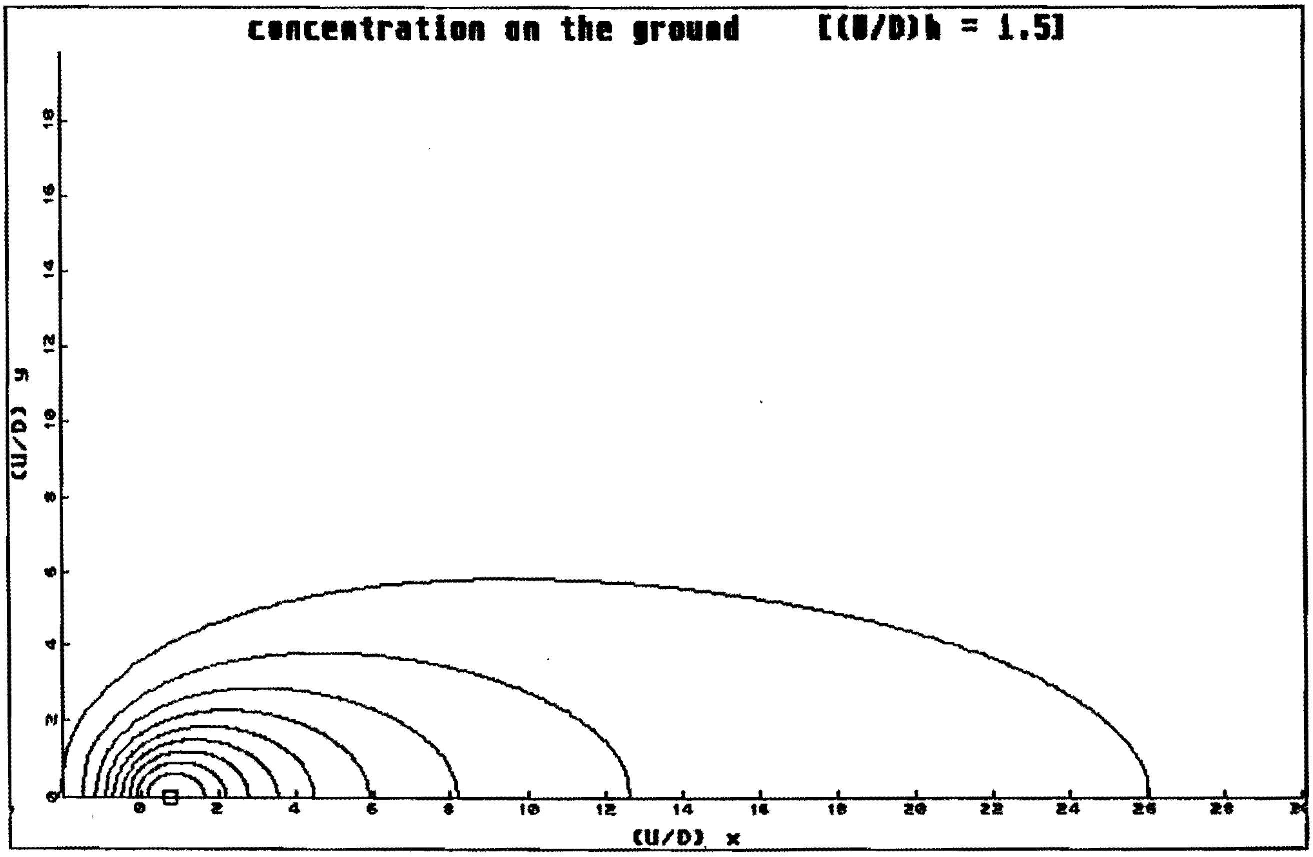
$$\alpha_\infty = \frac{1}{4}$$

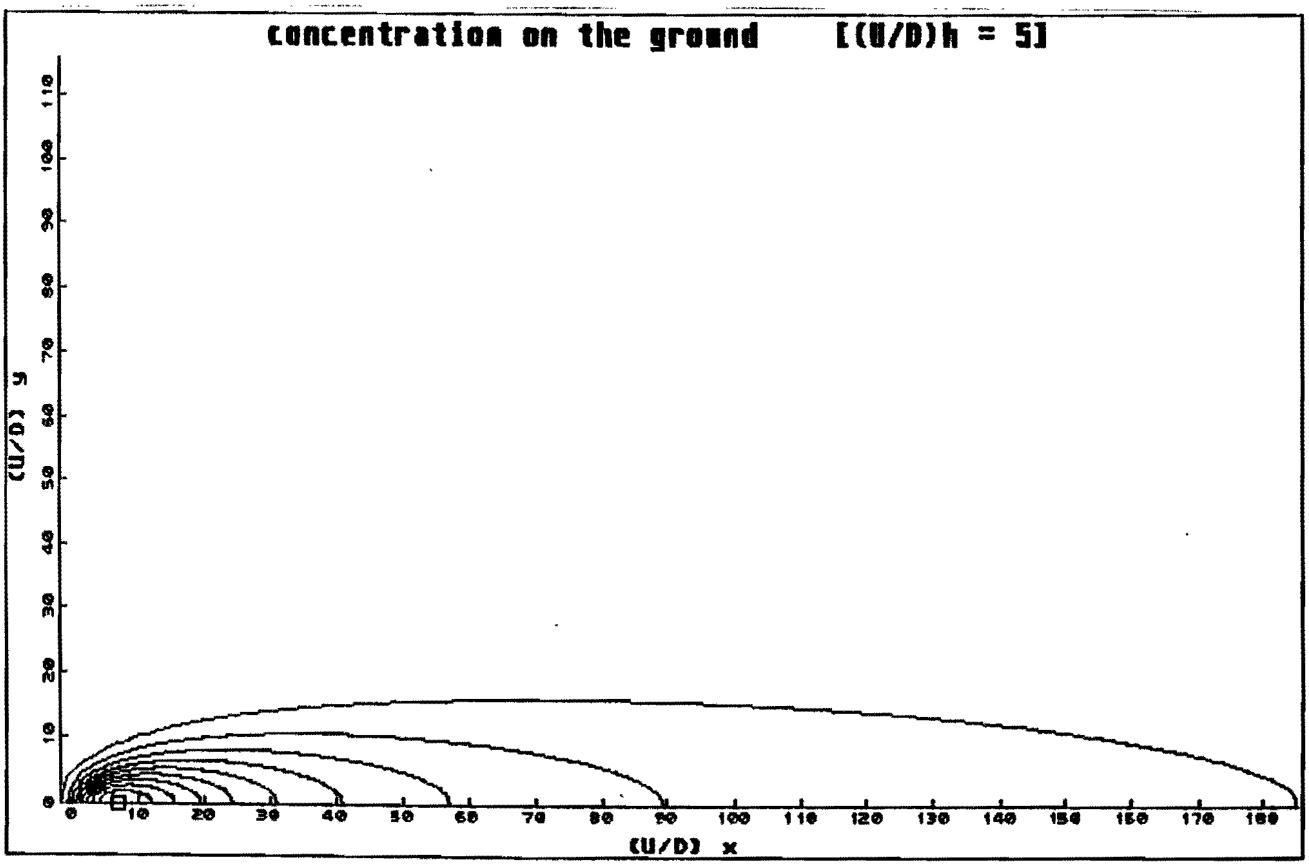
To prove (11) insert $(\frac{h^2}{4}, 0, 0)$ into (8) and let h tend to infinity.

5.2 Numerical Analysis

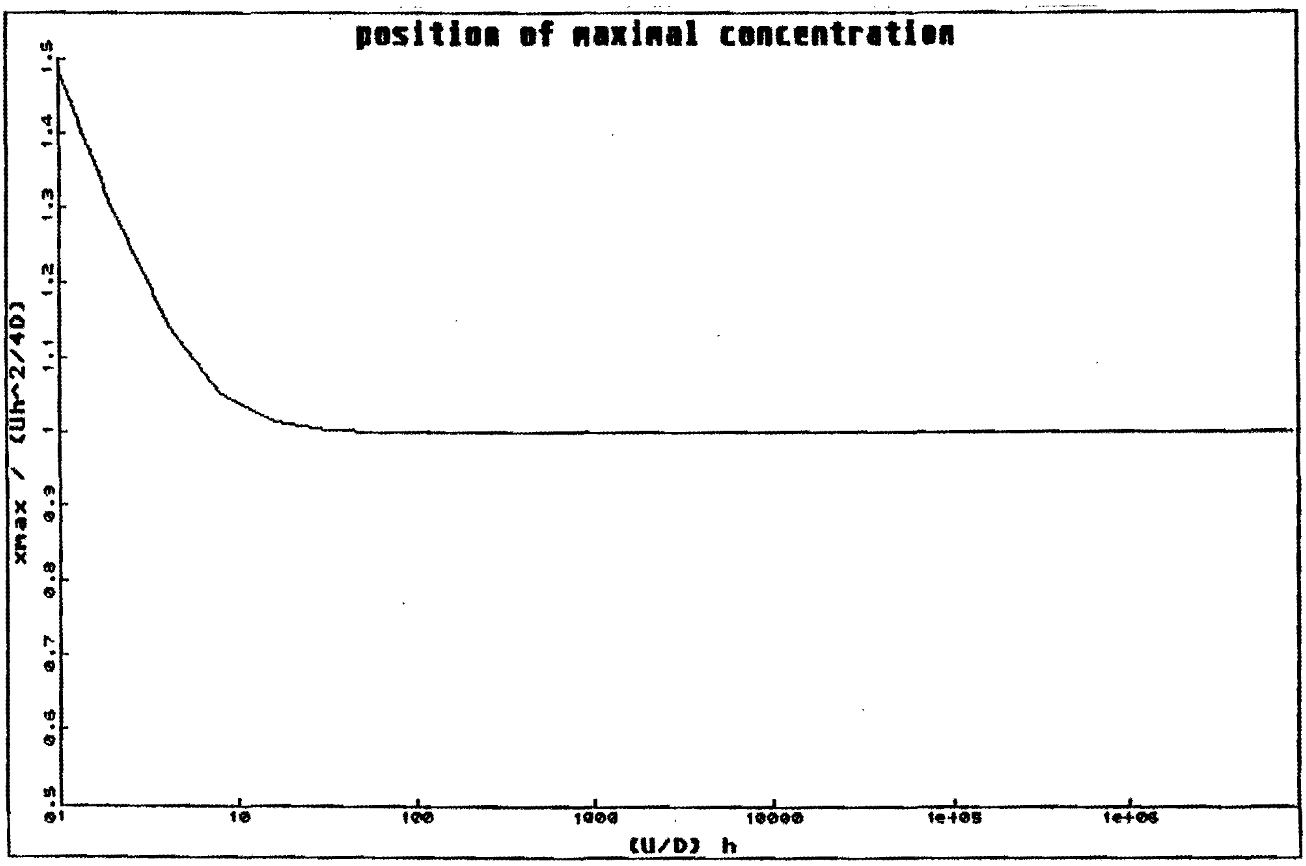
In the following figures you can see the contours of the concentration on the ground corresponding to different heights of the chimney. The little square marks the maximum concentration. The difference between two curves is one tenth of the maximum value. The contours can be computed by solving equation (8) for x using $R = \sqrt{x^2 + y^2 + h^2}$ as parameter.




concentration on the ground [(U/D)h = 1.5]
(U/D) y
(U/D) x


concentration on the ground [(U/D)h = 5]
(U/D) y
(U/D) x

The next figure shows the position of the maximal concentration x_{max} as a function of the chimney height. It illustrates that x_{max} converges asymptotically towards $\frac{Uh^2}{4D}$ for large values of h
(c.f. 10) .



6 3-Dimensional Ground-Flux - Model

6.1 Model 1

We want to extend the model of the last chapter to the case where there is flux into the ground (i.e. absorption on the ground). This means that $\lambda \neq 0$. Further we assume the same simplifications as before :

Simplifications

- restriction to the influence of diffusion and wind
- constant diffusion coefficient D
- horizontal winds of constant velocity U
- constant emission Q per unit time

Generalizations w.r.t. the model of chapter 3

- We allow absorption on the ground which is proportional to the concentration.
- Low winds are permitted (no restriction to $\frac{D}{hU}$).
- The problem is solved for the 3-dim. upper halfspace $H^+ = \{(x,y,z) \in \mathbb{R}^3 \big| z \geq 0)\}$.

Using the non-dimensional variables of chapter 4 and $\lambda' = \frac{D}{U}\lambda$ we get the following model:

$$c_x - c_{xx} - c_{yy} - c_{zz} - \delta(z-h)\delta(x)\delta(y) = 0$$

$$c_z = \lambda c$$

Applying the Fourier-transform twice we get : (where $\hat{c}$ is the Fourier-transform of c)

$$\hat{\hat{c}}_{zz} - \underbrace{(\xi^2 + \eta^2 - i\xi)}_{:=\omega}\hat{\hat{c}} = -\delta(z-h)$$

Using the Laplace-transform we get :

$$\bar{\hat{\hat{c}}} = \frac{(\lambda + \xi)\hat{\hat{c}}_{z=0} - e^{h\xi}}{\lambda^2 - \omega^2}$$

The inverse Laplace-transform can be found just as in chapter 3 :

$$\hat{\hat{c}} = \frac{e^{-\omega h}}{\lambda + \omega}$$

To get c as a function of (x,y,z) we have to apply the inverse Fourier transform twice :

$$\mathbf{c(x,y,0)} = \frac{\mathbf{1}}{\mathbf{2}\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathbf{e}^{-\mathbf{h}\sqrt{\xi^2+\eta^2-\mathbf{i}\xi}}}{\lambda + \sqrt{\xi^2 + \eta^2 - \mathbf{i}\xi}} \mathbf{e}^{-\mathbf{i}(\xi\mathbf{x}+\eta\mathbf{y})} \mathbf{d}\xi \mathbf{d}\eta \qquad (12)$$

6.2 Numerical Analysis

The integral (12) contains complex arguments which is not very convenient for numerical calculations. Therefore we will try and find a representation of (12) with purely real integrand:
Let

$$p(\xi,\eta) := \sqrt{\frac{\sqrt{(\xi^2+\eta^2)^2+\xi^2} + (\xi^2+\eta^2)}{2}} \qquad (13)$$

$$q(\xi,\eta) := \sqrt{\frac{\sqrt{(\xi^2+\eta^2)^2+\xi^2} - (\xi^2+\eta^2)}{2}} sign(-\xi) \qquad (14)$$

Then holds:

$$\sqrt{\xi^2+\eta^2-i\xi} = p + iq$$

$$c(x,y,0) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \underbrace{\left\{ \int_{-\infty}^{\infty} \frac{e^{-hp}}{(\lambda+p)^2+q^2} e^{-i(hq+\xi x)}(\lambda + p - iq) d\xi \right\}}_{=:I(\eta)} e^{-i\eta y} d\eta$$

Substituting ξ by $-\xi$ we find with
$q(\xi,\eta) = -q(-\xi,\eta)$ and $p(\xi,\eta) = p(-\xi,\eta)$:

$$I(\eta) = \overline{I(\eta)}, \quad \text{i.e. } I(\eta) \in \mathbb{R} \;\; \forall \eta \in \mathbb{R}$$

Since $I(\eta) = I(-\eta)$ we get in a similar way:

$$c(x,y,0) = \overline{c(x,y,0)}$$

Therefore it is enough to take the real parts of the integrands. By the symmetry of the inner integrand w.r.t. ξ and η follows:

$$c(x,y,0) = \frac{1}{\pi^2}\int_0^\infty\int_0^\infty \frac{e^{-hp}}{(\lambda+p)^2+q^2} \times \{(\lambda+p)cos(hq+\xi x) - qsin(hq+\xi x)\}\, cos(\eta y) d\xi d\eta \tag{15}$$

where p and q were defined in (13) and (14).

By rough estimations you find:

$$\begin{aligned} \frac{1}{\pi^2}\int_T^\infty\int_T^\infty |...|\, d\xi d\eta &\leq \frac{1}{\pi^2}\int_T^\infty\int_T^\infty \frac{e^{-h\sqrt{\xi^2+\eta^2}}}{\lambda} d\xi d\eta \\ \leq \frac{1}{\pi^2}\int_0^{2\pi}\int_T^\infty \frac{e^{-hr}}{\lambda} r dr d\phi &= \frac{2}{\pi\lambda h^2}(Th+1)e^{-hT} \\ &=: \Psi_{\lambda,h} \end{aligned}$$

So if you approximate c(x,y,0) by the finite integral over
$[0,T]\times$ [0,T] the absolute error will be less than $\Psi_{\lambda,h}$.
We have now reached the stage where we can commence numerical calculations.

6.3 Model 2

Since the results of model 1 are not easy to handle, we will now restrict ourselves to those cases in which diffusion in x-direction is dominated by wind effects. The other generalizations of model 1 are retained :

Generalizations w.r.t. the model of chapter 3

- We allow absorption on the ground which is proportional to the concentration
- The problem is solved for the 3-dimensional upper halfspace H^+

We use the same non-dimensional variables as in chapter 4 :

$$x' = \frac{D}{Uh^2}x, \; y' = \frac{y}{h}, \; z' = \frac{z}{h},$$

$$c' = \frac{D}{Qh^2}c, \; \lambda' = h\lambda$$

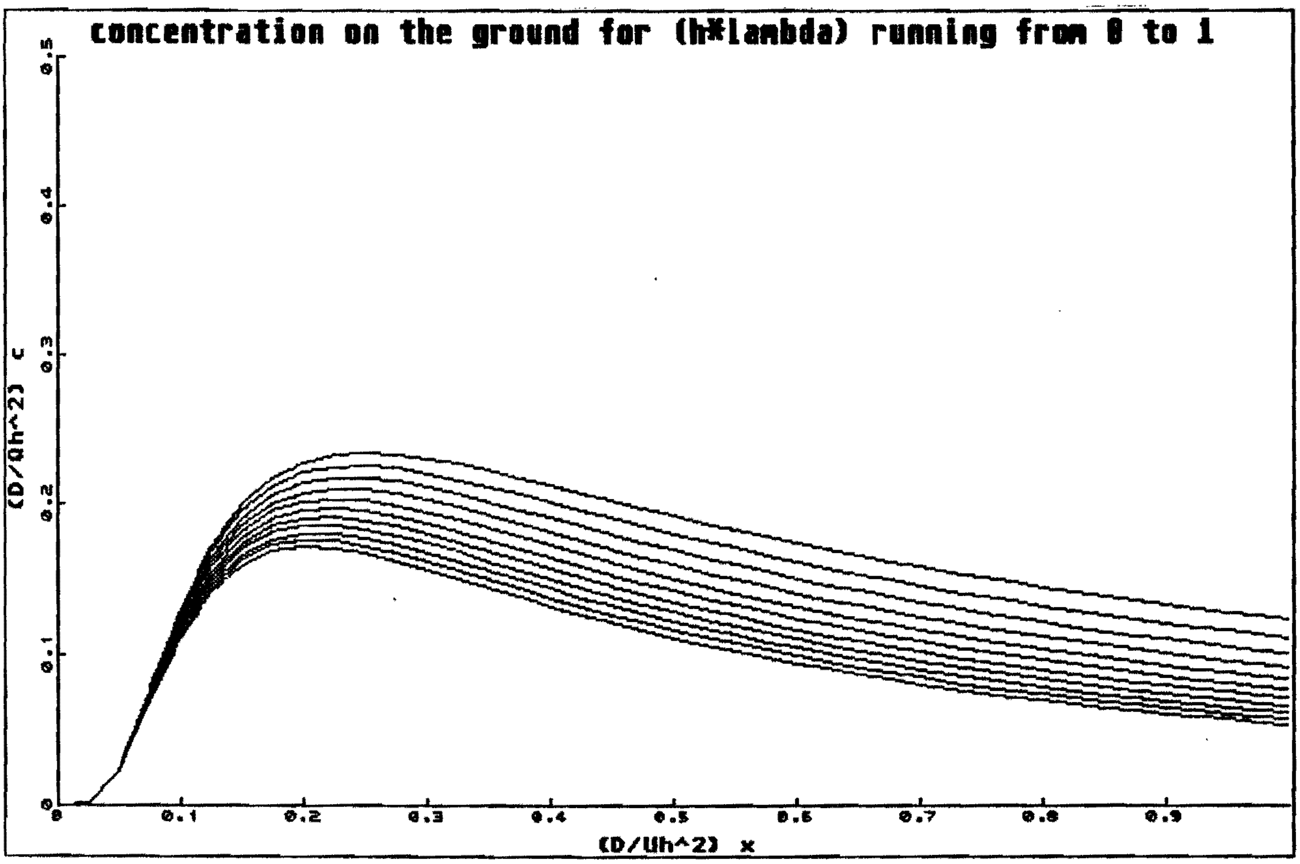
We neglect $\left(\frac{D}{Uh}\right)^2 \frac{\partial^2 c}{\partial x^2}$ in the corresponding differential equation. Just as in model 1 we apply forward and backward Fourier-transforms. If we denote the phase-space variables referring to x and y by ξ and η respectively, we find :

$$c(x,y,0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\left(\frac{1}{2\pi i} \int_{\eta^2 - i\infty}^{\eta^2 + i\infty} \frac{e^{-\sqrt{\tau}}}{\lambda + \sqrt{\tau}} e^{\tau x} d\tau \right)}_{I_1} e^{-(\eta^2 x + i\eta)} d\eta$$

where $\tau = \eta^2 - i\xi$

For s $= \eta^2 > \lambda^2$ I_1 has already been computed in chapter 4. It turned out to be independent of η^2. This, however, is true for all $\eta^2 > 0$, because the residual which corresponds to the singularity in λ^2 is zero.
Therefore we can place I_1 in front of the whole integral. The remaining integral can be solved analytically, such that we get :

$$c_\lambda(x,y,0) = \left(\frac{2}{\pi}\int_0^\infty \frac{u\cos u + \lambda \sin u}{\lambda^2+u^2} e^{-u^2 x} u du\right) \frac{e^{-\frac{y^2}{4x}}}{2\sqrt{\pi x}} \tag{16}$$



7 Conclusion

We have considered four different models which are intended to describe how the emission of a chimney is distributed by diffusion and wind. All these models have in common the following simplifying assumtions:

- restriction to the influence of diffusion and wind
- constant diffusion coefficient D
- horizontal winds of constant velocity U
- constant emission Q per unit time

The more advanced models refer to the 3-dimensional space instead of the plain of the prevailing wind. They take into account absorption on the ground and / or allow low winds which do not necessarily dominate diffusion effects.

For typical values of the parameters U (wind velocity) , D (diffusion coefficient) and h (chimney height) the assumption $\frac{D}{Uh} \ll 1$ turns out to be valid :

diffusion coefficient of air at T= 0°C, P = $1atm$	: $1.33 \times 10^{-5} \frac{m^2}{s}$
usual wind velocity	: 1 - $10 \frac{m}{s}$
usual chimney height	: $100m$
usual value of $\frac{D}{Uh}$	: $10^{-8} - 10^{-7}$

Switching from two to three dimensions has the following effect:
Under the condition $\frac{D}{Uh} \ll 1$ the formulas for the concentration at y=0 differ only by a factor proportional to $\frac{1}{\sqrt{x}}$. The two dimensional model gives the correct position of the maximum except for a factor of 2.

Since we have not found any data about the absorption coefficient, we cannot estimate the influence of the coefficient λ.

chap.	dim	λ	$\frac{D}{Uh}$	concentration on ground	position of max.	value of max.
3	2	0	$\ll 1$	$\frac{Qh^2}{D}\sqrt{\frac{Uh^2}{\pi Dx}}e^{-\frac{Uh^2}{4Dx}}$	$\frac{Uh^2}{2D}$	$\frac{Qh^2}{D}\sqrt{\frac{2}{\pi e}}$
4	2	>0	$\ll 1$	$\frac{Qh^2}{D}\frac{2}{\pi}\int_0^\infty \frac{\xi cos\xi+(h\lambda)sin\xi}{(h\lambda)^2+\xi^2}e^{-\frac{Dx}{Uh^2}\xi^2}\xi d\xi$	/	/
5	3	0	/	$\frac{D^2}{U^3}\frac{Q}{2\pi\sqrt{x^2+y^2+h^2}}e^{-\frac{U}{2D}\left(\sqrt{x^2+y^2+h^2}-x\right)}$	/	/
6.3.	3	0	$\ll 1$	$\frac{Qh^2}{D}\frac{Uh^2}{2\pi Dx}e^{-\frac{U(h^2+y^2)}{4Dx}}$	$\frac{Uh^2}{4D}$	$\frac{Qh^2}{D}\frac{2}{\pi e}$
6.3.	3	>0	$\ll 1$	$\frac{Qh^2}{D}\frac{2}{\pi}\int_0^\infty \frac{\xi cos\xi+(\lambda h)sin\xi}{(\lambda h)^2+\xi^2}e^{-\frac{Dx}{Uh^2}\xi^2}\xi d\xi \times \frac{1}{2}\sqrt{\frac{Uh^2}{\pi Dx}}e^{-\frac{Uy^2}{4Dx}}$	/	/

However, it is not at all clear that wind and diffusion are really the dominating effects on particle motion. To make the waste gas leave the chimney, it has to be given a certain velocity and tempertature [5, pages 322-375] . Each chimney exerts a thermal force. The type of chimney will determine how significant these effects are. In reality you can observe that the whole of the waste gas is rising to heights h_1 which may reach several times the chimney height h_0 . Let us assume that at this height the vertical motion of the gas has stopped by friction and cooling, and that from there on our wind-diffusion models are valid. But then the position of the maximal concentration x_{max} will depend on h_1 instead of h_0 . Keep in mind that x_{max} is proportional to h^2 !

Other effects that should be considered, in a more precise model, are :

- inhomogenious gases (different influence of gravitation)
- variable wind velocity, diffusion coefficient, emission and absorption rate
- topology of the environment
- etc...

The importance of these phenomena, however, will depend on the special type of chimney and cannot be estimated in general.

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