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A CONTINUOUS FLOW MODEL FOR THREE PRODUCTION UNITS IN SERIES WITH BUFFERS

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Summary: In this paper we study three-stage production lines with intermediate buffers. The goods flow is supposed to be continuous and machine speeds may be different. Production units may be subject to stochastic failure and repair. We calculate the average line production rate for some cases. Furthermore we prove some production lines can be reduced to simpler two-stage ones.

Zusammenfassung: In dieser Arbeit untersuchen wir Produktionslinien mit drei Maschinen getrennt durch Puffer. Die Maschinen können verschiedene Geschwindigkeiten haben und abwechselnd operationsfähig und operationsunfähig sein. Wir berechnen die erwartete Liniendurchsatz für verschiedene Fälle. Ausserdem zeigen wir dass einige dreistations Produktionslinien gleichwertig sind mit einfachen zweistations Linien.

1. Introduction

In Wijngaard (1979) a technique is introduced for analytically treating two production units in series with interstage buffer storage. In that model the goods flow is supposed to be continuous and the production units are subject to stochastic failure and repair. In this paper we will apply the same techniques in order to obtain results concerning the stationary behaviour of the three-stage production line. This technique uses regeneration points. The idea behind this is, that the average output rate of a production line can be written as the quotient of the expected production per cycle and the expected cycle length, where a cycle is defined as the time between subsequent regenerations.

In section 2 we work out the model. It will appear that in the general case (that is all machines behave stochastically and all machine speeds are different) partial differential equations are involved, which cannot be solved analytically. However, in section 3 we show that in case two production units never fail and only one production unit breaks down from time to time, the expected output rate can be calculated in some cases. In section 4 we will work out an example.

Finally, in section 5 we discuss the case where exactly two production units break down from time to time.

2. Model of the production line

The three stage production line subject of our study in this paper, is shown in figure 2.1.

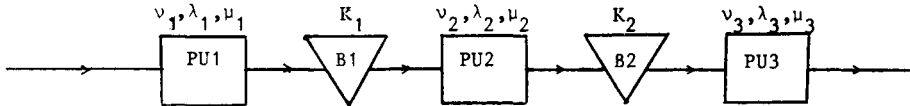


Fig. 2.1.

We suppose production is continuous, with different production rates. The production rate of production unit i (PU_i) is v_i ($i=1,2,3$). The PU 's are subject to machine failure; the time to the first failure for PU_i is exponentially distributed with parameter λ_i . We assume failures to be time dependent, that is, they occur at the same rate when the station is forced not to run or to run at a lower speed. For operation dependent failures the analysis can be carried out quite similarly. Since failure rates will only change in case a machine is blocked or starved, only the boundary conditions will change. Duration of repair of a PU which is broken down is also exponentially distributed, with parameter μ_i , respectively. We suppose PU_1 is never starved, that is, it has always items to work on. In a similar way PU_3 is never blocked by lack of storage capacity for finished items. Now suppose PU_i is working with rate v_i , buffer $(i-1)$ ($B(i-1)$) is empty and $v_i > v_{i-1}$ ($i=2,3$), then PU_i has to slow down to rate v_{i-1} , if $PU(i-1)$ is working. If $PU(i-1)$ is down then PU_i is forced down. In the same way if $v_i > v_{i+1}$ and B_i is full ($i=1,2$), then PU_i has to slow down to rate v_{i+1} , if $PU(i+1)$ is working. Otherwise, if $PU(i+1)$ is down, then PU_i is forced down. The capacities of the buffers are K_1 and K_2 , respectively.

For this model, the state of the system can be described completely by the quintuple (a,b,c,x,y) . The meaning of $(1,0,1,x,y)$ for instance is that PU_1 is up, PU_2 is under repair because of a failure, PU_3 is up, the inventory level in B_1 is $x \in [0, K_1]$ and in B_2 $y \in [0, K_2]$.

The different machine states are defined as in table 2.2.

state PU \	1	2	3	4	5	6	7	8
1	1	0	0	1	1	0	0	1
2	0	1	0	1	0	1	0	1
3	0	0	1	0	1	1	0	1

Table 2.2 Possible machine states

Now let $\alpha_i(x,y)$ be the cost rate in machine state i , given that the inventory level in B1 is x and in B2 is y ($i=1,\dots,8$). If $\alpha_i(x,y) = 1$, for all $x \in [0, K_1]$ and $y \in [0, K_2]$, then the expected cost per cycle, C_T , equals the expected cycle length, T . If $\alpha_i(x,y)$ equals the real production rate of PU3 in machine state i , with inventory levels x and y , respectively, then C_T equals the expected production per cycle, P_T . P_T/T is the net production rate of the line.

In order to calculate C_T we define

$f_i(x,y)$ = expected cost till the end of the cycle, if we are now in machine state i , with inventory levels x and y , respectively.

Now C_T can be expressed in the functions f_i , depending on the regeneration point we choose. For instance, if we choose for a certain parameter setting the regeneration point to be $(0,1,1,0,0)$, then

$$C_T = \frac{\alpha_6(0,0)}{\mu_1 + \lambda_2 + \lambda_3} + \frac{\mu_1}{\mu_1 + \lambda_2 + \lambda_3} f_8(0,0) + \frac{\lambda_2}{\mu_1 + \lambda_2 + \lambda_3} f_3(0,0) + \frac{\lambda_3}{\mu_1 + \lambda_2 + \lambda_3} f_2(0,0)$$

Here $\frac{1}{\mu_1 + \lambda_2 + \lambda_3}$ is the expected duration of a stay in $(0,1,1,0,0)$ and,

for instance, $\frac{\mu_1}{\mu_1 + \lambda_2 + \lambda_3}$ is the conditional probability of a transition

to state $(1,1,1,0,0)$.

By distinguishing between the cost in a certain small time interval Δ and the cost during the rest of the cycle, we can deduce the following system of equations for the f_i 's. We neglect some terms of $o(\Delta)$.

$$f_1(x,y) = \alpha_1(x,y)\Delta + (1-\lambda_1\Delta)(1-\mu_2\Delta)(1-\mu_3\Delta)f_1(x+v_1\Delta,y) + \lambda_1\Delta(1-\mu_2\Delta)(1-\mu_3\Delta)f_7(x,y) + (1-\lambda_1\Delta)(1-\mu_2\Delta)\mu_3\Delta f_5(x,y) + (1-\lambda_1\Delta)\mu_2\Delta(1-\mu_3\Delta)f_4(x,y)$$

By letting $\Delta \rightarrow 0$ we obtain

$$-v_1 f_{1x}(x,y) = \alpha_1(x,y) - (\lambda_1 + \mu_2 + \mu_3)f_1(x,y) + \lambda_1 f_7(x,y) + \mu_2 f_4(x,y) + \mu_3 f_5(x,y), \quad x \in [0, K_1] \quad y \in [0, K_2] \tag{2.1}$$

where $f_{1x}(x,y)$ is shorthand for $\frac{\partial f_1}{\partial x}(x,y)$. The equations for the other states can be derived in the same way.

In matrix-form they can be rewritten as

$$A \underline{f}_x(x,y) + B \underline{f}_y(x,y) = \underline{\alpha}(x,y) + C \underline{f}(x,y), \tag{2.2}$$

where $\underline{f}(x,y)$ is the vector $(f_1(x,y), f_2(x,y), \dots, f_8(x,y))^T$, $\underline{\alpha}(x,y)$ is the vector $(\alpha_1(x,y), \alpha_2(x,y), \dots, \alpha_3(x,y))^T$, A is the diagonal matrix $\text{Diag}(-v_1 v_2, 0, v_2^{-v_1}, -v_1 v_2, 0, v_2^{-v_1})$, B is the diagonal matrix $\text{Diag}(0, -v_2, v_3, -v_2, v_3, v_3^{-v_2}, 0, v_3^{-v_2})$ and

$$C = \begin{bmatrix} -\lambda_1 - \mu_2 - \mu_3 & 0 & 0 & \mu_2 & \mu_3 & 0 & \lambda_1 & 0 \\ 0 & -\lambda_2 - \mu_1 - \mu_3 & 0 & \mu_1 & 0 & \mu_3 & \lambda_2 & 0 \\ 0 & 0 & -\lambda_3 - \mu_1 - \mu_2 & 0 & \mu_1 & \mu_2 & \lambda_3 & 0 \\ \lambda_2 & \lambda_1 & 0 & -\lambda_1 - \lambda_2 - \mu_3 & 0 & 0 & 0 & \mu_3 \\ \lambda_3 & 0 & \lambda_1 & 0 & -\lambda_1 - \mu_2 - \lambda_3 & 0 & 0 & \mu_2 \\ 0 & \lambda_3 & \lambda_2 & 0 & 0 & -\mu_1 - \lambda_2 - \lambda_3 & 0 & \mu_1 \\ \mu_1 & \mu_2 & \mu_3 & 0 & 0 & 0 & -\mu_1 - \mu_2 - \mu_3 & 0 \\ 0 & 0 & 0 & \lambda_3 & \lambda_2 & \lambda_1 & 0 & -\lambda_1 - \lambda_2 - \lambda_3 \end{bmatrix}$$

The dimension of the system (2.2) can be reduced by one, by substituting the seventh equation in the other ones.

It is straightforward to derive the boundary conditions for these equations. For instance, the boundary condition for equation (2.1) is as follows

$$0 = \alpha_1(K_1, y) - (\lambda_1 + \mu_2 + \mu_3)f_1(K_1, y) + \lambda_1 f_7(K_1, y) + \mu_2 f_4(K_1, y) + \mu_3 f_5(K_1, y), \quad y \in [0, K_2] \tag{2.1a}$$

However, it is far from straight forward to solve the system (2.2) with the corresponding boundary equations. It can only be solved analytically in certain special cases (see Denneweyer (1968)) and also a numerical approach is not very attractive. Therefore we will restrict ourselves to cases with two or one perfect machine(s). For simpler two-stage lines a solution for the average line production rate is mentioned by Koenigsberg (1959), in case both PU's have the same production rate and are completely balanced (that is also the net production rates are equal). For unbalanced machines with the same speed an analytical solution is given by Wijngaard (1979) and Malathronas, Perkins and Smith (1983).

3. Two production units perfect, one production unit stochastic

In this and the next two sections we simplify the three-stage production line, by assuming one or more PU's are perfect. We first suppose one PU is stochastic, the other two are perfect. To obtain the net average output rate we have to distinguish between the thirteen possibilities for the machine speeds as represented in table 3.1.

v_3	$v_3 < v_1$			$v_3 = v_1$	$v_3 > v_1$		
v_2	$v_2 < v_3$	$v_2 = v_3$	$v_2 > v_3$	$v_2 < v_3$	$v_2 < v_3$		
$v_2 < v_1$	$v_2 < v_3$	$v_2 = v_3$	$v_2 > v_3$	$v_2 < v_3$	$v_2 < v_3$		
$v_2 = v_1$	$v_2 > v_3$			$v_2 > v_3$	$v_2 < v_3$		
$v_2 > v_1$	$v_2 > v_3$			$v_2 > v_3$	$v_2 < v_3$	$v_2 = v_3$	$v_2 > v_3$

Table 3.1 Different possibilities for the machine speeds

However, depending on which PU is the stochastic one, many of these cases can be dealt with in the same way. We will give some examples. Suppose PU2 is stochastic with parameters λ_2 and μ_2 and $v_2 \leq v_1$, $v_2 \leq v_3$. PU1 and PU3 are not subject to machine failures. After some time B1 will be occupied and will remain filled, B2 will remain empty. Therefore, the expected output per unit of time will be $v_2 \cdot \frac{\mu_2}{\lambda_2 + \mu_2}$.

If $v_1 \geq v_2 \geq v_3$, then again B1 will be full after some time and the system will operate effectively as a system consisting of PU2, B2 and PU3 alone. It can be analysed as in Wijngaard (1979).

The case $v_3 = v_1 < v_2$ will be treated in section 4.

The case $v_1 \leq v_2 \leq v_3$ can be treated as in Wijngaard (1979), since B2 will remain empty.

The two remaining cases $v_3 < v_1 < v_2$ and $v_1 < v_3 < v_2$ could not be solved analytically.

Now suppose PU3 is stochastic and PU's 1 and 2 produce constantly with rates v_1 and v_2 , respectively. We will not discuss all the cases mentioned before, but suppose $v_2 \geq v_3 > v_1$. This system can be analysed by looking at the aggregate inventory level $u := x+y$. This is because PU3 has to slow down (to rate v_1) if and only if $u = 0$. A similar analysis is carried out in section 5. The production line will behave in the same way as a system consisting of two production units, namely PU3 and PU1, separated by a buffer of capacity $K_1 + K_2$.

4. A production line with one imperfect machine

Suppose we have a production line consisting of a number of perfect machines with speed v_1 and somewhere in the line a stochastic machine with speed $v > v_1$, immediately preceded and succeeded by a buffer. Such a line behaves exactly in the same way as our three-stage system, where PU1 and PU3 are perfect, $v_3 = v_1$, and PU2 is stochastic with $v_2 > v_1$. We are again interested in the average line production rate.

The system of equations (2.2) reduces to

$$-v_1 f_{5x}(x,y) + v_1 f_{5y}(x,y) = \alpha_5(x,y) - \mu_2 f_5(x,y) + \mu_2 f_8(x,y),$$

$$x \in [0, K_1], y \in (0, K_2] \quad (4.1)$$

$$-(v_1 - v_2) f_{8x}(x,y) + (v_1 - v_2) f_{8y}(x,y) = \alpha_8(x,y) - \lambda_2 f_8(x,y) + \lambda_2 f_5(x,y),$$

$$x \in (0, K_1], y \in [0, K_2] \quad (4.2)$$

As regeneration points we choose the entrances of the state $(1, 0, 1, K_1, 0)$. Note that if $\| (x,y) - (K_1, 0) \| \rightarrow 0$, then $f_5(x,y) \rightarrow 0$. Therefore, in order to make f_5 continuous, we define $f_5(K_1, 0) = 0$. (4.3)
Other boundary conditions are

$$v_1 f_{5y}(K_1, y) = \alpha_5(K_1, y) - \mu_2 f_5(K_1, y) + \mu_2 f_8(K_1, y), y \in (0, K_2] \quad (4.4)$$

$$-v_1 f_{5x}(x,0) = \alpha_5(x,0) - \mu_2 f_5(x,0) + \mu_2 f_8(x,0), \quad x \in [0, K_1] \quad (4.5)$$

$$0 = \alpha_8(0,y) - \lambda_2 f_8(0,y) + \lambda_2 f_5(0,y), \quad y \in [0, K_2] \quad (4.6)$$

$$0 = \alpha_8(x, K_2) - \lambda_2 f_8(x, K_2) + \lambda_2 f_5(x, K_2), \quad x \in [0, K_1] \quad (4.7)$$

$$\text{The expected cost per cycle, } C_T = \frac{\alpha_5(K_1, 0)}{\mu_2} + f_8(K_1, 0) \quad (4.8)$$

The system (4.1) - (4.2) can be reduced to a single partial differential equation in $W(x,y) := f_8(x,y) - f_5(x,y)$. The boundary equations (4.6) and (4.7) can be expressed in this function W . Since $f_5(K_1, 0) = 0$ we can also express C_T in W . Now we may solve the new obtained equation by standard solution methods e.g. as indicated by Denne Meyer (1968). In order to obtain T we have to substitute $\alpha_5(x,y) = \alpha_8(x,y) = 1$, for all $x \in [0, K_1]$, $y \in [0, K_2]$. To obtain P_T we have to substitute $\alpha_5(x,y) = \alpha_8(x,y) = v_1$, for all $x \in [0, K_1]$, $y \in (0, K_2]$ and $\alpha_5(x,0) = 0$, $\alpha_8(x,0) = v_1$, for all $x \in [0, K_1]$.

For unbalanced lines, that is $\frac{v_2 \mu_2}{\lambda_2 + \mu_2} \neq v_1$, we obtain

$$P_T = \frac{v_1 v_2 \lambda_2 \mu_2 + v_1 (v_1 - v_2) \mu_2 (\lambda_2 + \mu_2) e^{p \min \{K_1, K_2\}}}{\lambda_2 \mu_2 (\lambda_2 v_1 + (v_1 - v_2) \mu_2)} \quad (4.9)$$

$$T = \frac{v_1 \lambda_2 (\lambda_2 + \mu_2) + (v_1 - v_2) \mu_2 (\lambda_2 + \mu_2) e^{p \min \{K_1, K_2\}}}{\lambda_2 \mu_2 (\lambda_2 v_1 + (v_1 - v_2) \mu_2)} \quad (4.10)$$

where $p := \frac{\lambda_2}{v_1 - v_2} + \frac{\mu_2}{v_1}$ and hence

$$\frac{P_T}{T} = \frac{v_1 v_2 \lambda_2 \mu_2 + v_1 (v_1 - v_2) \mu_2 (\lambda_2 + \mu_2) e^{p \min \{K_1, K_2\}}}{v_1 \lambda_2 (\lambda_2 + \mu_2) + (v_1 - v_2) \mu_2 (\lambda_2 + \mu_2) e^{p \min \{K_1, K_2\}}} \quad (4.11)$$

We may check formula (4.11) for the limiting cases to find

$$P_T/T \rightarrow \frac{v_1 \mu_2}{\lambda_2 + \mu_2}, \quad \text{if } \min \{K_1, K_2\} \rightarrow 0. \quad \text{If } \frac{v_2 \mu_2}{\lambda_2 + \mu_2} > v_1, \text{ then}$$

$$P_T/T \rightarrow v_1, \text{ if } \min \{K_1, K_2\} \rightarrow \infty. \text{ If } \frac{v_2 \mu_2}{\lambda_2 + \mu_2} > v_1, \text{ then}$$

$$P_T/T \rightarrow \frac{v_2 \mu_2}{\lambda_2 + \mu_2}, \text{ if } \min \{K_1, K_2\} \rightarrow \infty$$

For balanced lines, that is, $\frac{v_2 \mu_2}{\lambda_2 + \mu_2} = v_1$, we may solve the system for

this new case or, alternatively, we may let $\frac{v_2 \mu_2}{\lambda_2 + \mu_2} \rightarrow v_1$ in (4.9)-(4.11).

We find

$$P_T = \frac{1}{\mu_2 (v_2 - v_1)} [v_1^2 + v_2 \mu_2 \min \{K_1, K_2\}] \quad (4.12)$$

$$T = \frac{v_2}{v_1 \mu_2 (v_2 - v_1)} [v_1 + \mu_2 \min \{K_1, K_2\}] \text{ and hence} \quad (4.13)$$

$$\frac{P_T}{T} = \frac{v_1 (v_1^2 + v_2 \mu_2 \min \{K_1, K_2\})}{v_1 v_2 + v_2 \mu_2 \min \{K_1, K_2\}} \quad (4.14)$$

If $\min \{K_1, K_2\} \rightarrow 0$, then $P_T/T \rightarrow \frac{v_1^2}{v_2}$ if $\min \{K_1, K_2\} \rightarrow \infty$, then

$$P_T/T \rightarrow v_1$$

The reason why the system of partial differential equations can be solved in this case is that the problem is actually one-dimensional. In case $K_1 = K_2$ the aggregate inventory level will equal K_1 , after some time and from that moment on the aggregate buffer contents will only move along the line $y = K_1 - x$, $x \in [0, K_1]$.

It is interesting to compare the performance of this production line, which we will denote by A, with the performance of the production lines B and C of figure 4.1.

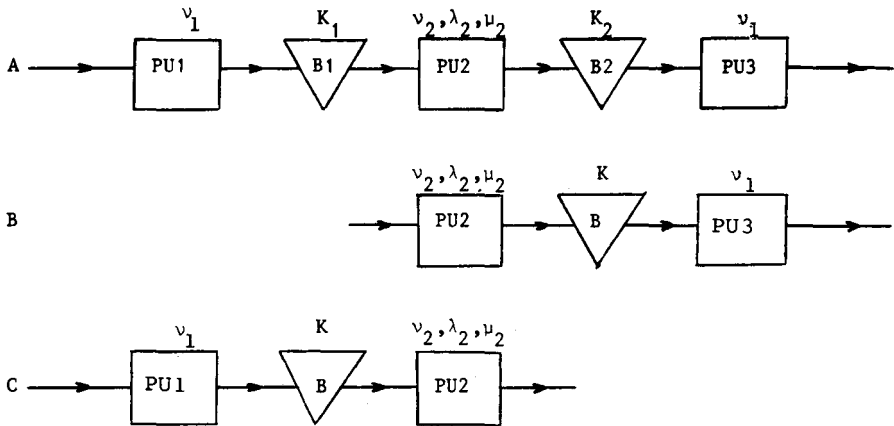


Fig. 4.1 Three production lines

We suppose, as always, that the first production unit in a line is never starved and the last production unit is never blocked. Furthermore, since the case $v_2 \leq v_1$ is unrealistic we suppose $v_2 > v_1$. For B and C we choose as regeneration points the entrances of the state $(0,1,0)$ and $(1,0,K)$ respectively. The last entry stands for the inventory level in B. After some straightforward calculations, quite similar to the calculations just done, it appears that, in case $\min\{K_1, K_2\} = K$, for all production lines A, B and C, the expected cycle time is the same, and this also holds for the expected production per cycle!

That this indeed has to be the case can be seen as follows. Let $a(t) =$ state of PU2 at time t , $a(t) = 1$, if PU2 is up

$$a(t) = 0, \text{ if PU2 is down.}$$

Suppose $a(0) = 0$ and the buffer contents at time $t = 0$ is 0 for line B, and K for line C. Let $\{(t, a(t)) | t \in \mathbb{R}_+\}$ be a stochastic realisation of PU2. The graphs of the buffer contents for lines B and C, corresponding to such a realisation, are sketched in figure 4.2. In case $K_1 = K_2 = K$ it is also the graph of the contents of B2 and B1, respectively, of line A, provided we start with B2 empty and B1 filled. The picture can easily be adapted for line A if $K = \min\{K_1, K_2\}$, $K_1 \neq K_2$.

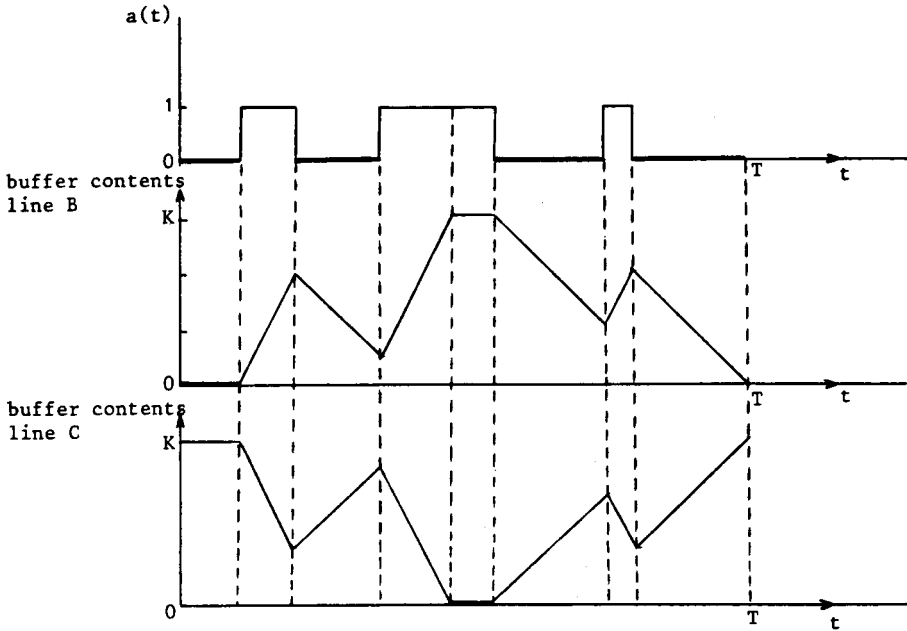


Fig. 4.2 Buffer contents for the lines B and C as function of time

We see that, because of the choice of the different regeneration points, the cycle time distribution is the same for all the production lines A, B and C. Now the only time line B cannot produce in a cycle is when the buffer is empty and PU2 is down. The only time in a cycle line C has no input is when the buffer is filled and PU2 is down. Since this only happens once in a cycle, by definition, we see that the input distribution of line C is equal to the output distribution of line B. We see also that the input distribution of line A is equal to its output distribution (and equal to the input distribution of line C). In particular, since the average input equals the average output, per cycle, for all lines, we have that the average output per cycle is the same for all lines ($= v_1(T - \frac{1}{\mu_2})$).

For more information about the reversibility of production lines see also the paper of Muth (1979).

5. Two production units stochastic, one production unit perfect

In this section we suppose that the production rate for every PU

equals 1. However, it is easy to see that some results also hold for more general cases.

If PU1 never fails and PU's 2 and 3 are stochastic, then the inventory level in B1 will increase monotonically. After the moment this inventory level equals K_1 , the system will behave as a system consisting of PU2 and PU3, separated by B2, only. This system has been analysed by Wijngaard (1979). In the same way, if only PU's 1 and 2 are stochastic, then B2 always remains empty. Therefore, the average line production rate equals the average line output per unit of time of PU1, B1 and PU2.

If PU2 is perfect and PU's 1 and 3 are stochastic the system can be analysed as follows. If x and y are the inventory levels in B1 and B2, respectively, then let $u := x+y$, $u \in [0, K_1+K_2]$. The state of the system can be described completely by the quadruple (a, l, c, u) , where a is the state of PU1, c the state of PU3. This is so, because starvation of PU3 is possible only if $u = 0$.

For instance, if PU3 is up, B2 is empty and B1 not, then PU3 still produces with rate 1. Only if also B1 is empty and PU1 is down, then PU3 is forced down. Note that this argument also holds if $v_2 > 1$, but not if $v_2 < 1$. Therefore, if $v_2 \geq 1$ we can solve this model completely analogous to Wijngaard (1979), with solution

$$\text{for } r_\kappa \neq r_\ell \left(r_\kappa := \frac{\lambda_1 \mu_3}{\mu_1 + \mu_3} + \frac{\lambda_1 \mu_3}{\lambda_1 + \lambda_3}, \quad r_\ell := \frac{\lambda_3 \mu_1}{\mu_1 + \mu_3} + \frac{\lambda_3 \mu_1}{\lambda_1 + \lambda_3} \right)$$

$$T = \frac{1}{\lambda_3 + \mu_1} \left(\frac{(\lambda_1 + \mu_1)(\lambda_3 + \mu_3)}{\lambda_3 \mu_1 - \lambda_1 \mu_3} \left(\frac{\lambda_3 \mu_1}{\lambda_1 \mu_3} e^{(r_\ell - r_\kappa)(K_1 + K_2)} - 1 \right) \right)$$

$$P_T = \frac{1}{\lambda_3 + \mu_1} \left(\frac{\mu_1 \mu_3}{\lambda_3 \mu_1 - \lambda_1 \mu_3} \left(\frac{\lambda_3 \mu_1}{\lambda_1 \mu_3} e^{(r_\ell - r_\kappa)(K_1 + K_2)} - 1 \right) + \frac{\lambda_3 \mu_1}{\lambda_3 \mu_1 - \lambda_1 \mu_3} \left(e^{(r_\ell - r_\kappa)(K_1 + K_2)} - 1 \right) \right)$$

For $r_\kappa = r_\ell$ we get

$$T = \frac{1}{\lambda_3 + \mu_1} \frac{(\lambda_3 + \mu_3)^2}{\lambda_3 \mu_3} \left(1 + r_\ell (K_1 + K_2) \right)$$

$$P_T = \frac{1}{\lambda_3 + \mu_1} \left(\frac{\mu_3}{\lambda_3} \left(1 + r_\ell(K_1 + K_2) \right) + r_\ell(K_1 + K_2) \right).$$

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