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## INSTANTANEOUS FOUR-BAR KINEMATICS BASED ON THE CARTER-HALL CIRCLE

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**Abstract** Kinematic subjects such as Function-Generation, straight-line approximation and also the uniform motion of coupler-points are easily accessible by way of the so-called Carter-Hall circle. Formula's for the derivatives of the four-bars' transmission-ratio, for the curvatures of the crank-polodes, and for other invariants in connection with four instantaneous positions of the four-bar, are easily obtained by studying the 'collineation-angles' and the way they vary. Examples of application demonstrate the power of this method. At the end of the paper, two mechanisms are shown to have coupler points nearly uniformly moving along an approximate straight-line.

**Keywords** Four-bar function generation, transmission-ratio, derivatives of collineation-angles, substitute linkage for gear-wheels, crank-polodes, location of Ball's point, cycloidal motion, constant-velocity of coupler-point.

**Introduction** An elementary course on instantaneous kinematics may very well begin with the merits and possibilities of the planar four-bar linkage. To the advantage of the designer, the four-bar also appears to function as an excellent example of application of general kinematic laws and theorems. Contemplated as a quadrilateral, the figure shows an important role in the geometry of quadratic curves, leading to, for instance, a particular case of the hexagon-theorem of Pascal in which the so-called 'collineation-axis' was fundamental. This axis, defined as the line connecting the intersections of opposite sides, turned the quadrilateral into a complete one. The collineation-axis of the quadrilateral was fundamental also in Bobillier's Theorem (1870). Later, Freudenstein's Theorem (1956) about the four-bar's angular accelerations, used this axis to study the motion of the collineation-axis with respect to the four sides of the four-bar, therefore, seems to play a fundamental role in the four-bar motion itself. It is for this reason, that we aim to derive expressions for the derivatives of the four collineation-angles ( $\beta_j$ ).

<sup>\*)</sup> Any conic section, drawn through the joints of a four-bar, has 'opposite tangents' intersecting at the collineation-axis. (Opposite tangents are tangent lines, drawn to the conic at opposite joints of the four-bar) The hexagon of Pascal's Theorem turns into a quadrilateral by taking two opposite sides infinitesimally small. Pascal's line then turns into the collineation-axis of the quadrilateral.

### The derivatives of the collineation-angles ( $\beta_j$ )

Figure 1 shows an arbitrary four-bar  $A_0ABB_0$ , the sides of which are numbered 0, 1, 2, 3 respectively. The turning-joints of the four-bar simultaneously act as instantaneous centers of rotation and are trivial as such. Thus, the turning-joints are trivial velocity-centers or ditto poles. The remaining velocity-centers that exist between the four links of a four-bar are the poles  $P_{20}$  and  $P_{31}$ . If connected, they constitute the collineation-axis  $PQ$ , by which the point  $P_{31}$  is sometimes called the collineation-point  $Q$ .<sup>3)</sup> (after Bobillier)

By definition, the collineation-angles are assumed to be positive, if they turn from the collineation-axis to the two cranks in an anti-clockwise direction, and from the frame and the coupler to the collineation-axis. (See figure 1) We further define the input-(crank) angle, which is the motion-variable as

$$\varphi_{10} = \angle AA_0B_0 \quad (1)$$

and, similarly, the output-angle as

$$\varphi_{30} = \pi - \angle A_0B_0B \quad (2)$$

Further, the coupler rotates with the angle

$$\varphi_{20} = \angle AQA_0 = \angle BQB_0 \quad (3)$$

Thus,

$$\psi_{10} = \beta_0 + \beta_1 \quad (4)$$

$$\psi_{20} = \beta_0 - \beta_2 \quad (5)$$

$$\psi_{30} = \beta_0 + \beta_3 \quad (6)$$

Hence, their derivatives with respect to the motion variable yield

$$1 = \beta'_0 + \beta'_1 \quad (7)$$

$$\psi'_{20} = \beta'_0 - \beta'_2 \quad (8)$$

$$\psi'_{30} = \beta'_0 + \beta'_3 \quad (9)$$

These formulas are in fact expressions for the transmission ratios:

$$i_{31} = \frac{d\psi_{30}}{d\psi_{10}} = \frac{\omega_{30}}{\omega_{10}} = \frac{P_{31}P_{10}}{P_{31}P_{30}} = \frac{P_{31}P_{10} \cdot P_{31}P_{20}}{P_{31}P_{20} \cdot P_{31}P_{30}} = \frac{\sin \beta_1 \cdot \sin(\beta_0 + \beta_3)}{\sin(\beta_0 + \beta_1) \cdot \sin \beta_3} = \frac{t_0 + t_3}{t_0 + t_1} = 1 \quad (10)$$

in which

$$t_j = \cotan \beta_j \quad \text{for } j = 0, 1, 2, 3 \quad (11)$$

In a similar way, we find that

$$i_{21} = \psi'_{20} = \frac{t_0 - t_2}{t_0 + t_1} \quad (12)$$

The latter may also be obtained from (10) by observing the input-link as the fixed link, followed by a cyclic interchange of numbers afterwards.

Further, by combining the eqns. (9) and (10) we get

$$\beta'_0 + \beta'_3 = \frac{t_0 + t_3}{t_0 + t_1} \quad (13)$$

and, similarly, (8) and (12) gives

$$\beta'_0 - \beta'_2 = \frac{t_0 - t_2}{t_0 + t_1} \quad (14)$$

According to Freudenstein's formula for the angular accelerations, we have that

$$\frac{\delta_{30}}{\omega_{10}^2} = i(1-i)\cotan \beta_2 + i \frac{\delta_{10}}{\omega_{10}^2}$$

If, however, we observe the derivatives with respect to the motion variable instead of to the time, we get the more simple expression:

$$i' = i(1-i)\cotan \beta_2 \quad (15)$$

The formula may be derived by differentiation of the Aronhold-Kennedy expression

$$\frac{\omega_{30}}{\omega_{10}} = \frac{P_{31}P_{10}}{P_{31}P_{30}} \quad \text{and then using the expression for the velocity } v_Q \text{ by which the collineation-point } Q \text{ runs along the fixed link:}$$

$$v_Q = \frac{d\psi_{10}}{dt} \cdot QA_0 \cdot \cotan \beta_2.$$

However, on the other hand, we may also differentiate equation (10). In carrying this out, we obtain the expression

$$i' = \frac{t'_0(t_1 - t_2) - t'_1(t_0 + t_3) + t'_2(t_0 + t_1)}{(t_0 + t_1)^2} \quad (16)$$

In here,  $t'_1$  as well as  $t'_3$  may be expressed in  $t'_0$ , using the equations (7) and (9) and the fact that  $t'_j = -(1+t_j^2)\beta'_j$  for  $j = 0, 1, 2, 3$ .

Equating (15) and (16) then gives a fairly simple result after some tedious calculation:

$$\beta'_0 = \frac{t_1 + t_3 - t_2}{t_0 + t_1} \quad (17)$$

The first derivative of the collineation-angle  $\beta_0$  has so been found. For the derivatives of the other angles  $\beta_j$  we simply use the eqns. (7), (8) and (9) again. This then results into the expressions:

$$\beta'_1 = \frac{t_0 + t_2 - t_3}{t_0 + t_1} \quad (18)$$

$$\beta'_2 = \frac{t_1 + t_3 - t_0}{t_0 + t_1} \quad (19)$$

$$\beta'_3 = \frac{t_0 + t_2 - t_1}{t_0 + t_1} \quad (20)$$

It is fairly easy now to find the further derivatives, for instance

$$\beta''_2(t_0 + t_1)^2 = 3(t_1 - t_0 + t_3 - t_2) + t_0 t_1 (3t_3 - 2t_2) - t_2 t_3 (t_0 - t_1 + t_3) \quad (21)$$

found by derivation of (19) and substitution of the values for  $\beta'_0$ ,  $\beta'_1$  and  $\beta'_3$  from the expressions (17),

(18) and (20).

#### Function-Generation with four-bar linkages

Suppose the object is to design a four-bar linkage such that the rotation of the output complies with a prescribed function of the rotation of the input. It is therefore required that  $\psi_{30} = f(\psi_{10})$ .

As we are dealing with instantaneous solutions, we may assume to work with the known derivatives

$$i = \psi'_{30}, \quad i' = \psi''_{30}, \quad i'' = \psi'''_{30} \quad \text{and, if necessary } i''' = \psi^{IV}_{30}.$$

Then,  $i = \frac{QA_0}{QB_0}$ ,  $i' = i(1-i)\cotan \beta_2$  and by

differentiation

$$\frac{d^3 \psi_{30}}{d\psi_{10}^3} = i'' = i(1-i)(1-2i)t_2^2 - i(1-i)(1+t_2^2)\beta_2' \quad (22)$$

In here, we may substitute the value for  $\beta_2'$  according to eq. (19).

However, as  $i$ ,  $i'$  and  $i''$  are known quantities, so are the values  $i$ ,  $t_2$  and  $\beta_2'$ . Throughout the following, therefore, these values may be observed as given constants. Further, as long as the fixed link doesn't vary in length, the location of the collineation point  $Q$  at the fixed link, doesn't vary too, because of the given transmission ratio  $i$ . The design of the four-bar is further restricted by the given value for the angle  $\beta_2$  and the equation (19) that has to be met.

We will prove, that eq. (19) is represented by a locus for the poles  $P$ , that appears to be a circle, nowadays known as the Carter-Hall-circle. The diameter of this circle runs from  $Q$  to a point  $V$  at the fixed link. Thus,  $\sphericalangle VPQ = 90^\circ$ . Further,

$$\frac{1 - \beta_2'}{QA_0} + \frac{1}{QB_0} = \frac{3}{QV} \quad (23)$$

To prove this, we apply the Rule of Sine in  $\Delta A_0PQ$ :

$$\frac{QV \cdot \cos \beta_0}{QA_0} = \frac{QP}{QA_0} = \frac{\sin(\beta_0 + \beta_1)}{\sin \beta_1} \quad \text{Hence,}$$

$$\frac{QV}{QA_0} = \frac{t_1}{t_0} + 1 \quad (24)$$

Similarly,

$$\frac{QV}{QB_0} = \frac{t_2}{t_0} + 1 \quad (25)$$

Substituting these values, as well as the value  $\beta_1$ , according to (19), into the eq. (23) then indeed leads to an identity.

Eq. (19) and (23) are therefore equivalent equations. This proves the Carter-Hall-Theorem given above. Instead of (23), we may also write

$$\beta_2^1 = 1 + \frac{QA_0}{QB_0} - 3 \frac{QA_0}{QV} = 1 + 1 - 3I. \quad \frac{QB_0}{QV} \quad (23A)$$

For 4 infinitesimally near positions, (3rd order approximation) the design of the four-bar now runs as follows (See figure 2):

- Determine the fixed link and the collineation point Q according to the equation  $I = QA_0/QB_0$ .
- Calculate the value  $t_2$  from Freudenstein's formula (15) and also  $\beta_2^1$  from eq. (22).
- Then determine the point V at the fixed link according to eq. (23).
- Determine the Carter-Hall-circle, having  $\overline{QV}$  for diameter.
- Choose a random point P at this circle.
- Respectively connect the given frame-centers  $A_0$  and  $B_0$  with the pole P.
- Now draw the coupler-line QA such that  $\angle PQA = \beta_2 = \arctan t_2^{-1}$ .
- Intersect QA at the coordinated points A and B of the respective path-normals  $PA_0$  and  $PB_0$ .

#### The cubics of stationary curvature

Clearly, the points A and B are coordinated to the points P of the Carter-Hall circle. So, the locus for P leads to a corresponding locus for the points A as well as to one for the points B. Naturally, the loci for A and B correspond as well. They are named the cubics of stationary curvature, as simultaneously the distance AB remains stationary for four consecutive and infinitesimally near positions. (If simultaneously two pairs AB are connected by bars, a five-link structure ensues, having 3rd order instantaneous mobility.)

To obtain the equations of the cubics, we proceed as follows:

Let the coupler AB intersect PV at a point U for which

$$\overline{QU} = \overline{QP} \cdot \cos^{-1} \beta_2 \quad (26)$$

As further,

$$\overline{QV} = \overline{QP} \cdot \cos^{-1} \beta_0 \quad (27)$$

we have that

$$\frac{\overline{QU}}{\overline{QV}} = \frac{\cos \beta_0}{\cos \beta_2} \quad (28)$$

$$\text{Thus, } \overline{QU} = \overline{QV} \cdot \cos \beta_0 \text{ in which } \overline{QU}_0 = \overline{QV}_0 \cdot \cos^{-1} \beta_2 \quad (29)$$

Clearly,  $\overline{QU}$  simultaneously represents the pole-tangent of the relative motion  $V_1/V_3$ , as, according to Bobillier's Theorem, the pole-tangent happens to be the image of the collineation-axis with respect to the angle bisector of the coupler and the fixed link. We deduce that the locus for the points U will be a circle, joining the points  $Q, V$  and  $U_0$ . (See figure 2.) Once the point U on this circle has been chosen, the angle  $\beta_0$  will be known:

$$\beta_0 = \angle UQU_0 \quad (30)$$

The location of point A at the line QU is then established by the equation

$$\frac{QA}{QU} = \frac{t_2}{t_1 + t_2} \quad (31)$$

an equation that follows from the Rule of Sine, applied in  $\Delta QAP$ .

The equations (24), (28) and (31) then lead to the cubic for the points A:

$$\frac{1}{QA} = \left( \frac{1}{QA_0} - \frac{1}{QV} \right) \frac{\sin \beta_2}{\sin \beta_0} + \frac{\cos \beta_2}{QV \cdot \cos \beta_0} \quad (\text{locus (32) for A)}$$

Thus, if we take the origin at Q and the x-axis along  $\overline{QU}_0$ , the polar-coordinates  $(QA, \beta_0)$  of the points A are related as given by equation (32). The locus represents a cubic equation of a circular curve, having Q as a double point of the curve, whereas the double-

points' curvatures (2/m and 2/l) of this cubic are respectively determined by the equations

$$m = \overline{QU}_0 = \overline{QV}_0 \cdot \cos^{-1} \beta_2 \quad (33)$$

and

$$\frac{1}{l} = \left( \frac{1}{QA_0} - \frac{1}{QV} \right) \sin \beta_2 \quad (34)$$

(the curvature radius of the latter, being in a direction perpendicular to  $\overline{QU}_0$ .)

For the to A coordinated points B, we find a similar locus

$$\frac{1}{QB} = \left( \frac{1}{QB_0} - \frac{1}{QV} \right) \frac{\sin \beta_2}{\sin \beta_0} + \frac{\cos \beta_2}{QV \cdot \cos \beta_0} \quad (\text{locus for B}) \quad (35)$$

The curve has a similar shape as the locus for A. Also the tangents to the double point Q of the curve are perpendicular, whereas the curvature radii  $m/2$  and  $l/2$  at this point, are now represented by the equations (33) and (36):

$$\frac{1}{l_0} = \left( \frac{1}{QB_0} - \frac{1}{QV} \right) \sin \beta_2 \quad (36)$$

#### Determination of Ball's point to acquire approximate straight-line mechanisms

If we seek a point  $A_U$ , (usually called Ball's point), for which the corresponding point  $B_U$  lies at infinity, we find from (35) that for such a point:

$$\tan \beta_{0U} = \left( 1 - \frac{\overline{QV}}{\overline{QB}_0} \right) \tan \beta_2 \quad (\text{Ball's point } A_U) \quad (37)$$

This determines the direction under which to seek Ball's point. As further, the equations (25) and (37) lead to the simple equation

$$\beta_{3U} = -\beta_2$$

the intersection of Carter-Hall's circle and a 2<sup>nd</sup> circle with the top-angle  $(-\beta_2)$ , joining the points Q and  $B_0$ , leads to a singular auxiliary point  $B_U$  and so to a singular point of Ball  $A_U$ . (See figure

3.) Note, that all points of the 2nd circle lead to the inflection circle of crank 1 with respect to crank 3, as is demonstrated in figure 3A.

The Curvature Radii of the Crank-Polodes

In order to find the curvature radii ( $R_1$  and  $R_2$ ) of the polodes, respectively attached to the primary - and to the secondary crank of the four-bar, we may use an expression for the velocity  $v_Q$  by which the collineation-point  $Q$  runs along the fixed link:

$$v_Q = \omega_{10} \cdot \overline{QA}_0 \cdot \cotan \beta_2 \quad (38)$$

This expression, by the way, was fundamental for the derivation of Freudenstein's formula. For the increment length of arc,  $ds$ , measured along either one of the crank-polodes, we naturally find the equations:

$$R_1(d\beta_2 + d\varphi_{10}) = ds = R_3(d\beta_2 + d\varphi_{30}) \quad (39)$$

As  $ds$  runs along the pole-tangent  $QU_0$ , clearly,

$$\frac{ds}{dt} \cdot \cos \beta_2 = v_Q \quad (40)$$

Hence, by using eq. (38), we find that

$$R_1(\beta_2' + 1) = \overline{QA}_0 \cdot \sin^{-1} \beta_2 \quad (41)$$

and

$$R_3(\beta_2' + 1) = \overline{QA}_0 \cdot \sin^{-1} \beta_2 \quad (42)$$

Generally, therefore,  $R_1 \neq \overline{QA}_0$  and  $R_3 \neq \overline{QA}_0$ .

i.e. generally, the curvature centers of the crank-polodes do not coincide with the centers  $A_0$  and  $E_0$  of the four-bar.

Elimination of  $\beta_2'$  from the equations (23A) and (41) yield:

$$\frac{1}{R_1} = \sin \beta_2 \left( \frac{2}{\overline{QA}_0} + \frac{1}{\overline{QB}_0} - \frac{3}{\overline{QV}} \right) \quad (43)$$

and similarly

$$\frac{1}{R_3} = \left( \frac{1}{\overline{QA}_0} + \frac{2}{\overline{QB}_0} - \frac{3}{\overline{QV}} \right) \cdot \sin \beta_2 \quad (44)$$

or, by using the expressions for  $l$  and  $l_0$  in (34) and (36):

$$\frac{1}{R_1} = \frac{2}{l} + \frac{1}{l_0} \quad (45)$$

$$\frac{1}{R_3} = \frac{2}{l_0} + \frac{1}{l} \quad (46)$$

Special case, for which  $i' = 0$ , whence  $\beta_2 = 90^\circ$

In this case, eq. (33) yields

$$a^{-1} = 0 \text{ and } U_0 = U_0^{\infty} \text{ (See figure 4)}$$

The equations (32) and (35) show that both cubics are now degenerated into the line  $QU_0$  and into a circle touching the pole-tangent  $QU_0$  at the collineation point  $Q$ . The diameters of these circles are determined by the equations:

$$\frac{1}{\overline{QA}_0} - \frac{1}{\overline{QV}} = \frac{1}{\overline{QL}} \quad (47)$$

$$\frac{1}{\overline{QB}_0} - \frac{1}{\overline{QV}} = \frac{1}{\overline{QL}} \quad (48)$$

The loci for the respective points A, B and P are thereby related through the equation:

$$\frac{1 - \beta_2^2}{\overline{QL}} + \frac{1}{\overline{QL}_0} = \frac{1 + \beta_2^2}{\overline{QV}} \quad (49)$$

a relation that is derived from the equations (23A), (47) and ((48).

Instantaneous Reproduction of the transmission-ratio  $i$  of a gear-wheel pair

Like we did before, the angular-velocity ratio  $i$ , which is equal to the ratio  $\overline{QA}_0 / \overline{QB}_0$ , is assumed to be given. Further,

$i' = i'' = 0$ , but  $i''' \neq 0$ , as otherwise  $\beta_2'' = 0$

leads to an inverted slider crank by meeting equation (21).

As in this case  $\beta_2' = 0$ , according to eq. (22), equation (23A) yields that

$$\frac{\overline{QB}_0}{\overline{QV}} = \frac{1+i}{3i} \quad (50)$$

From the equations (47) and (50) we further see that

$$\frac{\overline{QL}}{\overline{QB}_0} = \frac{3i}{2i-1} \quad (51)$$

Similarly, the eqns. (48) and (50) yield:

$$\frac{\overline{QB}_0}{\overline{QL}_0} = \frac{2i-1}{3i} \quad (52)$$

So,

$$\frac{\overline{LL}_0}{\overline{QB}_0} = \frac{9i(1-i)}{(2i-1)(2i-1)} \quad (53)$$

Further,

$$d/\overline{QB}_0 = \overline{A_0B_0} / \overline{QB}_0 = 1 - i, \quad (54)$$

and so

$$\frac{\overline{LL}_0}{d} = \frac{9i}{(2i-1)(2i-1)} \quad (55)$$

From figure 4 we further derive that

$$\frac{\overline{AB}}{d} = \frac{\overline{LL}_0}{d} \sin \beta_0 = \frac{9i \sin \beta_0}{(2i-1)(2i-1)} = \frac{\overline{AB}}{d} \quad (56)$$

Exploiting the Rule of Sine in  $\Delta A_0AL$  gives additionally:

$$\frac{\overline{A_0A}}{d} = \frac{\overline{A_0A}}{\overline{A_0L}} \left( \frac{\overline{QL}}{\overline{QB}_0} - \frac{\overline{QA}_0}{\overline{QB}_0} \right) \frac{1}{d} = \frac{\sin \beta_0}{\sin \beta_1} \left( \frac{3i}{2i-1} - 1 \right) \frac{1}{1-i}$$

As in this case also

$$\tan \beta_1 = \frac{\overline{QA}}{\overline{QP}} = \frac{\overline{QL} \cdot \sin \beta_0}{\overline{QV} \cdot \cos \beta_0} = \frac{1+i}{2i-1} \tan \beta_0, \text{ we find that}$$

$$\frac{\overline{A_0A}}{d} = \frac{1}{1-i} \sqrt{\left( \frac{1+i}{2i-1} \right)^2 \sin^2 \beta_0 + \cos^2 \beta_0} \quad (57)$$

Similarly,

$$\frac{\overline{B_0B}}{d} = \frac{1}{1-i} \sqrt{\left( \frac{1+i}{2i-1} \right)^2 \sin^2 \beta_0 + \cos^2 \beta_0} \quad (58)$$

The angle  $\beta_0$  in these expressions may be used as a design-parameter, to be varied in order to meet other requirements, such as a given value for the minimum transmission-angle  $\mu_{min}$ , or otherwise.

Note that the dimension-formulas exhibited in this section, are valid also for negative angular velocity ratios.

Note that generally, based on the equations (41) and (42), the two centers of the gear-wheels, only coincide with  $A_0$  and  $B_0$  of the (3rd order) substitute four-bar, when  $\beta_2 = 90^\circ$  and  $\beta_2' = 0$ .

If, as in figure 4, the value  $\beta_2' \neq 0$ , the substitute wheels do not center at these points.

A straight stretch of the coupler curve traced by a coupler point with an unvariable speed

To attain this with a four-bar mechanism, only approximate solutions are to be expected. Then, to obtain a straight-part in the coupler curve, a coincident solution would certainly use Ball's point for coupler point. As further the speed of this point is related to the regular rotations of the input-crank, we need a specific Roberts' curve-cognate to change this into a regular rotating coupler. (Two curve-cognates out of Roberts' Configuration, namely, have exchanged angular velocities of crank and coupler.)

With the curve-cognate, the problem is then reduced to four-bars for which the ratio of the coupler-points' velocity  $v_K$  and the angular velocity  $\omega$  of the coupler, remains a constant. As

$v_K = \omega \cdot PK$ , we aim at a four-bar coupler motion having a constant distance between coupler point K

and velocity pole P.

The fixed pole, therefore, should be an equidistant to the straight-part of the coupler curve, traced by K. This only occurs, when a circle (i.e. the moving pole) rolls about a straight-line (i.e. the fixed pole.) The circle's center then runs parallel to the fixed pole.

So, in this case, we are not concerned with a given ratio for the angular velocities, but with a four-bar having opposite sides respectively attached to the above mentioned poles.

If the opposite sides are  $AA_0$  and  $BB_0$ , the rolling gear-wheel, having a radius  $R_1$  is then assumed to be attached to the "coupler"  $AA_0$ , whereas the rack, for which  $R_3^{-1} = 0$ , is thought to be attached to the "frame"  $BB_0$ .

According to the equations (45) and (46) we then have that

$$-3R_1 = 1_0 = -2l \quad (59)$$

As Ball's point,  $A_U$ , now coincides with the center of the rolling wheel,  $A_U$  joins the pole-normal, perpendicular to the pole-tangent  $QU_0$ . Since  $A_U$  is also a point of the cubic of stationary curvature (i.e. the circling point curve), the cubic falls apart into the pole-normal and into a circle, touching  $QU_0$ , and having  $QL = 2l$  for diameter.

(See figure 5.) The center points' curve degenerates too: This curve falls apart into the pole-normal and into another circle touching  $QU_0$ , having  $QL_0 = 1_0$  for diameter.

At Ball's point, not a four- but even a five-point contact between curve and tangent occurs, if we take the point  $A_0$  of the pole-normal at the point I.

As a consequence,  $B_0 = L_0$ , whence  $\beta_2 = 90^\circ$ ,

$i = \frac{QA_0}{QB_0} = -\frac{1}{2}$  and  $F = P^{00}$  if stretched four-bars are not observed.

Equations (34) or (43) further show that

$$\frac{1}{QV} = 0, \text{ whence } V = V^{00}.$$

From eq. (25A) we then deduce that  $\beta_2' = \frac{1}{2}$ , whence  $i = -\frac{1}{2}$ ,  $i' = 0$  and  $i'' = 3/8$ .

From the figure we also derive the formula's for the dimensions of the linkage:

$$\overline{AB} = 3l \sin \beta_0$$

$$\frac{B_0B}{A_0B} = 2l \cos \beta_0$$

$$\frac{A_0A}{A_0B} = -1 \cos \beta_0$$

$$\frac{A_0B}{A_0B_0} = -3l$$

$$\frac{A_0A_U}{A_0A_0} = -1/3 \text{ with } K = A_U$$

$$\overline{AA_U} = \frac{1}{3} l \sqrt{\sin^2 \beta_0 + 4 \cos^2 \beta_0}$$

Further  $\angle AA_0K = \beta_0$  and  $\tan \angle A_UA_0A = \frac{1}{2} \tan \beta_0$ .

Application of Roberts' Law by which we interchange the (angular velocities of the) sides  $AA_0$  and  $B_0A_0$ , then finally results into a four-bar mechanism having a coupler point moving along a straight-part of the coupler curve with an approx-

imate constant velocity. (See figure 6.)

A 2nd solution emerges if we take the other curve-cognate from Roberts' Law. (See figure 7.) To obtain the latter, the angular rotations of the sides  $AA_0$  and  $AB$  have to be exchanged.

Like the first solution, the dimensions may be varied by varying the angle  $\beta_0$ . So, totally, we still have two times an infinite number of solutions.

The solutions demonstrated in this section, represent the very best approximations that are instantaneously possible. Also Burmester Theory states, that 5 infinitesimally separated positions, that are involved here, do represent the maximum number possible in this case.

One may note though, that the cycloidal motion does not have any Burmester-points outside the pole-normal  $QA_0$ . As a consequence, the coordinated points

A and B of figure 5 do not represent a Burmester-pair for this motion. That is why a change in the design-parameter  $\beta_0$  will not result into better approximations.

### Conclusions

Replacing a (constant) gear-wheel transmission-ratio gives another outcome than the four-bars that result by replacing the gear-wheels themselves. Though in both cases  $i' = 0$ , her next derivative ( $i''$ ) differs.

Both cases, however, lead to some very practical applications: the first one to function-generators, generally; the 2nd one to timed path-generators.

The kinematic approach displayed in this paper, certainly has the advantage of reducing the problem of the design of function-, path- and even timed path-generators to a common denominator.

Litterature

- Hall, A.S. and Dunk, A.C.: Designing four-bar linkages to substitute for gears. Machine Design 25(1953)4, p. 232-236.
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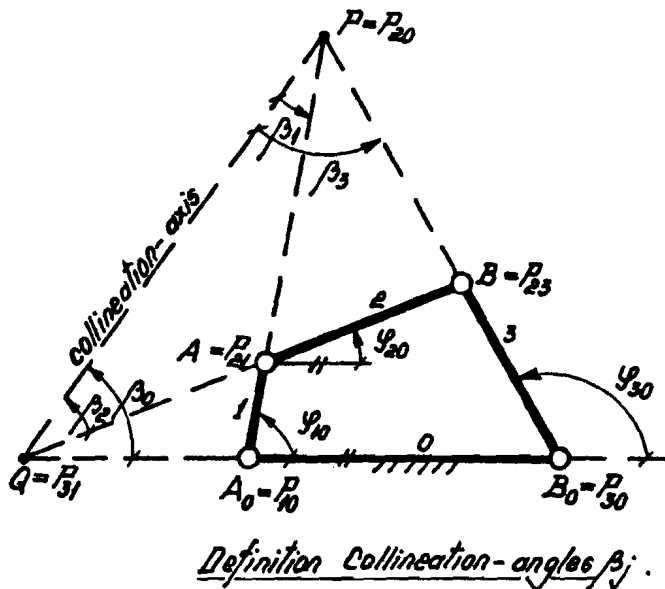
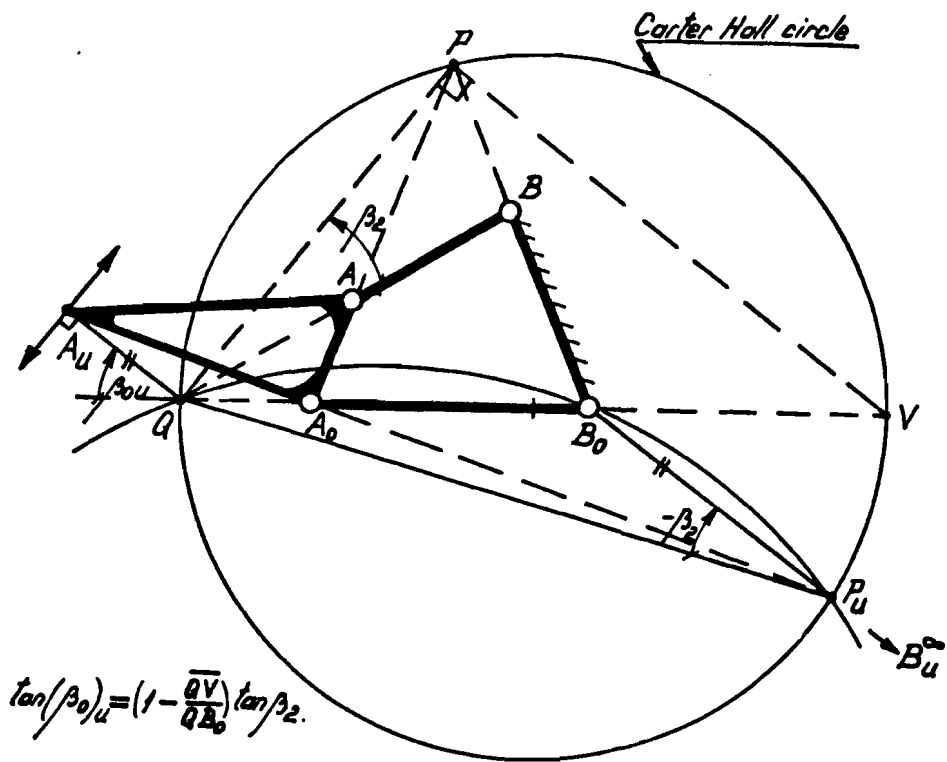


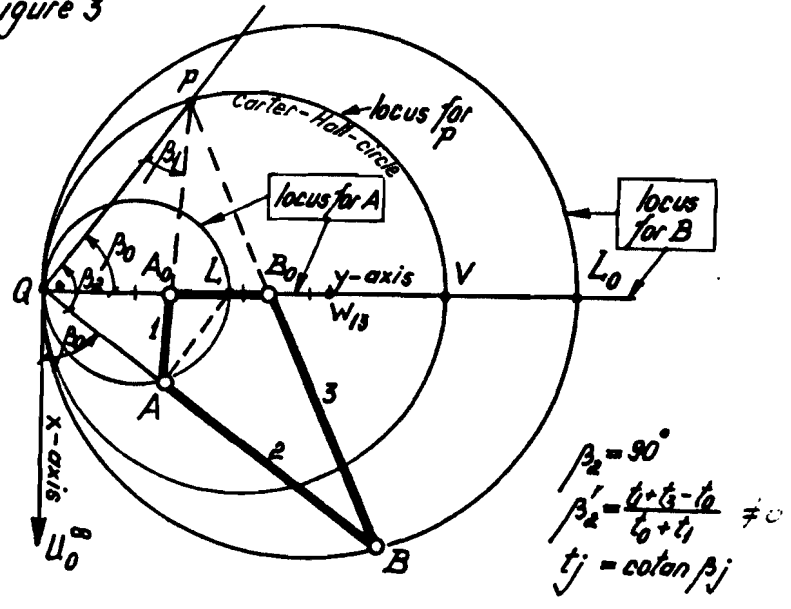
Figure 1





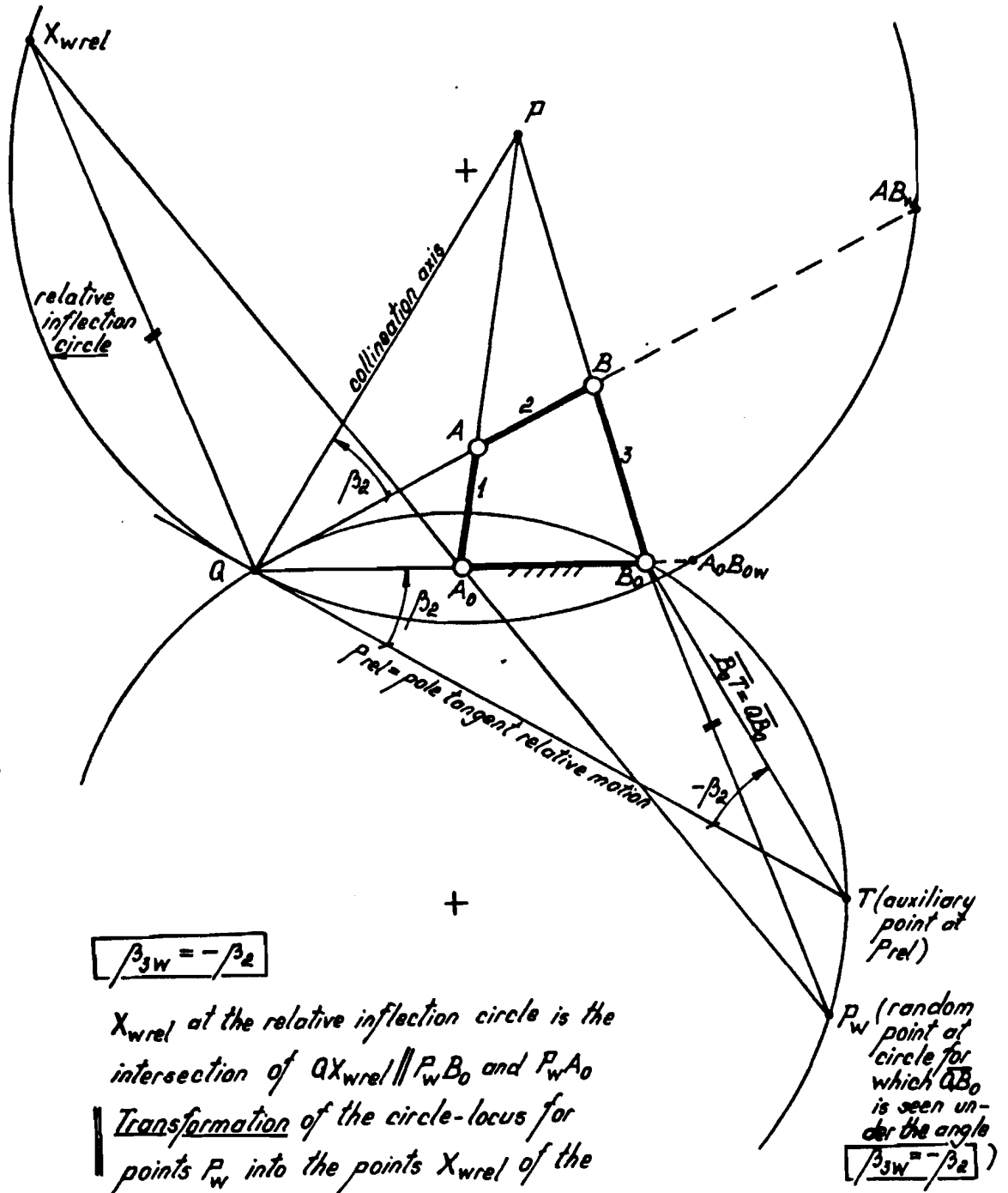


Determination of Ball's point  $A_u$   
Figure 3



Four-bars for which  $\overline{A_0B_0}$ ,  
 $\frac{d\beta_{30}}{d\beta_{10}} \equiv i = \frac{QA_0}{QB_0}$ ,  $i' = 0$  and  $i'' = -i(1-i)\beta_2''$   
have the same values

Figure 4



$$\beta_{3w} = -\beta_2$$

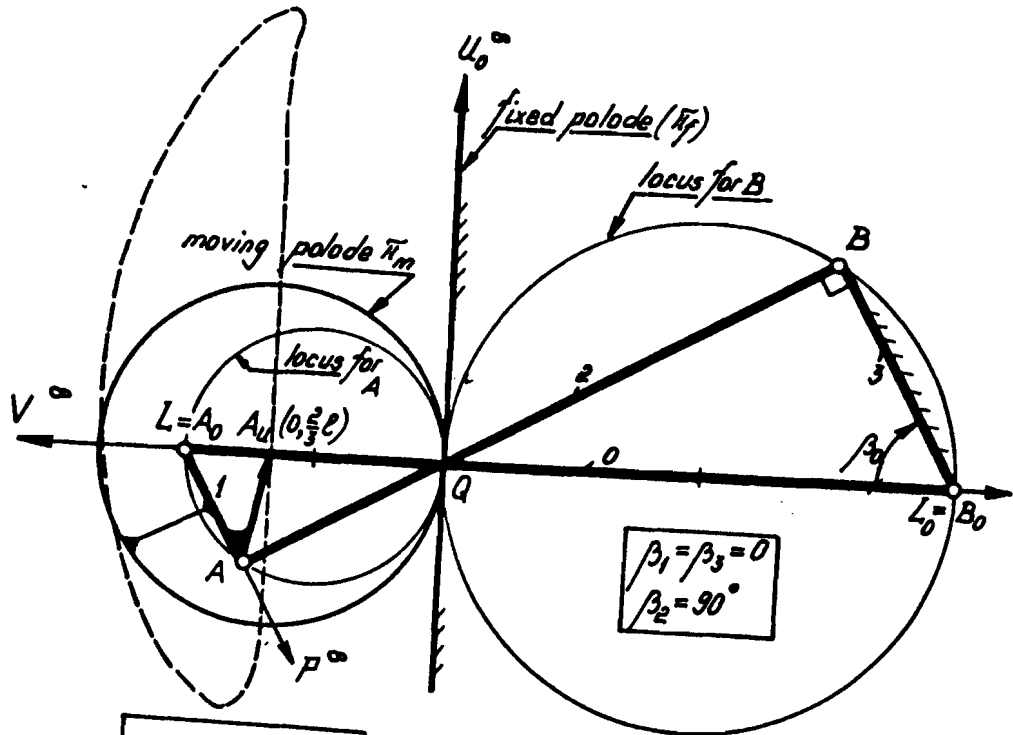
$X_{wrel}$  at the relative inflection circle is the intersection of  $AX_{wrel} \parallel P_w B_0$  and  $P_w A_0$

**Transformation of the circle-locus for points  $P_w$  into the points  $X_{wrel}$  of the relative inflection circle.**

New design for inflection circle

$P_w$  (random point of circle for which  $AB_0$  is seen under the angle  $\beta_{3w} = -\beta_2$ )

Figure 3A



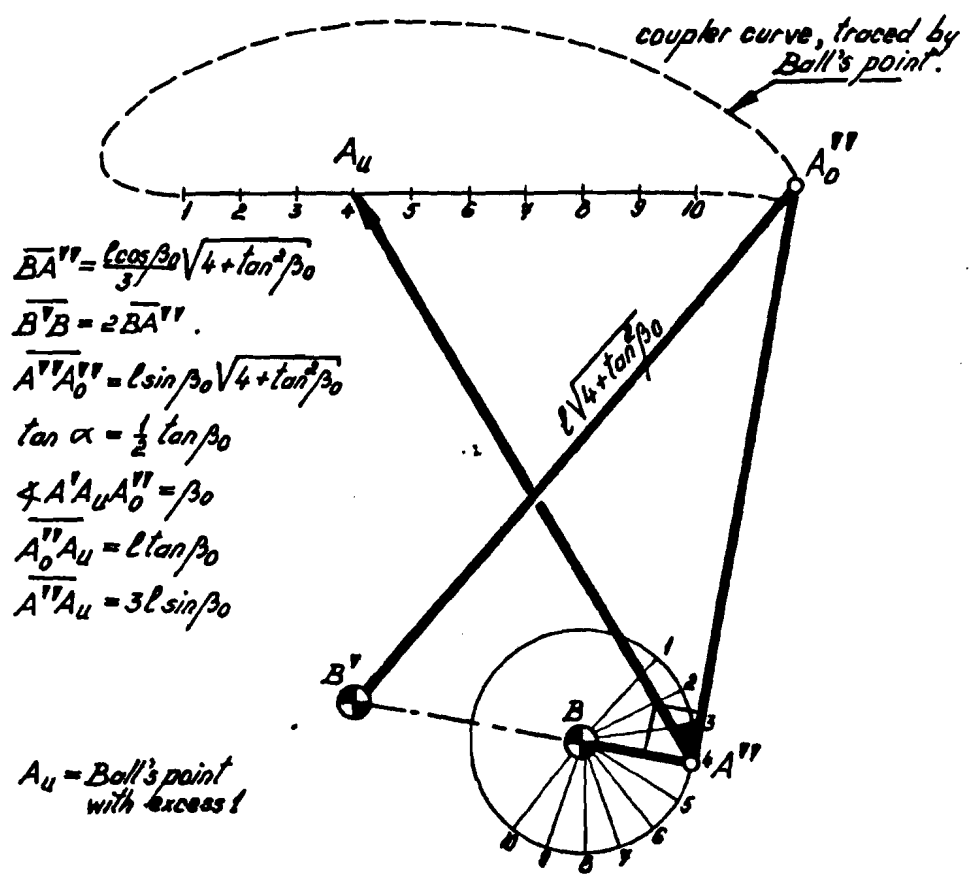
$$-3R_1 = l_0 = -2l$$

$$i = -\frac{1}{2}$$

Ball's point  $A_u$  = rolling wheel's center  $M_{T_m}$

Instantaneous replacement of cycloidal motion by a four-bar linkage.

Figure 5



2<sup>nd</sup> Cognate  
Four-Bar having an approximate constant  
velocity along her path.

Figure 7