The cell mapping method

The Cell Mapping Method: a Tool for Investigating Nonlinear Dynamic Systems

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Abstract

In this report, an introduction is given on the method of cell mapping, a tool for investigating nonlinear dynamic systems. The method is based on a discretisation of the system's state space in so-called cells. The cell mapping method is capable of determining the attractors and basins of attraction of a nonlinear dynamic system.

In Chapter 1, the importance of the cell mapping method is emphasized in the context of the necessary steps in the investigation of nonlinear dynamic systems.

In Chapters 2, 3, and 4, three different types of cell mapping are treated: simple cell mapping, generalized cell mapping, and interpolated cell mapping.

In Chapter 5, a recapitulation of the three types of cell mapping is given.

Contents

Notation

Symbols

a	forcing amplitude
C	cell mapping
d	system damping
h_i	cell size in i-th direction
M	total number of regular cells
M_i	number of cells in i-th direction
N	state space dimension
$\mathbf{p}$	cell probability vector
$\mathbf{P}$	transition probability matrix
p_i	i-th component of $\mathbf{p}$
p_{ij}	ij-th component of $\mathbf{P}$
t	time
T	forcing period
x	displacement (position)
$\dot{x}$	velocity
$\ddot{x}$	acceleration
$\mathbf{x}$	state vector
x_i	i-th component of $\mathbf{x}$
$x_i^{(l)}$	lower boundary of x_i
$x_i^{(u)}$	upper boundary of x_i
ξ	cell index
ω	forcing angular velocity
Ω	region of interest

Abbreviations

GCM	generalized cell mapping
ICM	interpolated cell mapping
MD	multiple domicile
MV	mean value
PG	persistent group
SCM	simple cell mapping
SD	single domicile

Chapter 1

Introduction

When studying the global behaviour of a nonlinear dynamic system, one usually focusses on the following steps:

- Formulation of the equations of motion
- Localization of stationary, periodic, quasiperiodic and chaotic solutions of the system
- Determination of the stability properties of the solutions
- Study of changes in the solutions (in form, magnitude, number, stability and type) when varying system parameters (bifurcation research).

The determination of the *basins of attraction* of all stable solutions should be the next step in the investigation of nonlinear dynamic systems. In nonlinear dissipative systems one often deals with coexistence of various stable solutions, also called *attractors*. Which attractor the system will lead to depends on the initial conditions of the system. So, for each attractor a subset of the state space exists, containing all initial values leading to that attractor, the so-called basin of attraction. For dissipative systems, these basins of attraction fill up the entire state space. One can imagine that in special dynamic problems it is important to know the position of the basins of attraction of all possible stable solutions. Determination of these basins should therefore be added to the abovementioned steps to complete the investigation of the global behaviour of a nonlinear dynamic system.

However, determination of the basins of attraction is a topic which is hardly encountered in literature on nonlinear dynamics. This is probably due to the lack of analytical techniques, when other than weakly nonlinear systems are concerned. On the other hand, numerical methods to solve this problem are very time-consuming.

Further, no standard methods are available when trying to locate chaotic solutions. Of course, the calculation of Liapunov exponents is a tool for proving the existence of chaos, but this requires some knowledge of the basins of attraction. When having found a chaotic solution, an additional task can be found in determining characteristic properties of the solution, such as its dimension.

Using the *cell mapping* method however, one can find all attractors—stationary, periodic and chaotic—as well as their basins of attraction. In some cases, also unstable and saddle solutions can be found. The cell mapping method was introduced by Hsu [1] and is based on a discretization of the system's state space.

In this report, three different variants of the cell mapping method are treated. In Chapter 2, an outline is given of the *simple cell mapping* method (SCM method). In Chapter 3, the *generalized cell mapping* method (GCM method), which deals with probabilities, is discussed. Using this method, statistic and probabilistic properties of chaotic solutions can be determined. Recently, an improvement on these methods was given by Tongue [4] in the form of the *interpolated cell mapping* method (ICM method), which is dealt with in Chapter 4. We finish this report with a recapitulation in Chapter 5.

Chapter 2

Simple Cell Mapping

2.1 Introduction

When using numerical techniques to solve (dynamic) problems, roundoff errors are introduced due to the computer's limited precision. Moreover, in experimental methods a limit of measurement accuracy exists. This means that in both numerical and experimental methods desired quantities cannot be obtained exactly. According to Hsu [1], this implies that a state variable, describing part of the state of a dynamic system, cannot be regarded as a continuum, which can take every possible value $x \in \mathbb{R}$. Instead, it is better to consider a state variable as a discrete magnitude, and its range as a collection of onedimensional intervals, determined by the computer's (or measurement's) precision. The complete state space $\mathbb{R}^N$ (where N is the system dimension) should therefore be considered as a discrete collection of N-dimensional cells, not as an N-dimensional continuum. Such a discretized space is called a *cell state space.* In the next section, a state space discretization will be executed and explained in detail.

2.2 State Space Discretization

We consider a dynamic system with Euclidian state space $\mathbb{R}^N$, $(N \in \mathbb{N},\ N \geq 2)$. Generally, the state of a dynamic system will be restricted to a bounded subset of the state space. For convenience, we take this subset, denoted by Ω, to be rectangular. Let $\mathbf{x} = (x_1, .., x_N)$ be the state vector, then for each state variable x_i a lower and upper boundary $x_i^{(l)}$ and $x_i^{(u)}$ exists. Hence,

$$x_i^{(l)} \leq x_i \leq x_i^{(u)}, \quad i = 1, .., N.$$

Having defined Ω, we divide it into cells. In principle, the cells can be of arbitrary form, as long as they fill up Ω. Practically, the choice of rectangular cells is obvious. The division of Ω in rectangular cells can be realised by dividing each interval $[x_i^{(l)}, x_i^{(u)}]$ into M_i intervals of equal length h_i, where

$$h_i = \frac{x_i^{(u)} - x_i^{(l)}}{M_i}, \quad i = 1, .., N.$$

In this way, Ω has been divided into M rectangular cells, with

$$M = \prod_{i=1}^{N} M_i.$$

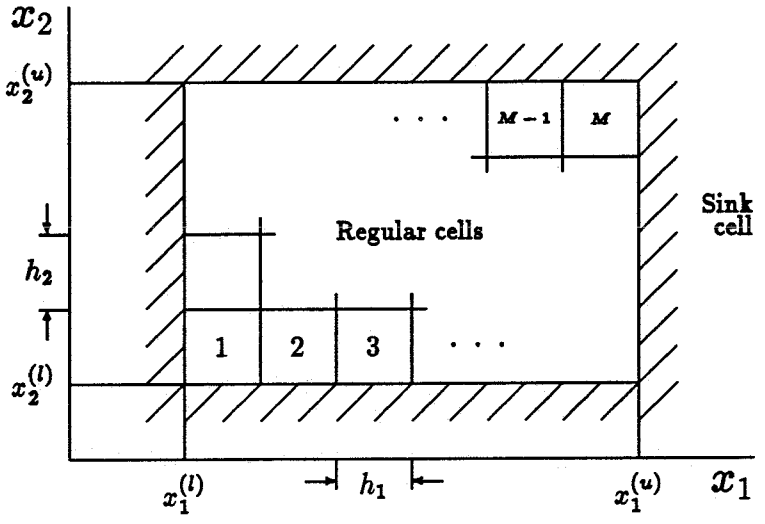

Figure 2.1: Discretization of a 2-dimensional state space.

We denote each cell by an index $j \epsilon \mathbb{N}$, going from 1 tot M. The region $\mathbb{R}^N \backslash \Omega$ is called the *sink cell*, denoted by index 0. The remaining cells are called *regular* cells. When the state of the system reaches the sink cell at a certain moment, the evolution of the system will no longer be followed. By definition, the system will stay in the sink cell forever. In Fig. 2.1, a cell discretization has been schematically drawn for $N = 2$.

The fundamental step in the cell mapping theory is the following: The state of the system at time t is no longer described by the state vector $\mathbf{x}(t)$, but by the index $\xi(t)$ of the cell containing the state vector. Hence,

$$\xi(t) = j \iff \mathbf{x}(t) \epsilon \text{ cell } j.$$

All possible states within one cell will be denoted by the same index and are therefore treated as one and the same state. Hence, a cell can be regarded as an indivisible entity of the state of the system.

2.3 System Evolution

Having described the state of the system by means of a cell, we now focus on how to describe the system evolution in time. If we inspect the state space at discrete equidistant times, the evolution of the system can be given by a sequence of positive integers $\xi(0), \xi(1), \xi(2), ...$ Here $\xi(n)$ corresponds to the cell containing the state of the system at $t = nT$ $(n = 0, 1, 2, ..)$ and T is the time between two state space inspections. This evolution process can be put in the form of a *simple cell mapping*

$$\xi(n+1) = C(\xi(n)), \tag{2.1}$$

with $C : \mathbb{N} \rightarrow \mathbb{N}$. According to equation (2.1), each cell has a single image cell in the evolution sequence[1]. In this report, we will only consider cell mappings which are independent of

[1] In the next chapter we treat the generalized cell mapping method, in which a cell can have several image cells.

the mapping step n. In the next section, it is shown how to create a simple cell mapping for a system governed by a set of ordinary differential equations.

To classify the system evolution, we define the following kinds of cells:
Equilibrium cell. A cell ξ satisfying

$$\xi = C(\xi)$$

is called an *equilibrium cell* of the system. By definition, the sink cell is an equilibrium cell.
Periodic cell. A cell ξ satisfying

$$\xi = C^m(\xi), \quad m \in \mathbb{N},$$

is called a *periodic cell* with *period* m, or simply, a $P-m$ cell. Here, C^m denotes the cell mapping C applied m times. If a cell ξ is a $P-m$ cell, then also the cells $C(\xi), .., C^{m-1}(\xi)$ are $P-m$ cells. Such a group of periodic cells is called a *periodic group* of m cells. Hence, an equilibrium cell is a $P-1$ cel.
Transient cell. A cell ξ which is not periodic is called a *transient cell.*

Starting from a regular cell ξ, we distinguish three possible system evolutions.

- Cell ξ is a periodic cell; the evolution of the system is a periodic motion.
- Cell ξ is mapped into the sink cell in a finite number of steps; by definition, the system will stay in this cell forever.
- Cell ξ is mapped into a periodic group in a finite number of steps; hereafter, the system will undergo a periodic motion.

Cells belonging to the latter type form the basins of attraction of the periodic groups.

We see from the above possibilities that in the context of simple cell mapping no aperiodic motions occur. Yet, the cell mapping approach is applicable to chaotic systems when taking the following assumptions for granted (see Kreuzer [3]):

- Chaotic motions are represented by periodic groups of relatively long period.
- A chaotic attractor is represented by a set of cells covering part of the attractor in state space.

Concerning the *stability* of a periodic group, we remark the following: when a periodic group possesses no basin of attraction (no attracting cells) then this group is called unstable. When a periodic group is surrounded by transient cells leading to this group, then we deal with a stable group. In all other cases (both attracting and non-attracting cells surround the periodic cells) we deal with a saddle-solution.

When the image cell of all cells in the region of interest has been found, one can determine to which periodic group each cell belongs, as a periodic or as a transient cell. Now, in the context of simple cell mapping, the dynamic behaviour in the region of interest has been fully determined.

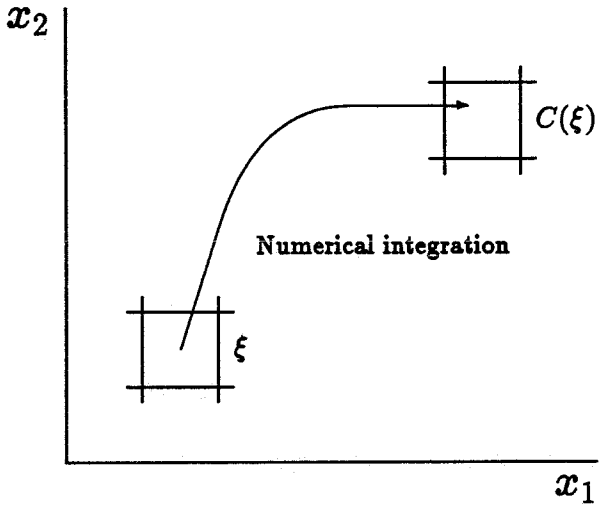

Figure 2.2: Center point method for a 2-dimensional system.

2.4 Center Point Method

For a system governed by a set of ordinary differential equations (ODE's), a simple cell mapping can be created by the center point method. According to this method, the image cell ξ^* of a regular cell ξ is determined as follows:

- Calculate the center point of cell ξ.
- Integrate the set of ODE's over a period T, using the center point as initial condition. Here, T is the time between two state space inspections.
- Determine cell ξ^* which contains the end point of the calculated trajectory.

In Fig. 2.2, this method is illustrated for a 2-dimensional system.

When the system under investigation is explicitly dependent on time in a periodic way, the interval T between the state space inspections should be chosen equal to the system's period. Otherwise, the cell mapping would be dependent on the mapping step n. For autonomous systems, T can be chosen arbitrarily, provided that it is not too small.

2.5 Example

As an example, we consider a modified Duffing equation:

$$\ddot{x} + d\dot{x} - x + x^3 = a\cos(\omega t). \tag{2.2}$$

This equation describes a harmonically forced beam with negative linear and positive cubic stiffness. For $d = 0.15$, $a = 0.3$, $\omega = 1.0$, a periodic and a chaotic attractor coexist. In Appendix A, state space projections and Poincaré sections of these attractors are shown, together with a picture of the corresponding basins of attraction. These results were obtained by means of numerical integration over a long integration time and for a large number of

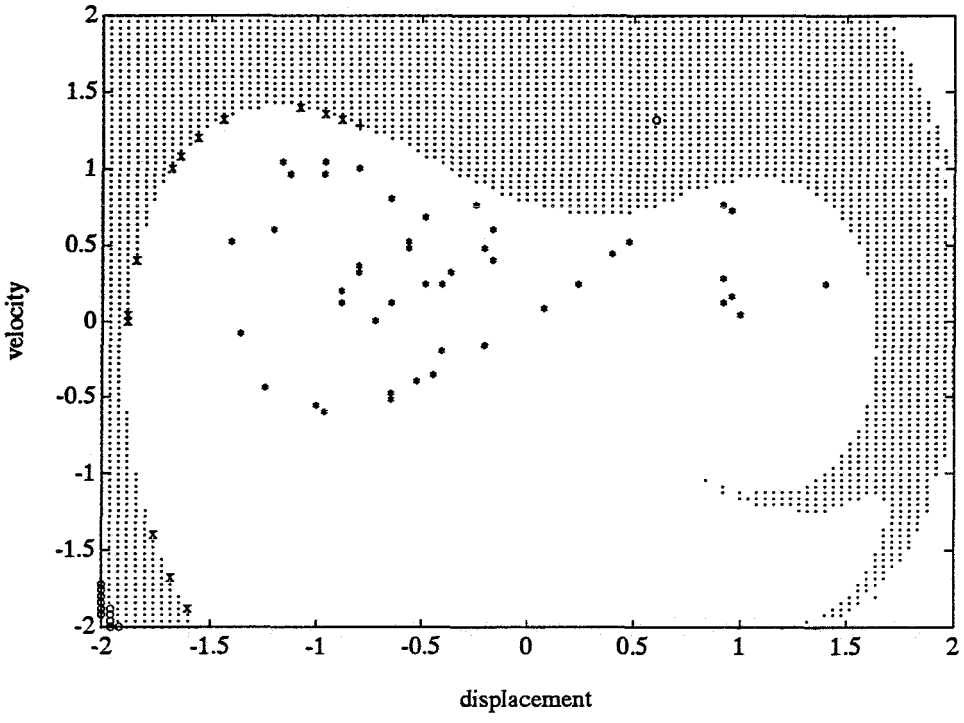

Figure 2.3: Attractors and basins of attraction of a modified Duffing equation with $d = 0.15$, $a = 0.3$, $\omega = 1$, found with the SCM-method.

initial states. The gain in computer-time when applying the SCM method, where for each cell an integration is required over only one period, is obvious.

To apply the SCM method to the modified Duffing equation, we first have to write (2.2) as a set of first-order equations:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -dx_2 + x_1 - x_1^3 + a\cos(\omega t). \end{aligned} \tag{2.3}$$

For our region of interest Ω, we take $-2.02 \leq x_1, x_2 \leq 2.02$. Further, we take $h_1 = h_2 = 0.04$, which implies $M_1 = M_2 = 101$. The total number of regular cells is then given by $M = 10201$. With the aid of the center point method, all image cells were determined. The integration time was chosen equal to the system's period $T = 2\pi/\omega$. A fourth order Runge-Kutta scheme with adjustable time step was used for the numerical integration. In Fig. 2.3, the results are shown for $d = 0.15$, $a = 0.3$, $\omega = 1.0$. We have:

- A $P-1$ cell (o) at $(0.6, 1.4)$ with a large basin of attraction ($\cdot$). The cell represents a stable periodic solution with period $T = 2\pi/\omega$.
- A $P-5$ and two $P-15$ groups ($*$), forming together a chaotic solution. The large basin of attraction (the white area) indicates that we are dealing with an attractor.

- A $P-1$ cell (+) at $(-0.8, 1.4)$, with a basin of attraction consisting of 13 cells (×), positioned at the boundary of the basins of attraction belonging to the two stable solutions. Hence, this cell represents a saddle solution.
- 14 Cells (o) which are mapped into to the sink cell.

When we compare the results with the exact solutions shown in Appendix A, we see that the periodic attractor as well as its basin of attraction have been found quite accurately. The chaotic attractor has not really been found, but is represented by three periodic groups with long period. The basins of attraction of these three groups approximate the real chaotic attractor's basin of attraction quite well. Additionally, we find a saddle solution, whose stable manifolds separate the two basins of attraction. The position of this saddle solution with respect to the attractors plays an important role in bifurcation research.

2.6 Conclusions and Remarks

From the above example we can conclude that the SCM method is a robust tool for obtaining a global picture of the behaviour of a nonlinear dynamic system.

Attractors and basins of attraction can be determined quite accurately. Existence of chaotic behaviour can be assumed when dealing with periodic groups of relatively long period.

For more detailed information about a particular solution, one can perform a regular numerical integration, starting in the corresponding basin of attraction. In this way, one can see whether one is dealing with a periodic or a chaotic attractor.

Whether unstable and saddle solutions will be found depends on the cell size and the actual position of the solution inside its cell.

In systems with low damping, a periodic solution of period m will in general not be represented by a $P-m$ group, but by m clusters of cells. Due to the low damping, not only the cells containing the actual periodic solution will be found as being periodic cells, but also a group of cells surrounding them.

When for a particular system the basins of attraction have fractal boundaries, the SCM method will not describe them very precisely, specially not when the fractal boundary layers are small with respect to the applied cell size. For those cases a *generalized* cell mapping method will be necessary to obtain accurate results. This method will be treated in the next chapter.

Chapter 3

Generalized Cell Mapping

3.1 Introduction

In the SCM theory, each regular cell is mapped into a *single* image cell. In reality, the image of a cell will be given by some bounded region, covering more than one cell, as shown in Fig. 3.1. This means that for each regular cell we should take into account more than one image cell.

In the theory of *generalized* cell mapping (GCM), a regular cell can have several image cells, each with a fraction of the total probability. The sum of the probabilities of all possible image cells of a cell equals one. The state of the system at $t = n$ is now no longer denoted by the cell index $\xi(n)$ but by the cell probabilility vector $\mathbf{p}(n) = [p_1(n), .., p_M(n)]^T$. Here M is the total number of regular cells and

$$p_i(n) = \text{Prob}\,[\,\xi(n) = i\,], \quad i = 1, .., M. \tag{3.1}$$

In words, $p_i(n)$ denotes the probability of the state of the system being in cell i at $t = n$.

Further, we introduce the *transition probability matrix*, or simply *transition matrix* $\mathbf{P} = (p_{ij})$ with

$$p_{ij} = \text{Prob}\,[\,\xi(n+1) = i \,|\, \xi(n) = j\,], \quad i, j = 1, .., M. \tag{3.2}$$

Here, p_{ij} is the *transition probability* from cell j at $t = n$ to cell i at $t = n + 1$. For a periodic system the time step between two state space inspections can be chosen such that p_{ij}, and hence P, is independent of n. The evolution of the system is given by

$$\mathbf{p}(n+1) = \mathbf{P}\mathbf{p}(n). \tag{3.3}$$

When we want to investigate the evolution of the system starting from a particular initial condition, we first have to determine the initial cell probability vector $\mathbf{p}(0)$. This vector is simply given by

$$p_i(0) = \begin{cases} 1 & \text{if cell } i \text{ contains the system's initial state,} \\ 0 & \text{otherwise.} \end{cases}$$

According to (3.3), the evolution of the system can be written as

$$\mathbf{p}(n) = \mathbf{P}^n\mathbf{p}(0). \tag{3.4}$$

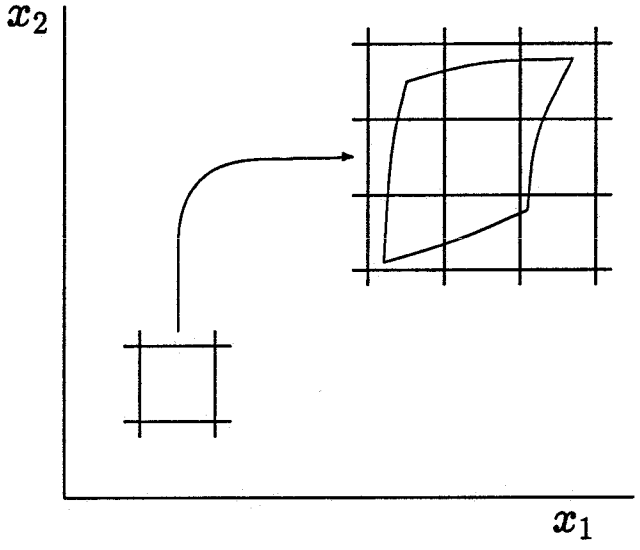

Figure 3.1: Real image of a cell, covering several cells.

The transition matrix **P** fully determines the dynamic behaviour of the system. Equation (3.3) defines a finite, discrete, stationary Markov chain (see e.g. [2]). The theory of Markov chains can be used to examine the properties of **P** and hence, to find periodic and chaotic solutions of the system. First, we will introduce some definitions of the Markov chain theory which later will be used to describe the system's behaviour.

3.2 Definitions

In the SCM theory treated in Chapter 2, we dealt with two kinds of regular cells, namely periodic and transient ones. In the context of GCM, we deal with *persistent* and transient cells, also called *recurrent* and *nonrecurrent* cells respectively. A regular cell i is called persistent when the probability of ever returning to i (when starting in it) equals one. A cell is called transient when it is not persistent. To give a mathematical description of these classifications we present the following definitions:

Definitions of $f_{ij}^{(n)}$ and f_{ij}^{*}. Starting from cell j, the probability of the system ever being in cell i is denoted by f_{ij}^{*}. The probability of being in cell i for the first time at the nth step is denoted by $f_{ij}^{(n)}$. We have,

$$f_{ij}^{*} = \sum_{n=1}^{\infty} f_{ij}^{(n)}. \tag{3.5}$$

According to these definitions, f_{ii}^{*} denotes for a cell i the probability of ever returning to i. Hence, by definition a cell i is persistent if $f_{ii}^{*} = 1$ and transient if $f_{ii}^{*} < 1$. The following relations hold between the f_{ij} and the transition probabilities p_{ij}:

$$p_{ij}^{(n)} = \sum_{k=1}^{n} p_{ii}^{(n-k)} f_{ij}^{(k)}, \tag{3.6}$$

$$f_{ij}^{*} = \lim_{N\to\infty} \{ \sum_{n=1}^{N} p_{ij}^{(n)} / \sum_{n=0}^{N} p_{ii}^{(n)} \}. \tag{3.7}$$

Here $p_{ij}^{(n)}$ is the probability of being in cell i after n steps, starting from cell j. It can be shown that $p_{ij}^{(n)}$ is the (i,j)th element of $\mathbf{P}^n$.

A cell j is said to *lead* to cell i, symbolically $j \rightarrow i$, if and only if there exists a positive integer m such that $p_{ij}^{(m)} > 0$. We say the cells i and j *communicate* if and only if $i \rightarrow j$ and $j \rightarrow i$, denoted by $i \leftarrow\rightarrow j$.

Essential and Inessential Cells. A cell that communicates with every cell it leads to is called *essential*, otherwise *inessential*. The property of being essential or inessential is a class property.

Period. If a cell i is essential, we can define the period of this cell as the greatest common divisor of the set of positive m such that $p_{ii}^{(m)} > 0$. The period of an essential cell i is denoted by d_i.

In the next section, we will use the above definitions to find the periodic and chaotic solutions of a dynamic system.

3.3 Persistent and Transient Cells

In the previous section, we remarked that the property of being essential is a class property. Since every persistent cell is essential, we can divide all persistent cells in disjoint groups, according to this class property. This means that a persistent cell communicates with all the cells in the same group, and does not communicate with any other cell. These groups are called *persistent groups* (PG's). When the system is in a PG, it will stay there forever. The PG's of a Markov chain, deduced from a dynamic system, correspond to the attractors of that system. We distinguish the following kind of PG's:

- Persistent groups having only one cell: A cell i for which $p_{ii} = 1$ forms a persistent group by itself. Such a cell is called *absorbing.*
- Persistent groups having more than one cell: When the period d of the group, as defined in the previous chapter, equals one, the group is called an *acyclic group* and the cells of this group *acyclic cells.* When $d \geq 2$ for a persistent group, it is called a *periodic group* and its cells *periodic cells.*

When a cell is not persistent, it is transient by definition. For finite Markov chains the system will leave the transient cells with probability one and will settle on one of the PG's. When a transient cell i leads to a particular PG, then that PG is called a *domicile* of

i. Transient cells can be divided in *single-domicile* (SD) and *multiple-domicile* (MD) cells, according to the number of domiciles (one or more respectively).

Starting in a SD cell, the system will lead to a particular PG with probability one. For each PG, one can form the set of SD cells leading to that PG. This set then corresponds to the basin of attraction of the attractor, which is represented by the PG.

Starting in a MD cell, the system can lead to several PG's with corresponding probabilities, the sum of which of course equals one. The MD cells form the separatrices, the boundaries between the basins of attraction.

3.4 Limiting Probability

For each cell i of a PG, one can determine the *limiting probability* p_i^*, which is the probability of the system being in i at $t = \infty$, under condition that it ever entered the PG to which i belongs. For an acyclic PG we have

$$p_i^* = \lim_{n\to\infty} p_{ij}^n.$$

This limit is independent of j, provided that j belongs to the same PG as i, as a persistent cell or as a single-domicile cell. If i belongs to a periodic PG with period d, its limiting probability is given by

$$p_i^* = \lim_{n\to\infty} p_{ij}^{nd},$$

where the same condition holds for j.

Especially for PG's consisting of a huge number of cells, the limiting probability can give some usefull information. Such PG's usually represent chaotic attractors; the limiting probability then gives an idea of which parts of the attractor are "visited" very frequently by the system and which parts very rarely. Thus, a better picture is obtained of the chaotic attractor as a whole.

3.5 Sampling Method

When applying the GCM method to systems of nonlinear ordinary differential equations, the question arises how to determine the transition matrix, that is, the cell transition probabilities. Normally, these cannot be determined analytically, but several ways exist to approximate them. We will discuss the simplest one here, called the *sampling method.*

Choose for a regular cell j L points, uniformly distributed over j, as initial conditions for numerical integration. Integrate the system over a time interval τ, where τ is arbitrary for an autonomous system and equal to the excitation period otherwise. In doing so, L trajectories are constructed, starting from the L points in cell j. The cells which contain the end points of these trajectories are taken to be the image cells of cell j. Now, let there be l image cells $i_1, .., i_l$, containing $L_{i_1}, .., L_{i_l}$ end points, respectively. The cell transition probability p_{ij} can now be defined as follows:

$$p_{ij} = \begin{cases} L_i/L & i \in \{i_1, .., i_l\}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

If we do this for each regular cell j, the transition probability matrix P has been determined.

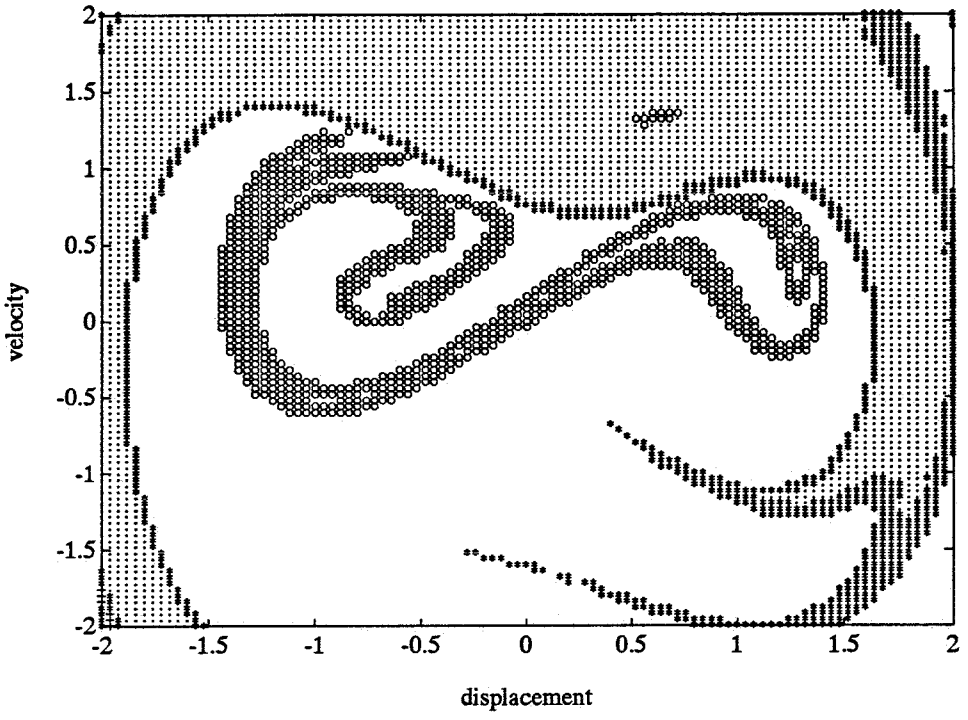

Figure 3.2: Attractors and basins of attraction of a modified Duffing equation with $d = 0.15$, $a = 0.3$, $\omega = 1.0$, found with the GCM method.

3.6 Example

We now apply the GCM method to the modified Duffing equation (2.3) with $d = 0.15$, $a = 0.3$, $\omega = 1.0$. For the region of interest Ω we again take $-2.02 \leq x_1, x_2 \leq 2.02$ and divide it into 101 x 101 cells. The transition probabilities are calculated with the sampling method, using 9 integration points for each cell. Algorithms given by Hsu [1] are used to determine the PG's, the SD and MD cells, and the limiting probability distribution. In Fig. 3.2 the results are shown. We find:

- A PG consisting of 10 cells (o), corresponding to the periodic solution. The basin of attraction of this PG is given by its SD cells ($\cdot$).
- A PG consisting of 966 cells, corresponding to the chaotic solution. The basin of attraction (the SD cells) is given by the white area in Fig. 3.2.
- 6 Cells (+) leading to the sink cell.

The multiple domicile cells are tagged by a ($*$) in Fig. 3.2.

Further, the limiting probability distribution has been calculated for both PG's. We expressed this probability in the mean value (MV) which equals 1/10 for the PG representing the periodic attractor and 1/966 for the PG representing the chaotic attractor (Fig. 3.3).

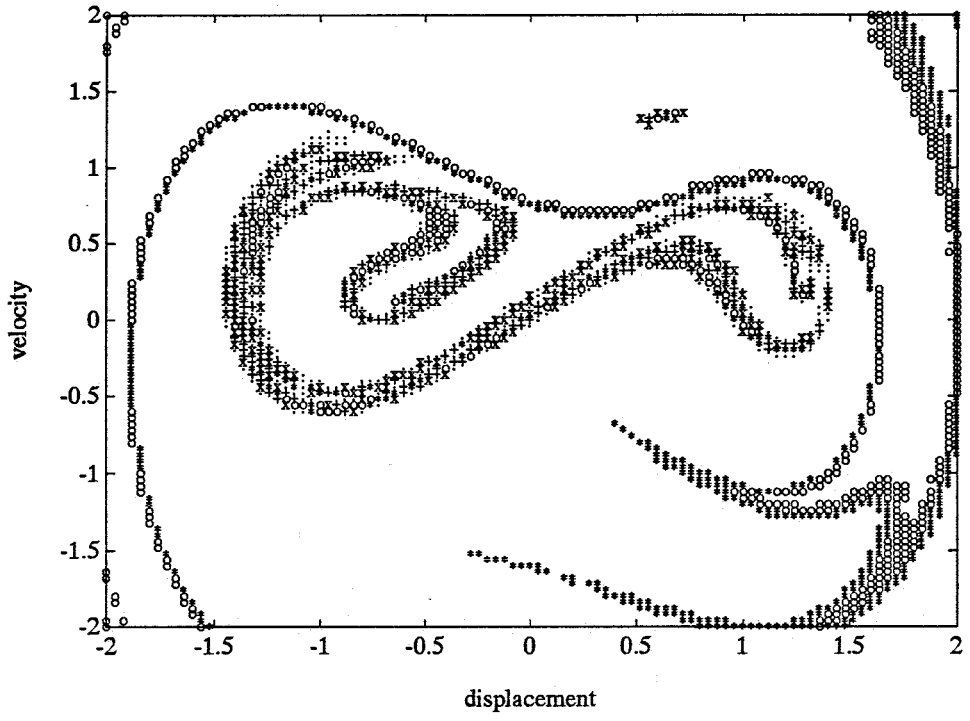

Figure 3.3: Probabilistic properties of the persistent groups and the multiple-domicile cells.

Cells denoted by $\circ$, have a limiting probability of $f*$MV, with $f > 2$. For cells denoted by $*, +, \times, \cdot$, we have $f\epsilon[1.5, 2.0]$, $f\epsilon[1.0, 1.5]$, $f\epsilon[0.5, 1.0]$, $f\epsilon[0.0, 0.5]$, respectively.

Finally, to obtain a better picture of the basins of attraction, the multiple domicile cells have been divided in two groups: cells leading to the periodic solution with probability $p > 0.5$ ($\circ$), and cells leading to the chaotic attractor with probability $p > 0.5$ ($*$) (Fig. 3.3).

When we compare these results with the SCM results, we see that the chaotic attractor is now represented by one large PG (in stead of several periodic groups, found with SCM) which gives a much better picture of the attractor. The periodic attractor is represented by a small PG, which is not as accurate as the $P - 1$ cell found with SCM. The domains of attraction are determined just as accurate as with SCM. The saddle solution has not been found with GCM.

3.7 Conclusions and Remarks

From the example treated in the previous section (and other examples not treated here) we can conclude the following:

The GCM method is very suited to find the chaotic attractors of a nonlinear dynamic system. The PG that represents a chaotic attractor will give a good picture of this attractor in general.

With GCM, the basins of attraction of all stable solutions can be determined more accu-

rately than with SCM. When two basins of attraction are separated by a thick layer of MD cells, additional research has to be done to obtain an accurate separatrix.

Further, additional properties of chaotic attractors can be determined, such as the limiting probability distribution.

However, periodic solutions will not be found as accurate as with SCM. Unstable and saddle-solutions will in general not be found at all.

Therefore, we can say that the GCM method is a very usefull addition to SCM, especially if one is interested in chaotic attractors and their properties.

Chapter 4

Interpolated Cell Mapping

4.1 Introduction

When applying cell mapping methods, one has to be aware of some drawbacks. First, because of the finite number of cells, only a finite number of solutions are possible. In the SCM method, only periodic solutions can be determined. Second, due to the discretisation of the state space errors have to be taken into account. In fact, for one excitation period a certain trajectory is calculated exactly, that is, as exact as a numerical integration routine can be. After that, the end point of the trajectory is moved to the midpoint of the corresponding cell. In this way an error is introduced, which magnitude is bounded by the distance between a cell corner and midpoint.

In spite of these drawbacks, satisfying results have been made with the cell mapping approach as discussed so far. In most cases the global picture of a system's dynamical behaviour is quite well found. However, in some situations where the basins of attraction have fractal boundaries, spurious results are found due to the discretisation error. To eliminate this error, Tongue[4] introduced the *interpolated* cell mapping method (ICM method).

4.2 ICM method

The first step in interpolated cell mapping is the same as in simple cell mapping: integration of each cell center point over one excitation period. Now, in stead of determining the cell which contains the end point of the calculated trajectory and adding this cell as image cell to the original one (as we did in SCM), we store the coordinates of the end point. Doing this for each cell, we obtain a $M \times N$ matrix containing the coordinates of the image points of all cell center points. Here M and N are the number of regular cells and the dimension of the state space, respectively. For convenience, we take $N = 2$ in this section.

For a certain cell center point p the trajectory is now calculated as follows: The image point p^* is determined by numerical integration as described above. We now determine the four cell center points p_i, $i = 1,..,4$, that surround p^*. To obtain the second image point p^{**}, that is, the trajectory point after two excitation periods, we use the—already known—image points p_i^* of p_i, $i = 1,..,4$. The point p^{**} is now determined by bilinear interpolating between the points $p_1^*,..,p_4^*$. This procedure is schematically illustrated in Fig. 4.1. Repeated application of this technique gives us the total trajectory starting from point p.

Of course, the question arises how many interpolation steps to make. How many steps

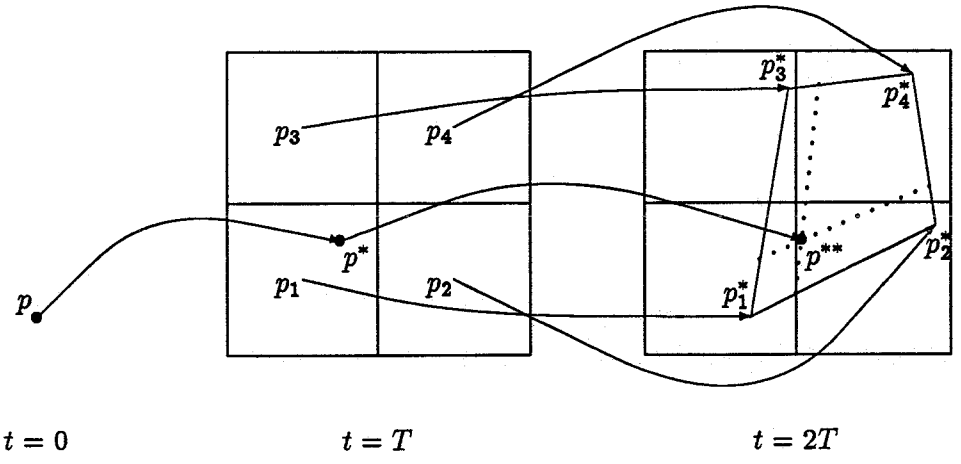

Figure 4.1: Procedure of Interpolated Cell Mapping. Integration step, followed by one or more interpolation steps.

are necessary to classify the evolution started from a certain initial value? This is quite arbitrary. Following Tongue, a trajectory is considered to be periodic if it passes within 10^{-3} of a previously mapped position. If no periodicity has been shown within 20 iterates, then the motion is considered to be chaotic. Of course other criteria can be applied.

This brings us to an important difference between ICM and SCM. Under SCM, a periodic motion is represented by a repeated pattern of cells. The highest period that can be sustained is equal to the total number of regular cells. A chaotic motion can therefore only be represented by a periodic motion with relatively long period. Under ICM however, no explicit restriction is placed on the length of the period. A chaotic motion is represented by a trajectory that is really never repeating. In practice, there is of course a limit in the form of the maximum number of interpolation steps.

Another advantage of ICM is the possibility of chosing the interpolation grid to be finer than the integration grid. For instance, when tackling a problem with 101 x 101 integration points, one can afterwards use a 1001 x 1001 interpolation resolution to obtain the basins of atraction. This is an interesting aspect of ICM, keeping in mind that the interpolation part requires much less CPU-time than the integration part.

4.3 Example

As an example, we again consider the modified Duffing equation (2.3), with $d = 0.15$, $a = 0.3$, $\omega = 1.0$. For the region of interest Ω, we take: $-2 \leq x_1, x_2 \leq 2$. Gridpoints are given by (ih_1, jh_2), with $i, j = -50, .., 50$, $h_1 = h_2 = 0.04$ (notice that these points correspond to the cell center points of the examples in Chapters 2 and 3). Treating this equation as described in the previous section (using the same criteria), the following results were found (see Fig. 4.2):

- A P-1 point (o) at $(0.638, 1.341)$ with a large basin of attraction ($\cdot$).

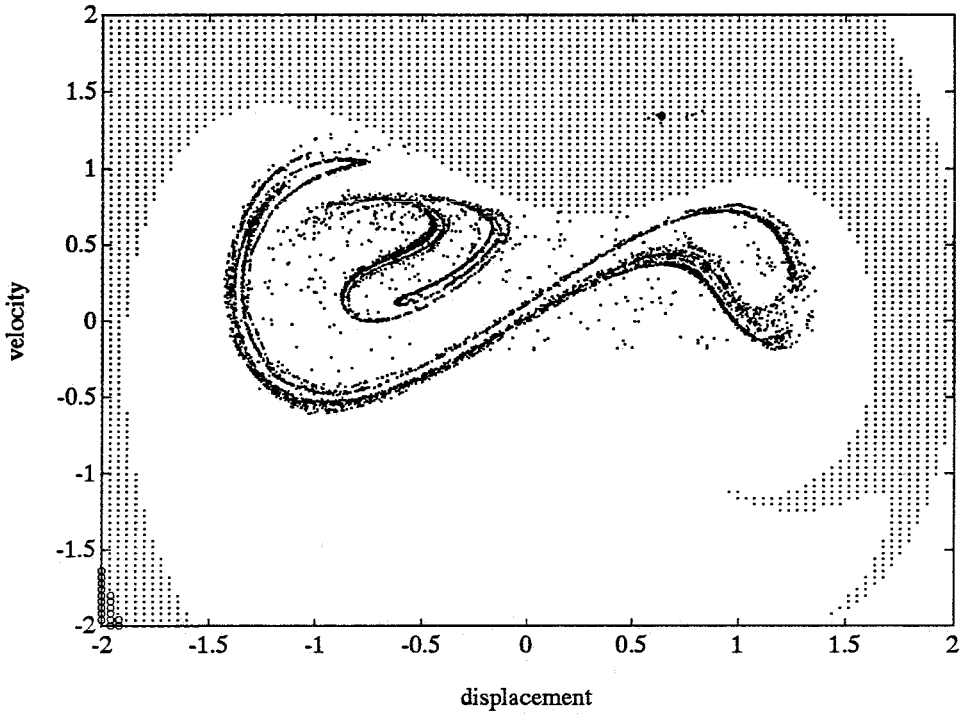

Figure 4.2: Attractors and basins of attraction for a Duffing system with $d = 0.15$, $a = 0.3$, $\omega = 1$, found with the ICM method.

- A chaotic attractor ($\cdot$), formed by the end points of trajectories that did not show any periodicity within 20 interpolation steps. The white area contains all initial conditions leading to this attractor.
- Some periodic groups (not tagged in Fig. 4.2), lying on the chaotic attractor with only a few transient points. These groups are artifacts of the ICM method, caused by the periodicity criterium. Since a chaotic attractor is "dense," some chaotic trajectories can be regarded as being periodic, due to this criterium.
- 17 Cells (o) leading to the sink-cell.

Comparing these results with those obtained by SCM and GCM, we can conclude the following: The periodic attractor has been localized more precisely now. The chaotic attractor looks more like the real attractor, shown in Appendix A, than the one produced by GCM. However, part of the dots forming the chaotic attractor are not lying on the real attractor (see Appendix A); in fact, some of them are lying close to the $P-1$ solution. These *transient dots* represent end points of trajectories that have not settled on one of the attractors yet. A larger maximum number of interpolation steps will reduce the number of transient dots. Further, just like in GCM, the saddle-solution has not been found now.

4.4 Conclusions and Remarks

The ICM method has been proved to be an attractive tool for investigating dynamic behaviour. We give a recapitulation of the main advantages with respect to SCM and GCM:

- No discretization error is being dealt with.
- No restriction is placed on a periodic motion's period length.
- Chaotic motion is represented by non-periodic motion (as it should be).
- The interpolation grid can be chosen much finer than the integration grid.

However, the following critical remarks have to be made as well.

- Unstable and saddle solutions will not be found in general.
- When a trajectory is found not to repeat itself within the maximum number of interpolation steps, it is regarded as being chaotic. The initial values of all chaotic trajectories are now assumed to form the basin of attraction of a chaotic attractor. This however, may not always be true, for instance when more than one chaotic attractor exist. Storing the end points of all chaotic trajectories and plotting them afterwards gives a good idea of the form of the chaotic attractor, and also (the only) information with respect to its uniqueness.
- When a trajectory is found to repeat itself, its interpolation points are stored as the Poincaré points of a periodic motion. But how do we know that this periodic solution has not already been found? Starting from a previous set of initial values, a periodic trajectory may have been found which is quite "close" to the one just found. Do we then consider these two trajectories to be different periodic motions or not? Obviously, we have to make certain rules in advance to handle this kind of problems.
- When the maximum number of interpolation steps is chosen too small, a spurious chaotic attractor will be found.

Concluding, we can say that the ICM method is an efficient addition to SCM, taking only a small amount of additional CPU-time. The method's drawbacks require some improvisation and ad-hoc thinking of the user.

Chapter 5

Recapitulation

We finish this report with a short recapitulation of the three types of cell mapping methods.

The simple cell mapping method gives a global overview of a dynamic system under investigation. Periodic attractors will be found, chaotic attractors will be represented by periodic groups of relatively long period. Basins of attraction can be determined quite accurately, provided that they do not have fractal boundaries. The capability of finding unstable and saddle solutions depends on the actual position of these solutions with respect to the cell boundaries.

The generalized cell mapping method is particularly suited for the determination of chaotic attractors and their properties. Further, when dealing with fractal boundaries, the basins of attraction will be determined more accurately than with SCM.

The interpolated cell mapping method gives a precise location of periodic solutions. Chaotic attractors can be determined very accurately, provided that the maximum number of interpolation steps is large enough.

These three cell mapping methods should not be seen as concurrents but should be combined into one method, thus taking advantage of each method's specific strong points. This should not be to hard: starting with SCM we already have the basis (the integration part) for ICM and have already done part of the work for GCM. With the SCM results in hand, one should decide wether to do an additional ICM or GCM or both.

Bibliography

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Appendix A

The Modified Duffing Equation

In this appendix we will present the exact solutions of the modified Duffing equation:

$$\ddot{x} + d\dot{x} - x + x^3 = a\cos(\omega t), \tag{A.1}$$

for $d = 0.15$, $a = 0.3$, $\omega = 1.0$. For these parameter values a periodic and a chaotic attractor coexist. In Fig. A.1, a time-history graph and a state space projection are shown for both the periodic and the chaotic attractor.

To obtain the exact basins of attraction of both attractors in the area $-2 \leq x, \dot{x} \leq 2$, a complete numerical integration was executed for 101×101 initial conditions. Initial conditions leading to the periodic attractor are shown in Fig. A.2 ($\cdot$). The Poincaré section of the periodic attractor itself is given by $(x, \dot{x}) = (0.638712, 1.341599)$ ($\circ$). Trajectories that showed no periodicity within 40 excitation periods (80π seconds) were considered to be chaotic. The initial conditions of these trajectories form the chaotic attractor's basin of attraction (the white area in Fig. A.2), the end points of these trajectories are assumed to lie on the chaotic attractor. In Fig A.2 these end points are shown ($\cdot$), giving a good picture of the chaotic attractor's Poincaré section.

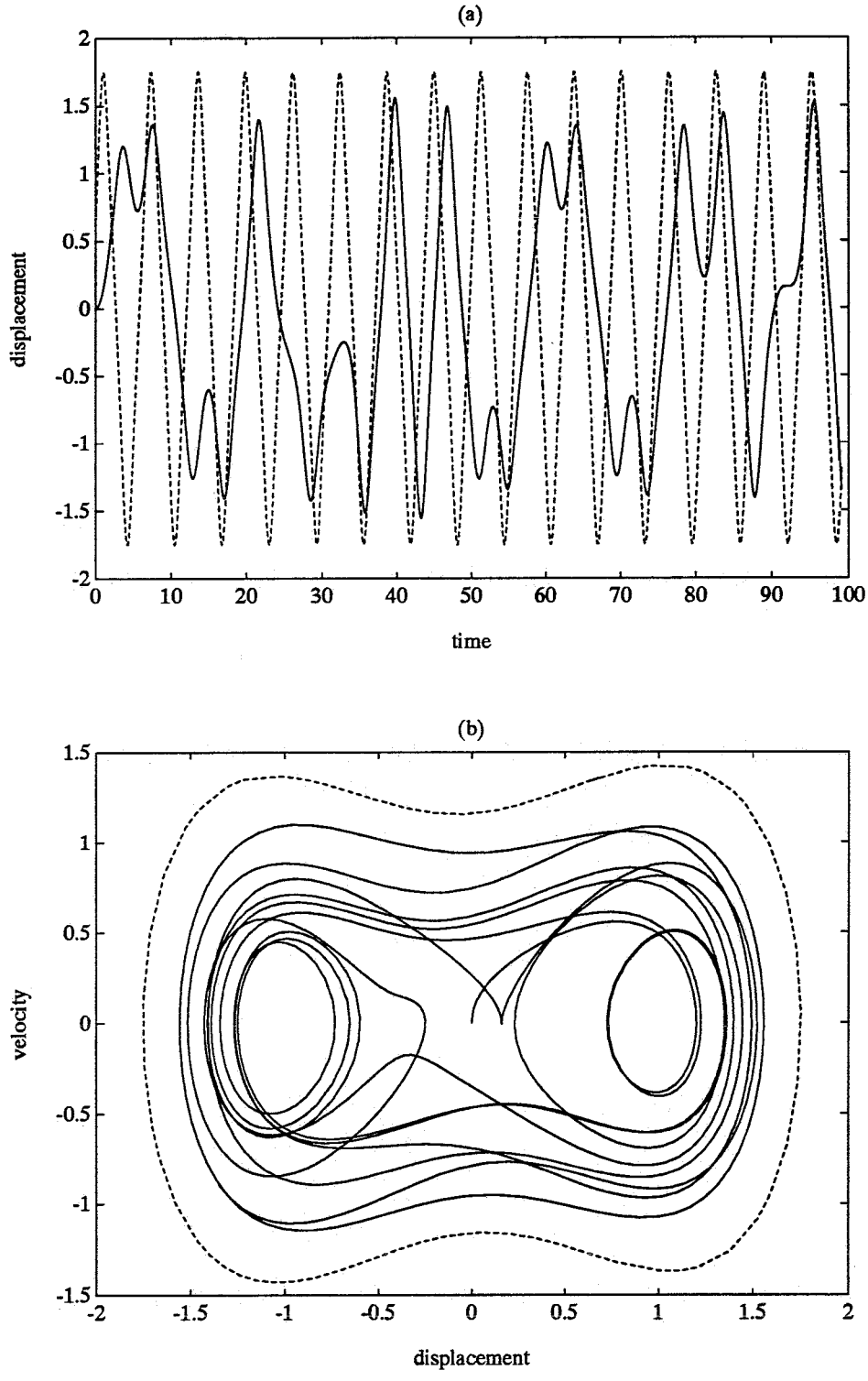

Figure A.1: Attractors of the modified Duffing equation with $d = 0.15$, $a = 0.3$, $\omega = 1.0$: (a) Time-history graph of periodic (dashed curve) and chaotic (continuous curve) attractor. (b) State space projection of periodic (dashed curve) and chaotic (continuous curve) attractor.

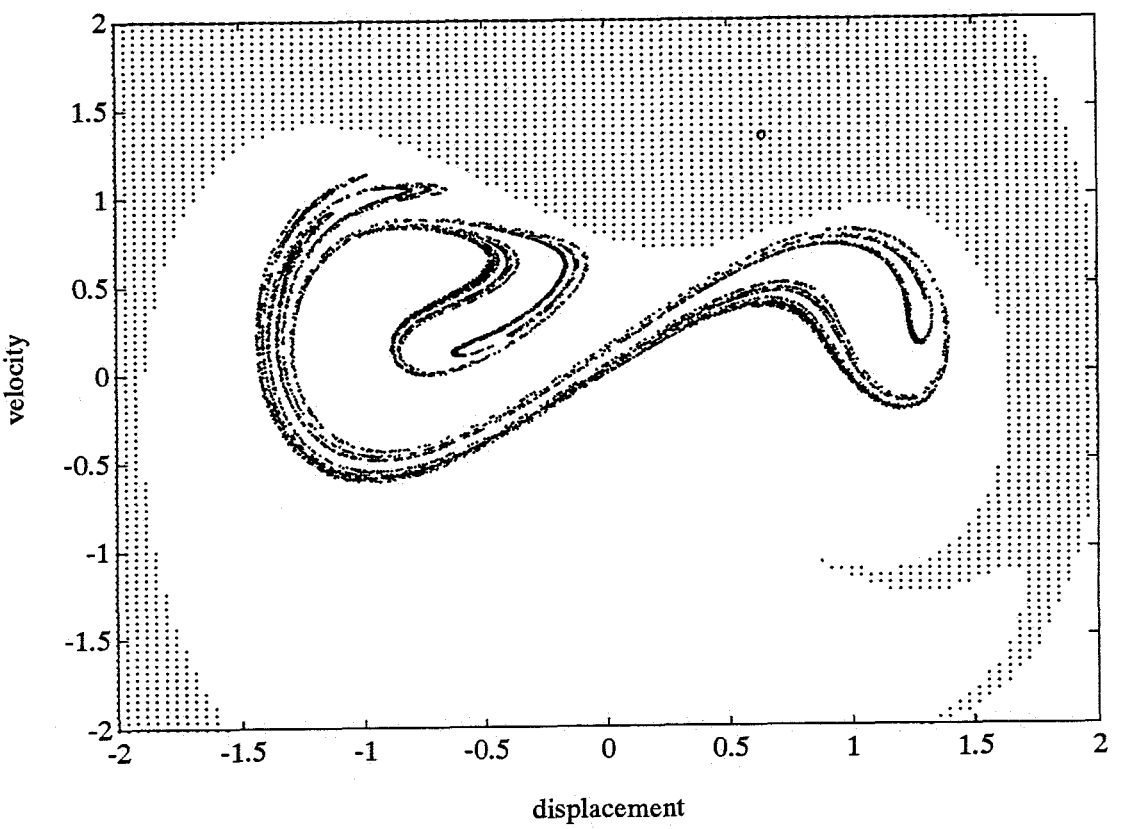

Figure A.2: Coexisting periodic and chaotic attractor and basins of attraction of the modified Duffing equation with $d = 0.15$, $a = 0.3$, $\omega = 1.0$.