On systems defined by implicit analytic nonlinear functional equations

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Abstract

Systems are often specified by an implicit functional relation between output and input variables. The problem arises as to the general form of solution for output in terms of input in this type of relation. In this report, this problem is formulated and solved for analytic nonlinear systems, a region being determined in which explicit analytic solution is valid. It is further shown that this region coincides with the region of convergence of the iterative method of solution of the implicit functional relation making use of the contraction mapping property. The method is illustrated by the solution of a system of analytic nonlinear differential equations in state-variable form.

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1. Introduction

The simplest system description is by an explicit functional relation between input u and output x of the form

$$x(t) = f(t\,;\, u(t'),\; t' \leq t) \tag{1.1}$$

or in discrete time,

$$x_k = f_k(u_k, u_{k-1}, \cdots) \tag{1.2}$$

Very often however, systems are described by an implicit functional relation of the form

$$x(t) = f(t\,;\, x(t'),\, t' < t\,;\, u(t''),\, t'' \leq t) \tag{1.3}$$

or in discrete time,

$$x_k = f_k(x_{k-1}, x_{k-2}, \cdots\;;\; u_k, u_{k-1}, \cdots) \tag{1.4}$$

Here the output at any time depends not only on previous and present inputs but also on previous outputs.

Examples

(a) Feedback:the distinction between the open- and closed-loop systems of figure 1 is the same as the distinction between explicit and implicit functional description.

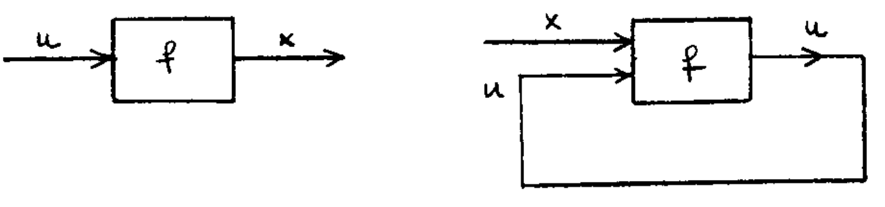

Fig.1 Open- and closed-loop systems illustrate explicit and implicit description.

(b) Åström-Bohlin description: a common way of describing discrete-time systems is

$$x_k = \sum_{r=1}^{\infty} a_r x_{k-r} + \sum_{r=0}^{\infty} b_r u_{k-r} \tag{1.5}$$

This type of relation may often be regarded as a linearization of a nonlinear relation of the form (1.4) about some fixed working level of the variables x and u (usually taken as x = 0, u = 0)

(c) State-space description: in discrete time, the equation will have the form

$$x_k = f(x_{k-1}, u_k) \tag{1.6}$$

which is just a special case of (1.4). In continuous time, the equation is

$$\dot{x}(t) = f(x(t), u(t), t) \tag{1.7}$$

and the relation to (1.3) is disguised. It becomes clear on writing the equation as an integral equation:

$$x(t) = x(t_0) + \int_{t_0}^{t} f(t', x(t'), u(t'))\, dt' \tag{1.8}$$

It is possible to consider in the same way equations with delays:

$$\dot{x}(t) = f(t\,;\, x(t), x(t-\alpha_1), x(t-\alpha_2), \ldots ;\, u(t), u(t-\beta_1), \ldots) \tag{1.9}$$

The discrete-time form of such an equation is of the type (1.4).

In the case of linear systems there is no important difference between systems described in the explicit form (1.1) and systems described in the implicit form (1.3): provided stability conditions are satisfied, one form is easily converted to the other. In the case of nonlinear systems the same is not so and the implicit type of description is in general only locally equivalent to the explicit form.

The present report will investigate the explicit solution of implicitly defined relationships and determine a region in which the solution is valid. The answer found has various applications e.g. to the input-output stability of implicitly defined systems.

In order to make the methods and conclusions as general as possible, the problem is formulated and solved abstractly. Input u and output x are considered to be elements of a Banach space and the system is described by a nonlinear functional relation between these spaces. As indicated in the title of the report, attention is restricted to analytic functional relations.

2. Preliminary remarks on analytic functions between Banach spaces.

Let $\mathcal{U}$ and $\mathcal{X}$ be two Banach spaces and $u \in \mathcal{U}$, $x \in \mathcal{X}$. A function between u and x will be denoted by the notation

$$x = f[u] \tag{2.1}$$

the square bracket being a convenient way of distinguishing between functions between Banach spaces, i.e. nonlinear operators, and the ordinary functions between scalars and vectors occurring in the applications of the theory.

An analytic function between elements of a Banach space will be defined in the following way. It will be assumed that the function has a representation

$$f[u] = f_0[u] + f_1[u] + f_2[u] + \tag{2.2}$$

in terms of functions of degrees 0,1,2,... and that the term of degree n can be written

$$f_n[u] = \frac{1}{n!} a_n[u, u, \cdots, u] \tag{2.3}$$

where $a_n\left[u^{(1)}, u^{(2)}, \ldots, u^{(n)}\right]$ is a completely symmetrical multilinear function of $u^{(1)}, u^{(2)}, \ldots, u^{(n)}$, i.e. it is

(a) unchanged by a permutation of $u^{(1)}, u^{(2)}, \ldots, u^{(n)}$.

(b) linear in each of these functions.

The factor $1/n!$ is inserted for convenience.

Boundedness, majorants: attention will be restricted to bounded analytic functions in the sense of the following definition.

Definition. An analytic function $f[u]$ will be said to be bounded if non-negative constants A_n, $n = 0,1,2,\ldots$ exist such that

(a) the multilinear function $a_n[\ ]$ is bounded in the usual sense:

$$\|a_m[u^{(1)}, u^{(2)}, \ldots, u^{(m)}]\| \leq A_m \|u^{(1)}\| \cdots \|u^{(m)}\| \tag{2.4}$$

(b) the series

$$A_0 + \frac{A_1}{1!} U + \frac{A_2}{2!} U^2 + \ldots \tag{2.5}$$

is convergent for some positive range of values of U.

A series (2.5) having these properties is called a majorant series for the analytic function $f[u]$. The associated function

$$F(U) = A_0 + \frac{A_1}{1!} U + \frac{A_2}{2!} U^2 + \ldots \tag{2.6}$$

defined by a particular majorant series in its region of convergence is called a majorant function for $f[u]$ (majorant for short)

It is clear that if $\|u\| \leq U$ where U lies within the range of convergence of the majorant, then

(a) the series for $f[u]$ is absolutely and uniformly convergent.

(b) $f[u]$ satisfies the inequality

$$\| f[u] \| \leq F(U) \tag{2.7}$$

If a majorant exists, it is always possible to define least majorant as the majorant obtained by putting

$$A_m = \| a_m \| \tag{2.8}$$

where $\| a_n \|$, the norm of the multilinear function $a_n[\ ]$, is defined in the usual way as the lower bound of all A_n for which the inequality (2.4) is satisfied.

Functions of two variables: this is a simple extension of the theory for functions of one variable. An analytic function of two variables u and x is defined as one with a representation

$$f[x,u] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_{mn}[x,u] \tag{2.9}$$

where $f_{mn}[x,u]$ is a function of degree m in x and n in u with the explicit form

$$f_{mn}[x,u] = \frac{1}{m!\,m!}\, a_{mn}[x,x,\ldots,x;u,u,\ldots,u] \tag{2.10}$$

where $a_{mn}[x^{(1)},x^{(2)},\ldots,x^{(m)};u^{(1)},u^{(2)},\ldots,u^{(n)}]$ is a multilinear function which is completely symmetrical separately in the x's and in the u's.

The boundedness condition is that there exist constants $A_{mn} \geq 0$ such that

(a) $a_{mn}[\ ]$ is a bounded multilinear function:

$$\| a_{mn}[x^{(1)}, x^{(2)}, \ldots, x^{(m)}; u^{(1)}, u^{(2)}, \ldots, u^{(n)}] \leq A_{mn} \| x^{(1)} \| \| x^{(2)} \| \ldots \| x^{(m)} \| \| u^{(1)} \| \ldots \| u^{(n)} \| \tag{2.11}$$

(b) the series

$$\sum_{m=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!\,n!} A_{mn} X^m U^n \tag{2.12}$$

is convergent for some positive range of values of X and U.

3. Statement of the problem and properties of the comparison equation.

Suppose given an implicit functional equation

$$x = f[x,u] \tag{3.1}$$

between two Banach space elements $u \in \mathfrak{U}$ and $x \in \mathfrak{X}$ where $f[x,u]$ is a bounded analytic function. The problem is to find the conditions under which it is possible to solve this equation explicitly in the form

$$x = g[u] \tag{3.2}$$

where $g[u]$ is also a bounded analytic functional.

The known answer to this problem is, of course, the classical implicit function theorem proved for ordinary analytic functions by Cauchy in 1827 (see Goursat /1/). Cauchy's theorem was extended to analytic functions between Banach spaces by Michal and Clifford in 1933 (ref. /2/. See also Michal 1958, ref./4/) The limitation of this theorem is that it only gives a local solution valid "for sufficently small values",or at least,in a very restricted range. Hille (ref./5/) extended the classical result of Cauchy showing the solution obtained by the Cauchy method of series substitution is valid over a wider range provided an additional condition was satisfied. The extension of this idea to the Banach space theory gives a suitable result for system-analysis applications. The extension will be given in the next section; it has not, to the author's knowledge, yet been given in the mathematical literature. The present section will be devoted to certain preliminary topics.

The solution of the equation (3.1) will be studied in the neighbourhood of an assumed solution

$$x = 0 \, , \, u = 0 \tag{3.3}$$

which involves no loss of generality. Consequently $f[x,u]$ will be

$$f[x,u] = a_{10}[x] + a_{01}[u] + \frac{1}{2!}\left(a_{20}[x,x] + 2a_{11}[x,u] + a_{02}[u,u]\right) + \ldots\ldots \tag{3.4}$$

The majorant of f[x,u] will correspondingly be taken in the form

$$F(X,U) = A_{10}X + A_{01}U + \frac{1}{2!}\left(A_{20}X^2 + 2A_{11}XU + A_{02}U^2\right) + \ldots \tag{3.5}$$

The case when f[x,u] and its majorant have no linear terms in x and X respectively is basic to the theory and will be referred to as the <u>normal form</u>. In this case

$$a_{10}[x] \equiv 0 \tag{3.6}$$

and also

$$A_{10} = 0 \tag{3.7}$$

The solution of the implicit functional equation (3.1) is found to be closely related to the solution of the associated implicit func- -tion equation

$$X = F(X,U) \tag{3.8}$$

which will be called the <u>comparison equation</u> for the implicit func- -tional equation (3.1) formed from the majorant F(X,U).

If the implicit functional equation (3.1) is in normal form, then by (3.7) the associated comparison equation (3.8) will be in normal form also, i.e. without linear terms in X.

The remaining part of this section will discuss the properties of the comparison equation.

The graph of the comparison equation: noting that all coefficients in the series expansion of $F(X,U)$ are non-negative, it is easy to verify that, unless $F(X,U)$ is linear in both X and U, the graph of the comparison equation will have the form shown in fig.2 below ($X \geq 0$, $U \geq 0$ assumed)

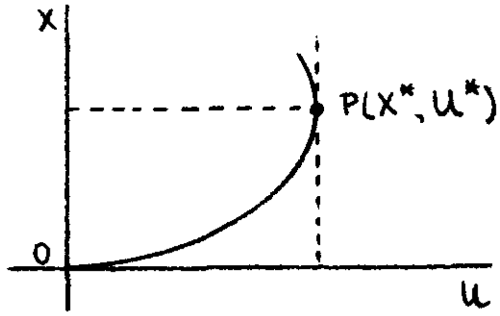

Fig.2 The graph of the comparison equation.

Starting at the origin, the graph curves upward and then turns back--ward. There is a vertical tangent at a point P with coordinates (X^*,U^*) determined as the unique solution of the equations:

$$\left.\begin{aligned} X^* &= F(X^*, U^*) \qquad (3.9) \\ 1 &= F'_X(X^*, U^*) \qquad (3.10) \end{aligned}\right\}$$

where $F'_X(X,U)$ denotes the partial derivative of $F(X,U)$ with respect to X.

The value U^* will be important in what follows. It will be referred to as the <u>turning value</u> of U on the graph of the comparison equation.

On the range $0 \leq U \leq U^*$, the branch OP of the graph defines a single--valued function of U which will be denoted by

$$X = G(U) \qquad (3.11)$$

It is well known that, at least for sufficiently small values of U, the function $G(U)$ may be determined by the method of <u>series substi--tution</u>. This method is as follows for equations in normal form.

Starting from the power series representation of the implicit equation:

$$X = A_{01} U + \frac{1}{2!}\left(A_{20} X^2 + 2 A_{11} X U + A_{02} U^2 \right) + \ldots \qquad (3.12)$$

it is assumed that

$$X = B_1 U + \frac{B_2}{2!} U^2 + \frac{B_3}{3!} U^3 + \ldots \qquad (3.13)$$

By substitution and the equating of coefficients of powers of u , there follow the equations

$$\begin{aligned} B_1 &= A_{01} \\ B_2 &= A_{02} + 2 A_{11} B_1 + A_{20} B_1^2 \\ B_3 &= A_{03} + 3 A_{12} B_1 + 3 A_{21} B_1^2 + \\ &\quad + A_{30} B_1^3 + 3 A_{11} B_2 + 3 A_{20} B_1 B_2 \end{aligned} \qquad (3.14)$$

. . . . etc

from which the coefficients $B_1, B_2, B_3, \ldots$ may successively be calculated. That the series (3.13) thus obtained is convergent for sufficiently small values of U is the classical result of Cauchy:

Theorem (Cauchy) The method of series substitution applied to an implicit function equation in normal form gives a series solution which is convergent for sufficiently small values of the independent variable.

Proof: see e.g. Goursat / 1 /

From this result follows the analytic representation of the curve OP sufficiently near to the origin, i.e.

$$G(U) = B_1 U + \frac{B_2}{2!} U^2 + \frac{B_3}{3!} U^3 + \ldots \qquad (3.15)$$

for sufficiently small values of U.

It is most important for the present report that a stronger result than this is obtainable if the function F(X,U) has a series expansion which is convergent for all (positive) values of X and U. This is the result given by Hille (ref./5 /) The use of Hille's theorem in problems of system-analysis is due to Kiełkiewicz /6 /

Theorem (Hille) If an analytic function with series expansion

$$F(X,U) = A_{01}U + \frac{1}{2!}\left(A_{20}X^2 + 2A_{11}XU + A_{02}U^2\right) + \dots \qquad (3.16)$$

has non-negative coefficients and is convergent for all positive values of X and U, then the implicit function equation

$$X = F(X,U) \qquad (3.17)$$

when solved by the method of series substitution, gives an analytic solution

$$X = G(U) \qquad (3.18)$$

which has a power series convergent over the range $0 \leq U \leq U^*$ and which represents the branch OP of the graph in fig.2

Proof: see Hille /5 / for a proof of the theorem by complex variable theory. A few inessential modifications have been made to Hille's statement of the theorem to fit in better with the present context.

Note: in the statement of the theorem, the trivial case when F(X,U) is linear (and therefore no turning value U exists) will be excluded.

4. Solution of implicit functional equations by series substitution.

Consider an analytic implicit functional equation in normal form:

$$\begin{aligned} x &= f[x,u] \\ &= a_{01}[u] + \frac{1}{2!}\left(a_{20}[x,x] + 2a_{11}[x,u] + a_{20}[u,u]\right) + \cdots \end{aligned} \tag{4.1}$$

and assume a series solution

$$\begin{aligned} x &= g[u] \\ &= b_1[u] + \frac{1}{2!} b_2[u,u] + \frac{1}{3!} b_3[u,u,u] + \cdots \end{aligned} \tag{4.2}$$

This is found to satisfy the equation algebraically if the following relations hold among the multilinear functions:

$$b_1[u] = a_{01}[u]$$

$$\begin{aligned} b_2[u_1,u_2] = {} & a_{02}[u_1,u_2] + \\ & a_{11}[b_1[u_1];u_2] + a_{11}[b_1[u_2];u_1] + \\ & a_{20}[b_1[u_1], b_1[u_2]] \end{aligned}$$

$$\begin{aligned} b_3[u_1,u_2,u_3] = {} & a_{03}[u_1,u_2,u_3] + \\ & a_{12}[b_1[u_1];u_2,u_3] + a_{12}[b_1[u_2];u_3,u_1] + a_{12}[b_1[u_3];u_1,u_2] + \\ & a_{21}[b_1[u_2],b_1[u_3];u_1] + a_{21}[b_1[u_1],b_1[u_3];u_2] + a_{21}[b_1[u_1],b_1[u_2];u_3] + \\ & a_{30}[b_1[u_1], b_1[u_2], b_1[u_3]] + \\ & a_{11}[b_2[u_1,u_2];u_3] + a_{11}[b_2[u_1,u_3];u_2] + a_{11}[b_2[u_2,u_3];u_1] + \\ & a_{20}[b_1[u_1], b_2[u_2,u_3]] + a_{20}[b_1[u_2], b_2[u_1,u_3]] + a_{02}[b_1[u_3], b_2[u_1,u_2]] \end{aligned} \tag{4.3}$$

$$\ldots\ldots$$

and generally

$$\ell_N[u_1,\ldots,u_N] = \sum a_{rn}\Big[b_{r_1}[u_{\sigma_1},\ldots,u_{\sigma_{r_1}}],\, b_{r_2}[u_{\sigma_{r_1+1}},\ldots],$$

$$\ldots,\, b_{r_m}[\ldots,u_{\sigma_m}];\, u_{\sigma_{m+1}},\ldots,u_{\sigma_N}\Big] \quad (4.4)$$

where the summation on the right is over all

(a) integers r,n such that r + n = N

(b) integers $m, r_1, \ldots, r_m$ such that $r_1 + \ldots + r_m = r$

(c) division $\{u_{\sigma_1},\ldots,u_{\sigma_m}\}\{u_{\sigma_{m+1}},\ldots u_{\sigma_N}\}$ of the variables $u_1,\ldots,u_N$ into two groups of m and n elements.

By using the symmetry of the multilinear "coefficients",the formulae may be written in shortened form:

$$\ell_1[u] = a_{10}[u]$$

$$\ell_2[u_1,u_2] = a_{02}[u_1,u_2] + 2a_{11}[\ell_1[u_1], u_2] + a_{20}[\ell_1[u_1], \ell_2[u_2]]$$

$$\begin{aligned}\ell_3[u_1,u_2,u_3] = {} & a_{03}[u_1,u_2,u_3] + 3a_{12}[\ell_1[u_1]; u_2,u_3] + \\ & 3a_{21}[\ell_1[u_1], \ell_1[u_2], u_3] + a_{30}[\ell_1[u_1], \ell_1[u_2], \ell_1[u_3]] + \\ & 3a_{11}[\ell_2[u_1,u_2], u_3] + 3a_{20}[\ell_1[u_1,u_2], \ell_1[u_3]]\end{aligned} \quad (4.5)$$

$\ldots$

On taking norms, the following inequalities are found:

$$\| b_1 \| = \| a_{01} \|$$

$$\| b_2 \| \leq \| a_{20} \| + 2 \| a_{11} \| \, \| b_1 \| + \| a_{02} \| \, \| b_1 \|^2$$

$$\| b_3 \| \leq \| a_{30} \| + 3 \| a_{21} \| \, \| b_1 \| + 3 \| a_{12} \| \, \| b_1 \|^2 +$$

$$\| a_{03} \| + 3 \| a_{11} \| \, \| b_2 \| + 3 \| a_{02} \| \, \| b_1 \| \, \| b_2 \|$$

$$\ldots \ldots \text{ etc} \tag{4 6}$$

On relating these inequalities to the equations (3 14) for the coefficients of the series solution of the comparison equation it is seen that since

$$\| a_{01} \| \leq A_{01} \, , \; \| a_{20} \| \leq A_{20} \, , \; \| a_{11} \| \leq A_{11} \, , \ldots \text{ etc} \tag{4 7}$$

there follows

$$\| b_1 \| \leq B_1 \, , \; \| b_2 \| \leq B_2 \, , \; \| b_3 \| \leq B_3 \, , \ldots \text{ etc} \tag{4 8}$$

Consequently, the series solution of the comparison equation gives a majorant for the series solution of the functional equation (4.1). From this follows the theorem:

Theorem For equations in normal form, the method of series substi--tution applied to an implicit functional equation

$$x = f[x, u] \tag{4 9}$$

where f[x,u] is a bounded analytic function, gives a solution which has a majorant obtained by series solution of the corresponding comparison equation

Corollary 1 For implicit functional equations in normal form, the method of series substitution gives a convergent analytic solution for sufficiently small values of the independent variable.

This is an immediate consequence of Cauchy's theorem.

Corollary 2 For implicit functional equations in normal form having majorant convergent for all values of the variables, the method of series substitution gives an analytic solution convergent for values of the independent variable with norm not exceeding the turning value given by the graph of the comparison equation.

This follows from Hille's theorem. Under the condition of this corollary it is thus asserted that the series (4.2) is convergent for

$$\|u\| \leq U^{*} \qquad (4.10)$$

Further it follows that

$$\|g[u]\| \leq G(\|u\|) \qquad (4.11)$$

Now consider equations not in normal form. In many cases, such equations may be converted to normal form as follows. Let the equation be written

$$(I - a_{10})[x] = \psi[x,u] \qquad 4.12$$

where I is the identity operation:

$$I[x] \equiv x \qquad (4.13)$$

and

$$\psi[x,u] = a_{01}[u] + \frac{1}{2!}\left(a_{20}[x,x] + 2a_{11}[x,u] + a_{02}[u,u]\right) + \dots \qquad (4.14)$$

Now suppose that

Assumption

$I - a_{10}$ has a bounded inverse.

On denoting this inverse by

$$h[x] = (I - a_{10})^{-1}[x] \tag{4.15}$$

the equation becomes

$$\begin{aligned} x &= h\psi[x,u] \\ &= h a_{01}[u] + \frac{1}{2!}\left(h a_{20}[x,x] + 2 h a_{11}[x,u] + h a_{20}[u,u]\right) + \cdot \end{aligned} \tag{4.16}$$

which is now in normal form and may be solved by the method above. The terms of the expansion up to second order are

$$\begin{aligned} x = h a_{01}[u] + \frac{1}{2!}\Big(&a_{20}[h a_{01}[u],\, h a_{01}[u]] + \\ &a_{11}[h a_{01}[u], u] + a_{11}[u, h a_{01}[u]] + a_{02}[u,u]\Big) + \end{aligned} \tag{4.17}$$

The comparison equation is

$$X = H\Psi(X,U) \tag{4.18}$$

where

$$H \geq \|h\|$$

$$\Psi(X,U) = A_{01}U + \sum_{m+n\geq 2}\sum \frac{A_{mn}X^m U^n}{m!\,n!} \tag{4.20}$$

The convergence of the series solution may be discussed by this comparison equation.

5. The contraction mapping property and solution by iteration.

Apart from the method of series substitution, the implicit functional equation can be solved by iteration within the region for which the contraction mapping property holds. It will be shown in this section that this region is the same as the region of convergence of the functional series solution obtained by series substitution under the conditions of Hille's theorem.

The contraction mapping property of a bounded analytic function is a consequence of the following lemma (Michal 1958):

Lemma Let f[x] be a bounded analytic function with majorant F(X). Then for any x,x' such that

$$\|x\|,\ \|x'\| \le X \tag{5.1}$$

it is true that

$$\| f[x'] - f[x] \| \le F'(X)\ \| x' - x \| \tag{5.2}$$

Proof: if f[x] is given by

$$f[x] = \sum_{m=0}^{\infty} \frac{1}{m!} a_m[x, x, \cdots, x] \tag{5.3}$$

then

$$f[x'] - f[x] = \sum_{m=0}^{\infty} \frac{1}{m!} \left(a_m[x', x', \ldots, x] - a_m[x, x, \ldots, x] \right) \tag{5.4}$$

and

$$a_m[x', x', \ldots, x'] - a_m[x, x, \ldots, x] =$$

$$\begin{aligned} &= (a_m[x', x', \ldots, x'] - a_m[x, x', \ldots, x']) \\ &\quad - (a_m[x, x', \ldots, x'] - a_m[x, x, x', \ldots, x']) \\ &\quad - \ldots\ldots\ldots \qquad \ldots \qquad \ldots\ldots \\ &\quad - (a_m[x, x, \ldots, x, x'] - a_m[x, x, \ldots, x]) \end{aligned} \tag{5.5}$$

$$
\begin{aligned}
= \; & a_m[x'-x, x', \ldots, x'] + \\
& a_m[x, x'-x, \ldots x'] + \\
& \ldots \quad \ldots \quad \ldots + \\
& a_m[x, x, \ldots, x'-x]
\end{aligned} \tag{5.6}
$$

from which

$$
\begin{aligned}
& \| a_m[x', x', \ldots, x'] - a_m[x, x, \ldots, x] \| \\
& \qquad \leq m A_m X^{m-1} \| x - x' \|
\end{aligned} \tag{5.7}
$$

From this now follows

$$
\begin{aligned}
\| f[x'] - f[x] \| &\leq \sum_{m=0}^{\infty} \frac{1}{m!} \| a_m[x', x', \ldots, x'] - a_m[x, x, \ldots, x] \| \\
&\leq \left(\sum_{m=0}^{\infty} \frac{m A_m X^{m-1}}{m!} \right) \| X - X' \| \\
&= F'(X) \, \| x' - x \|
\end{aligned} \tag{5.8}
$$

as required.

<u>Corollary</u> A bounded analytic function $f[x]$ of $\mathfrak{X} \to \mathfrak{X}$ gives a contraction mapping of any region $\|x\| \leq X$ such that

$$
F'(X) < 1 \tag{5.9}
$$

It will be necessary to apply the contraction mapping lemma to functions $f[x,u]$ in which u is entering as a parameter. For this, a straightforward extension of the last lemma gives:

<u>Lemma</u> Let f[x,u] be a bounded analytic function $\mathcal{X} \times \mathcal{U} \to \mathcal{X}$ with majorant F(X,U). Then for any $x, x' \in \mathcal{X}, u \in \mathcal{U}$ such that

$$\|x\| , \|x'\| \leq X , \quad \|u\| \leq U \tag{5.10}$$

it is true that

$$\| f[x',u] - f[x,u] \| \leq F'_X(X,U) \, \| x' - x \| \tag{5.11}$$

where $F'_X(X,U)$ denotes partial derivative of F(X,U) with respect to X.

Now let us interpret this result in connexion with the graph of the comparison equation given in fig.2. On the arc OP, i.e. for

$$0 \leq U \leq U^* , \quad 0 \leq X \leq X^* \tag{5.12}$$

it is easy to see that

$$F_X(X,U) \leq 1 \tag{5.13}$$

with equality only at P. In view of this, the contraction mapping property stated in the following theorem may be deduced.

<u>Theorem</u> With the notation used previously, for any u such that

$$\|u\| \leq U < U^* \tag{5.14}$$

the transformation

$$x \to f[x,u] \tag{5.15}$$

maps the spherical region

$$\| x \| \leq X = G(U) \tag{5.16}$$

into itself by contraction mapping with contraction constant

$$k \leq F_X(G(U), U) \tag{5.17}$$

Proof: that the region is mapped into itself follows from

$$\| f[x,u] \| \leq F(X,U) = X \tag{5.18}$$

That the mapping is a contraction with the contraction constant as stated follows from (5.13) combined with the last lemma.

<u>Corollary</u> If u satisfies $\|u\| \leq U^*$, the implicit functional equation

$$x = f[x,u] \tag{5.19}$$

may be solved by the iterative procedure:

$$x^{(i+1)} = f[x^{(i)},u] \quad i=0,1,2,\ldots \tag{5.20}$$

the initial value being arbitrary subject to the condition

$$\| x^{(0)} \| \leq G(\|u\|) \tag{5.21}$$

Proof: take $U = \|u\|$ in the last theorem. The spherical region

$$\|x\| \leq G(\|u\|) \tag{5.22}$$

is mapped into itself by contraction mapping and so by Banach's fixed point theorem, the iterative process converges to a unique value x satisfying the equation and (5.22)

<u>Note</u>: the use of Banach's fixed point theorem is the only use made of the completeness property of the Banach space $\mathcal{X}$. Otherwise the results of this report are valid for arbitrary normed spaces. This remark is important in applications of the theory to systems operat--ing over an infinite time interval where input and output variables belong to a normed space which does not necessarily have the complete--ness property.

6. Illustration: functional series solution of differential equations in state-variable form.

As an illustration of the previous theory it will be shown in this section how a set of analytic differential equations in state-variable form may be explicitly solved using functional series. This subject was previously reported on by the author in ref./8 /.

The input and output variables u and x will be taken to be vector functions of time t on the interval $t_0 \leq t \leq t_f$:

$$u = \{ u(t), t_0 \leq t \leq t_f \} \tag{6.1}$$

$$x = \{ x(t), t_0 \leq t \leq t_f \} \tag{6.2}$$

where u(t) and x(t) will be taken to be column vectors of dimensions r and s respectively:

$$u(t) = [u_1(t), u_2(t), \ldots, u_r(t)]^T \tag{6.3}$$

$$x(t) = [x_1(t), x_2(t), \ldots, x_s(t)]^T \tag{6.4}$$

Norms will be defined as:

$$\| u \| = \sup_{t_0 \leq t \leq t_f} \max_{j=1,\ldots,r} | u_j(t) | \tag{6.5}$$

$$\| x \| = \sup_{t_0 \leq t \leq t_f} \max_{i=1,\ldots,s} | x_i(t) | \tag{6.6}$$

u and x thus become elements of a normed space. As has been remarked, completeness of the space of functions is not required except in connexion with Banach's fixed point theorem when considering iterative solution. It is therefore not necessary, in the first place, to put any restrictive conditions on the class of functions considered.

The differential equations considered will have the following form:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), t) & t_0 \le t \le t_f \\ x(t_0) \text{ given} \end{cases} \tag{6.7}$$

where $f(x(t),u(t),t)$ is analytic in the first two variables and is given explicitly by

$$f(x(t), u(t), t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_{mn}(x(t), u(t), t) \tag{6.8}$$

$$f_{mn}(x(t), u(t), t)_i = \frac{1}{m!\,n!} \sum_{r_1=1}^{r} \cdots \sum_{r_m=1}^{r} \sum_{s_1=1}^{s} \cdots \sum_{s_m=1}^{s} \Big(a^{(m,n)}_{i;\,r_1,\ldots,r_m;\,s_1,\ldots,s_m}(t)\, x_{r_1}(t) \ldots x_{r_m}(t)\, u_{s_1}(t) \ldots u_{s_m}(t) \Big) \tag{6.9}$$

where the coefficients are assumed to be completely symmetrical in $r_1,\ldots,r_m$ and in $s_1,\ldots,s_n$.

The corresponding majorant has the form

$$F(U,X) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} F_{mn}(X,U) \tag{6.10}$$

$$F_{mn}(X,U) = \frac{A_{mn}}{m!\,m!} X^m U^n \tag{6.11}$$

where the coefficients A_{mn} satisfy

$$A_{mn} \geq \sup_{t_0 \leq t \leq t_f} \max_{i=1,\ldots,r} \sum_{r_1=1}^{r} \cdots \sum_{r_m=1}^{r} \sum_{s_1=1}^{s} \cdots \sum_{s_n=1}^{s} \Big(\Big| a^{(m,n)}_{i;r_1,\ldots,r_m;s_1,\ldots,s_n}(t) \Big| \Big) \tag{6.12}$$

It is of interest to find the functional series solution valid in the neighbourhood of a particular solution of the equation. Without loss of generality this particular solution may be taken as

$$u(t) = 0 \quad , \quad x(t) = 0 \tag{6.13}$$

and so correspondingly it is assumed that

$$a_i^{(0,0)} = 0 \qquad i = 1,\ldots,s \tag{6.14}$$

The equations considered then take the form

$$\dot{x}(t) = a_{10}(t)\, x(t) + \psi\big(x(t), u(t), t\big) \tag{6.15}$$

where $\psi(x(t),u(t),t)$ represents those terms of $f(x(t),u(t),t)$ which are either linear in $u(t)$ or nonlinear in $x(t)$ or $u(t)$. Assuming, for simplicity, that the initial conditions are zero, the differential equation may then be converted to the nonlinear integral equation

$$x(t) = \int_{t_0}^{t} h(t,t')\, \psi\big(x(t'), u(t'), t'\big)\, dt' \tag{6.16}$$

where $h(t,t')$ is the impulse response matrix of the linearized part. This has the form

$$x = h\,\psi[x,u] \tag{6.17}$$

already considered in connexion with the solution of implicit functional equations which can be transformed to normal form. The functional series solution (4.17) is immediately applicable to the present problem.

The comparison equation is

$$X = H\,\Psi(X,U) \tag{6.18}$$

where

$$H \geq \|k\|$$

$$= \sup_{t_0 \leq t \leq t_f} \; \max_{i=1,\ldots,r} \; \sum_{j=1}^{s} \int_{t_0}^{t_f} |k(t,t')|\, dt' \tag{6.19}$$

$$\Psi(X,U) = A_{01} U + \sum\sum_{m+n \geq 2} \frac{A_{mn} X^m U^n}{m!\, n!} \tag{6.20}$$

It should be remarked that $\Psi(X,U)$ is majorant both for the function $\psi(x(t),u(t),t)$ and for the nonlinear functional transform--ation $\psi[x,u]$ defined by it.

Apart from the functional series solution, a solution may also be found by the iterative process

$$x^{(i+1)}(t) = \int_{t_0}^{t} k(t,t')\, \psi\left(x^{(i)}(t'),\, u(t'),\, t'\right) dt' \tag{6.21}$$

with $x^{(0)}(t)$ determined as in the lemma on p. 20 It is simplest to put

$$x^{(0)}(t) = 0 \tag{6.22}$$

If the iterative process is used, a class of functions should be chosen to make the function space of x complete.

For both methods of solution the turning value U^* found from the graph of the relation (6.18) is critical and gives the criterion for convergence. This criterion was discussed in the context of the present problem in the author's report / 8 /. A similar criterion for analytic equations where the right hand side is linear in u(t) was earlier given by Halme in 1971 / 7 / along the lines of the early work of Brilliant /3/.

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