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ABSTRACT

The $1-D$ reflection of an arbitrary incident acoustic wave at an impedance wall is studied, resulting in the formulation of necessary conditions for physically possible impedance models. These results are applied to the problem of an impedance of Helmholtz resonator type, in which case the reflection of an incident pulse is obtained in explicit form, expressed in generalised Laguerre polynomials.

INTRODUCTION

For theoretical investigations of a harmonic (i.e., of a single frequency) sound wave reflecting at an impedance wall, no more information about the impedance is required than its complex value Z (Pierce [1]). However, for a general incident wave, with an extended frequency spectrum, one needs to know $Z = Z(\omega)$ as a function of frequency ω. Also in the context of a mean flow with inherent instabilities of the vortex sheet along the impedance wall, knowledge of the behaviour of $Z(\omega)$ in the (complex) frequency domain is essential (Rienstra [2]).

Not any model $Z(\omega)$ is physically possible, however. It may also occur that the same model allows several interpretations (in the sense that it is obtained via several limits) of which only one is the physical.

In the present paper we will discuss the general solution of an arbitrary one-dimensional sound wave reflecting at an impedance wall. Specifically, we will consider the necessary conditions for $Z(\omega)$ to obtain a real and causal sound field. These results are then applied to the problem of a pulse reflecting at an impedance wall of Helmholtz resonator type. An explicit solution is obtained, of which numerically evaluated examples are presented.

GENERAL PROBLEM

Consider the one-dimensional acoustic wave equations (made dimensionless on mean sound speed, mean density and some length) for potential ϕ, pressure p, and velocity v

$$\begin{aligned} \phi_{xx} - \phi_{tt} &= 0 \\ p &= -\phi_t \\ v &= \phi_x \end{aligned} \tag{1}$$

on $-\infty < x \leqslant 0$, $-\infty < t < \infty$, with the impedance boundary condition at $x = 0$

$$\hat{p}(0,\omega) = Z(\omega)\hat{v}(0,\omega) \tag{2}$$

where "^" denotes a Fourier transform in time:

$$\hat{p}(x,\omega) = \int_{-\infty}^{\infty} p(x,t)e^{-i\omega t}\,dt.$$

The complex impedance Z represents the reflection properties of the wall at $x = 0$. If the wall is passive, it has $\mathrm{Re}\, Z \geqslant 0$, while it absorbs acoustic energy if $\mathrm{Re}\, Z > 0$ (Pierce [1]). A hard wall corresponds to $|Z| = \infty$. Except for the case of Z = real constant, Z must vary (in some way) with ω. This will be further discussed in the next section.

GENERAL SOLUTION AND CONDITIONS FOR Z

The classical solution (Pierce [1]) of the one-dimensional wave equation is a sum of arbitrary functions of $t-x$ and $t+x$. So we have here the reflected field $g(t+x)$ of an incident field $f(t-x)$ satisfying

$$p(x,t) = f(t-x) + g(t+x)$$

$$v(x,t) = f(t-x) - g(t+x). \tag{3}$$

If we introduce (formally, for the moment)

$$z(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(\omega)e^{i\omega t}\,d\omega \tag{4}$$

we can transform the impedance boundary condition back to the time domain, leading to the convolution

$$p(0,t) = \int_{-\infty}^{\infty} z(\tau)v(0,t-\tau)\,d\tau, \tag{5}$$

and hence

$$f(t)+g(t) = \int_{-\infty}^{\infty} z(t-\tau)\{f(\tau)-g(\tau)\}\,d\tau, \tag{6}$$

which is, of course, just an integral equation in g for given f. A solution is found by Fourier transformation of (3), and then solving (2), yielding

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(\omega)e^{i\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{Z(\omega)-1}{Z(\omega)+1}\,\hat{f}(\omega)e^{i\omega t}\,d\omega. \tag{7}$$

This is rather formally, however. We will see in the next section that sometimes equation (6) is ideally suited to obtain, more directly, explicit results.

Apart from equation (6) there is another important result drawn from (5): not any Z is physically possible, and we can deduce necessary conditions for $Z(\omega)$:

- since p and v are real, so must be z, and therefore Z has to satisfy the *reality* condition

$$Z^*(\omega) = Z(-\omega) \tag{8}$$

  where Z^* denotes Z's complex conjugate;
- since $p(0,t)$ can only depend on values of $v(0,t)$ of the past, z has to satisfy the *causality* condition

$$z(t) = 0 \quad \text{for} \quad t < 0. \tag{9}$$

These conditions may, for example, be used to choose consistent parametrisations of measured impedances.We will apply these conditions here to select a correct interpretation of a non-uniform limit.

IMPEDANCE OF HELMHOLTZ RESONATOR TYPE

An impedance wall of Helmholtz resonator type with a thin facing sheet appears to be well described by

$$Z(\omega) = R + i\omega m - i\,\mathrm{cotg}(\omega L) \tag{10}$$

for positive parameters R, m and L (Panton and Miller [3]). $R + i\omega m$ is the impedance change across the porous facing sheet (in practice, determined experimentally), and L is the depth of the resonator cell. It is easily verified that Z satisfies reality condition (8). It is, however, not so trivial whether or how it satisfies the causality condition (9), since the integral for z is divergent. The integral does have a meaning, however, if we consider z as a generalised function (Jones [4]). Then a causal z is obtained if we interpret

$$-i\,\mathrm{cotg}(\sigma) = 1 + 2\lim_{\varepsilon\to 0}\sum_{n=1}^{\infty}\exp(-2in\sigma-\varepsilon n) \tag{11}$$

with the result

$$z(t) = R\delta(t) + m\delta'(t) + \delta(t) + 2\sum_{n=1}^{\infty}\delta(t_n) \tag{12}$$

where $t_n = t - 2nL$. For the general solution we obtain from (6) the equation

$$(R+2)g(t) + mg'(t) = Rf(t) + mf'(t) + 2\sum_{n=1}^{\infty}\{f(t_n)-g(t_n)\} \tag{13}$$

with solution $(m \neq 0)$

$$g(t) = f(t) - \frac{2}{m}\int_{-\infty}^{t} \exp(-\frac{R+2}{m}(t-\tau))[f(\tau)-\sum_{n=1}^{\infty}\{f(\tau_n)-g(\tau_n)\}]\,dt \tag{14.a}$$

and $(m=0)$

$$g(t) = \frac{R}{R+2}f(t) + \frac{2}{R+2}\sum_{n=1}^{\infty}\{f(t_n)-g(t_n)\}. \tag{14.b}$$

So the response $g(t)$ is expressed in $f(t)$ and in f and g of the past. If $f \equiv 0$ (and hence $g \equiv 0$) before some time, the series in (14.a) and (14.b) are finite and the solution g can be built up iteratively, in time intervals of $2L$. This can be made explicit for a pulse, and after some (tedious) algebra we obtain $(m \neq 0)$

$$f(t) = \delta(t)$$

$$g(t) = \delta(t) - \frac{2}{m}\sum_{n=0}^{[t/2L]} \exp(-\frac{R+2}{m}t_n)\, L_n^{(-1)}(2t_n/m) \tag{15}$$

with [ ] denoting the integral part, and where $L_n^{(-1)}$ is a suitably defined generalised Laguerre polynomial (Abramowitz and Stegun [5])

$$L_n^{(-1)}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}\binom{n-1}{k-1}x^k \tag{16}$$

with $\binom{n-1}{-1} = 0$ if $n \geqslant 1$, and $= 1$ if $n = 0$. Use is made of the relation

$$\frac{d}{dx}L_n^{(-1)}(x) = -\sum_{k=0}^{n-1} L_k^{(-1)}(x). \tag{17}$$

Convenient for numerical evaluation is the recurrence relation

$$(n+1)L_{n+1}^{(-1)} = (2n-x)L_n^{(-1)} - (n-1)L_{n-1}^{(-1)} \tag{18}$$

starting with $L_0^{(-1)} = 1$, $L_1^{(-1)} = -x$.

If $m = 0$ we have the expression

$$g(t) = \frac{R}{R+2}\delta(t) + \frac{4}{R(R+2)}\sum_{n=1}^{\infty}\left[\frac{R}{R+2}\right]^n \delta(t_n) \tag{19}$$

consisting of a row of δ-pulses. This can be derived directly from equation (14.b), or (for example as a check) from (15) in the limit $m \to 0$, using the relation

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} H(x)\exp(-\lambda x/\varepsilon) L_n^{(-1)}(x/\varepsilon) = \frac{1}{\lambda}\delta(x) \qquad \text{if} \quad n = 0$$

$$= -\frac{(\lambda-1)^{n-1}}{\lambda^{n+1}}\,\delta(x) \quad \text{if} \quad n \geqslant 1 \tag{20}$$

where $\lambda > 0$.

Although there is, for any $0 < R < \infty$, acoustic energy dissipated by the impedance wall (Pierce [1]), it is interesting to note that for the present impedance there is some conservation law valid, in the form of equal average values of incident and reflected waves:

$$\int_{-\infty}^{\infty} f(t)dt = \hat{f}(0) = \lim_{\omega\to 0} \frac{Z(\omega)-1}{Z(\omega)+1}\hat{f}(0) = \hat{g}(0) = \int_{-\infty}^{\infty} g(t)dt. \tag{21}$$

Of course, one might also say: the wall is hard ($|Z| = \infty$) if $\omega = 0$, and therefore reflects the zero-frequency component completely.

EXAMPLES

Some numerically evaluated examples of equation (15) are given in figure 1-8 and of equation (19) (the amplitudes) in figure 9-11. In all cases the time scale is $t/2L$, and the varied parameters are R and $m/2L$. (This is possible because there is in the problem no length scale other than L).

We see that R and $m/2L$ are in some sense counter effective: an increasing R as well as a decreasing $m/2L$ enhance the character of a train of isolated pulses (fig. 1,2,5,7), while a decreasing R or increasing $m/2L$ smooths the reflected signal (fig. 6). When R and $m/2L$ are small enough, the response takes the form of a train of a finite number of damped oscillations (fig. 1,2,3,4,8).

For $m = 0$ (equation (19)) it is not difficult to prove that always the first two pulses are higher in amplitude than the others, but which of the two is highest, depends on R. If $R > \sqrt{5}-1 = 1.236$ (fig. 9) the first pulse is the highest; if $R = \sqrt{5}-1$ (fig. 10) the first and second are equal, and if $0 < R < \sqrt{5}-1$ the second pulse is the highest. If $R \to \infty$, only the first pulse is present (hard wall reflection), and with $R = 0$ there is only the second pulse (reflection at the hard walled bottom of the resonator cell).

CONCLUSIONS

By studying the reflection of a one-dimensional acoustic wave at an impedance wall, we have derived necessary conditions for the impedance as a function of frequency. These conditions, telling us which impedance models are a priori physically impossible, have been applied to the problem of an impedance of Helmholtz resonator type. In this case the used impedance model has to be interpreted as a certain limit, selected by the above conditions. Then, the explicit solution could be derived for the reflection of a pulse, in terms of a finite series of generalised Laguerre polynomials. This solution was illustrated by some numerical examples.

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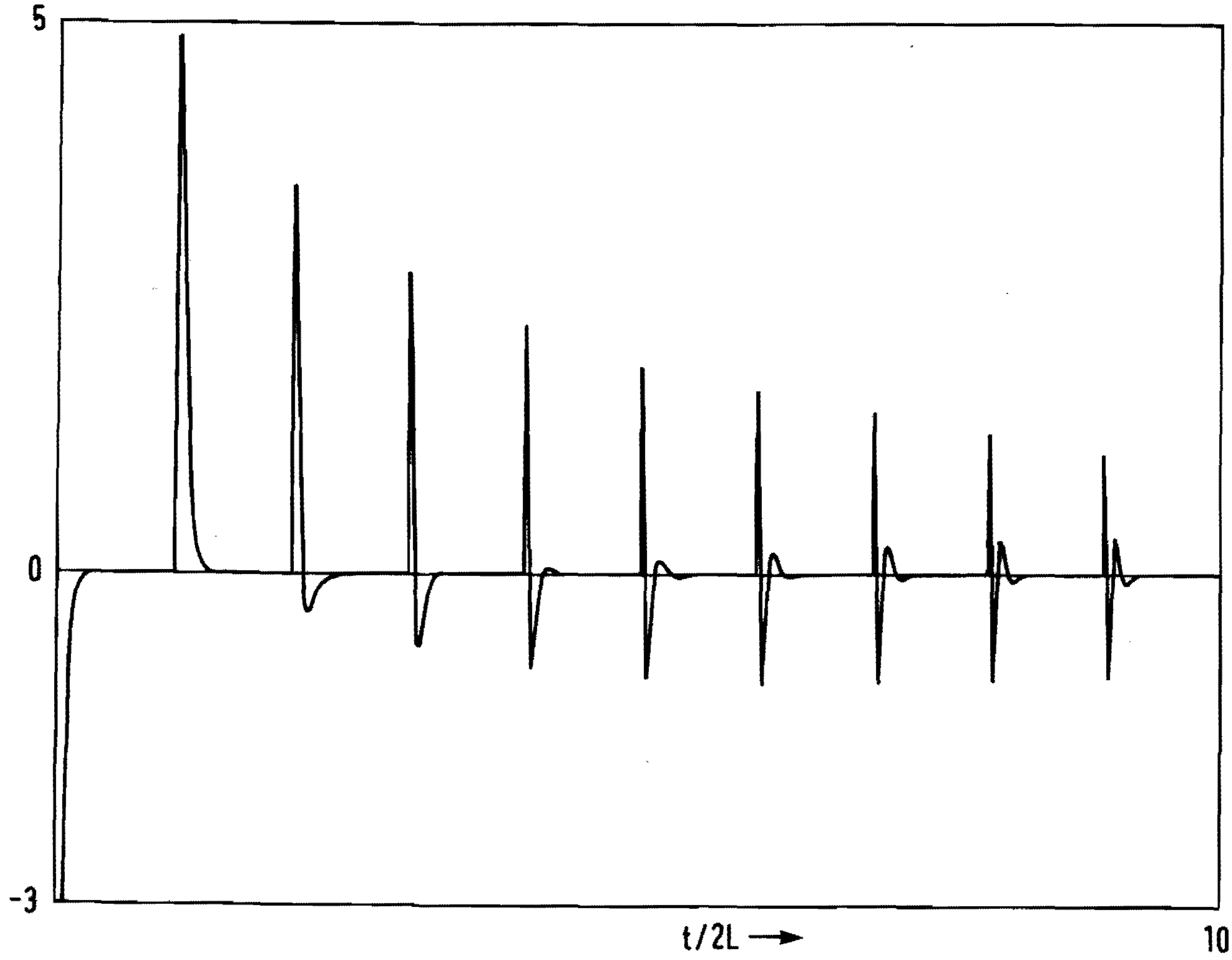

Figure 1. g(t) with R=1, m/2L=0.1

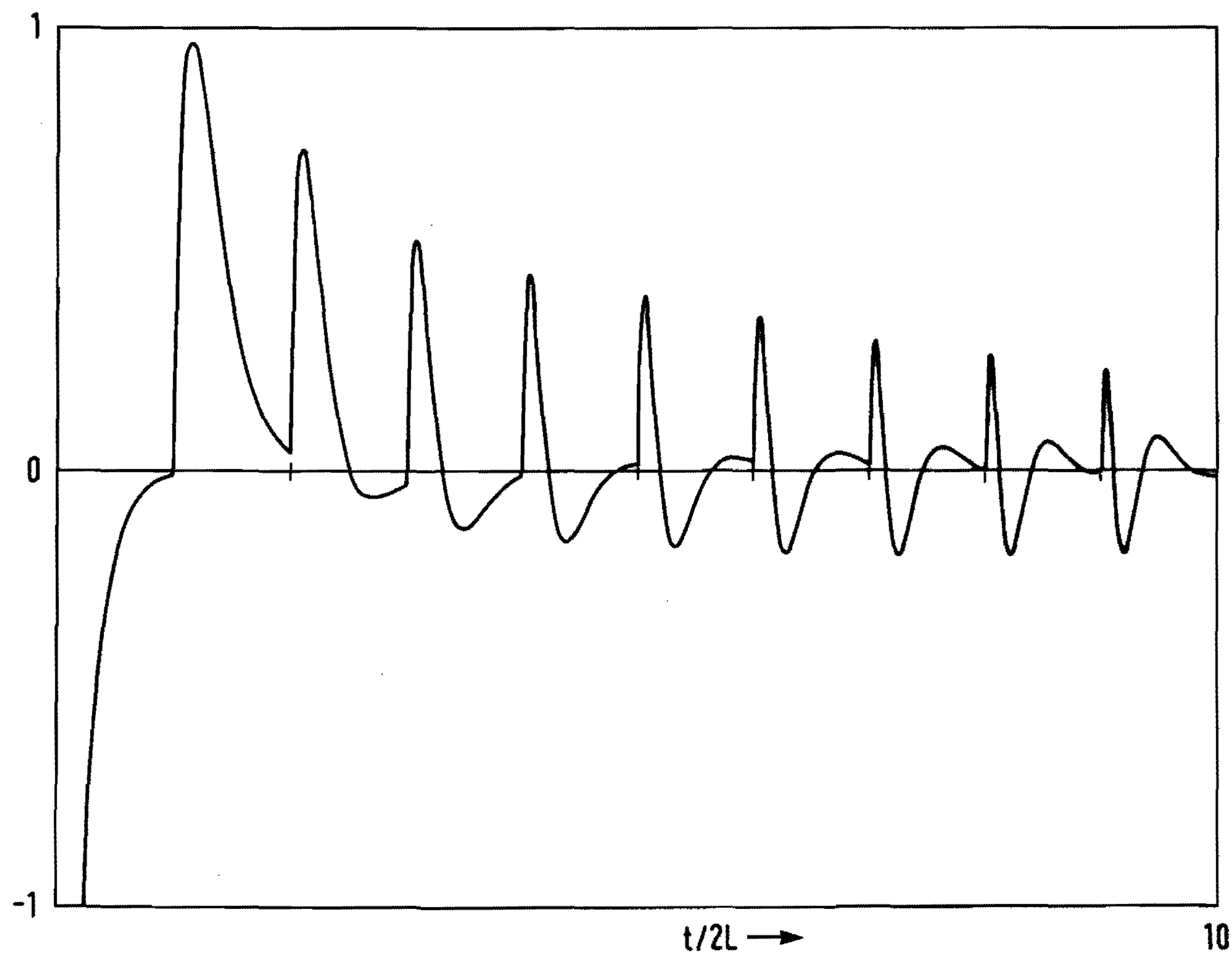

Figure 2. g(t) with R=1, m/2L=0.5

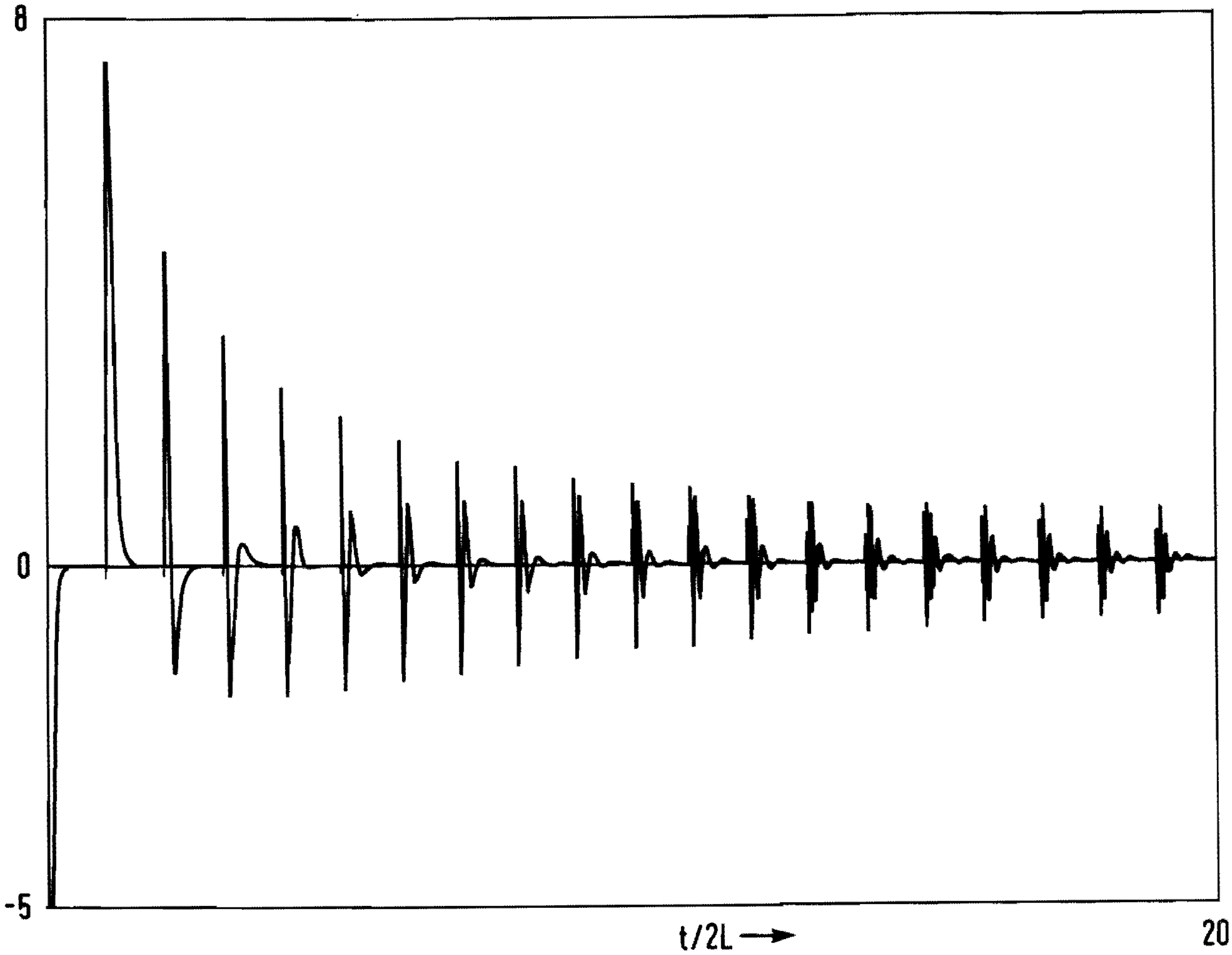

Figure 3. g(t) with R=0, m/2L=0.1

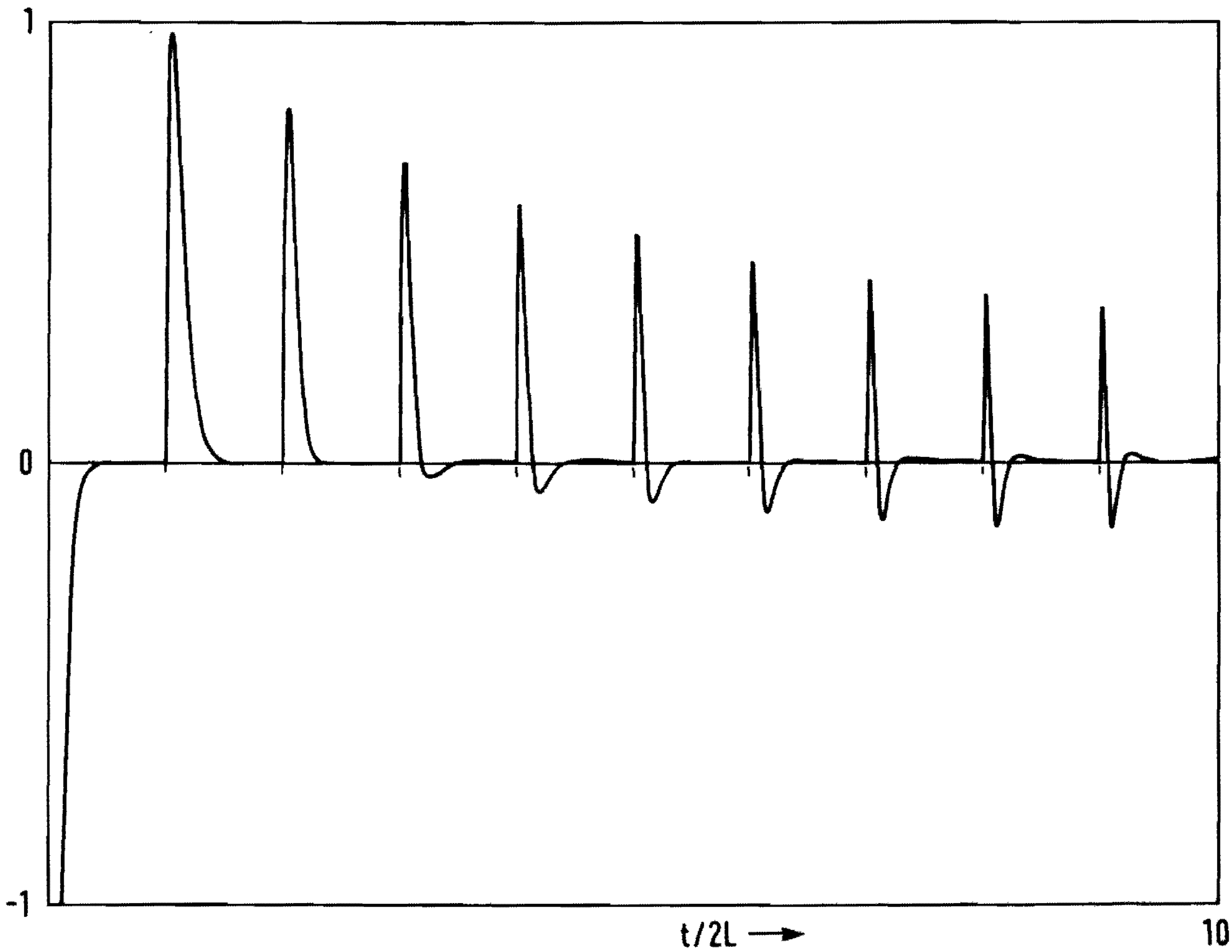

Figure 4. g(t) with R=3, m/2L=0.3

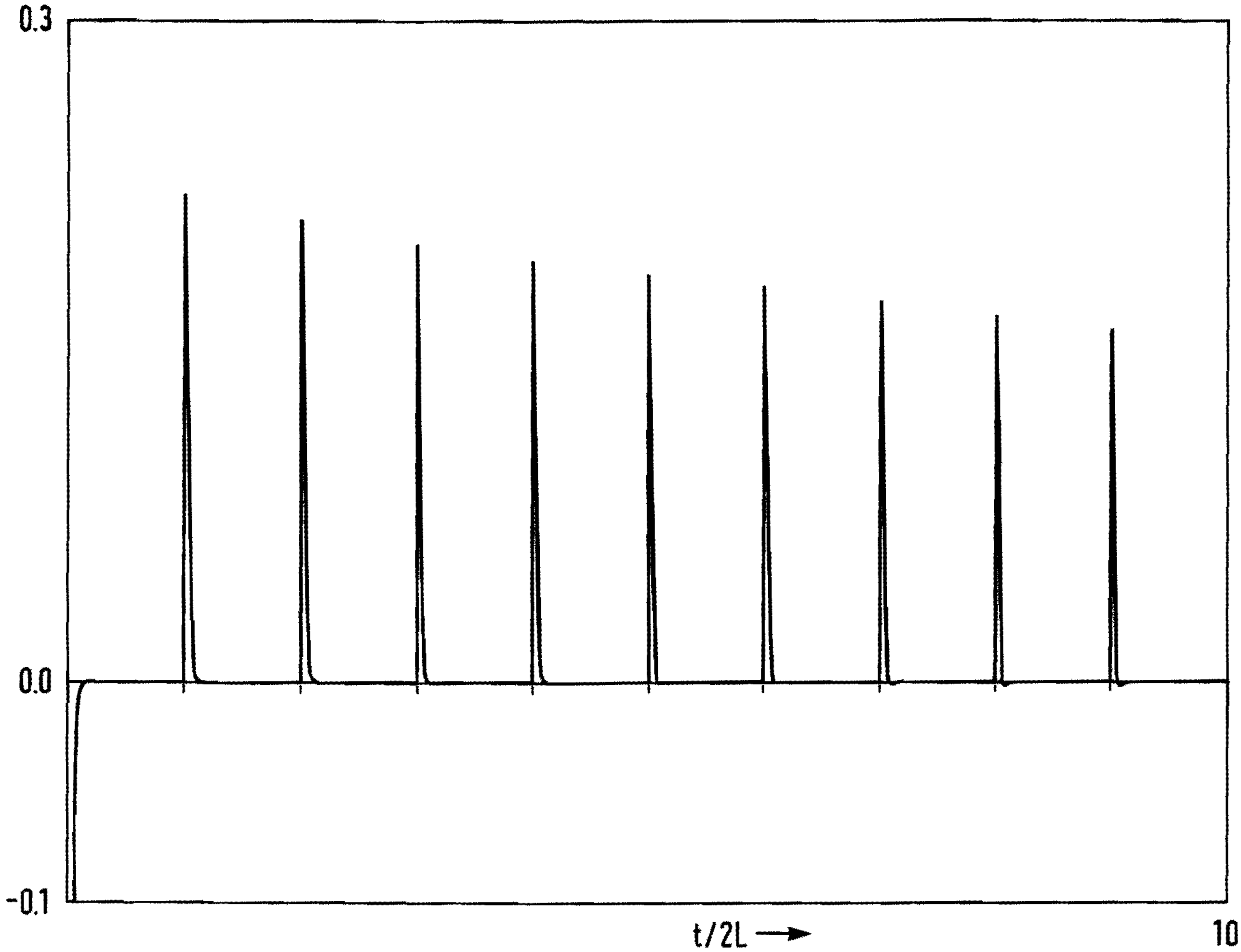

Figure 5. g(t) with R=20, m/2L=0.3

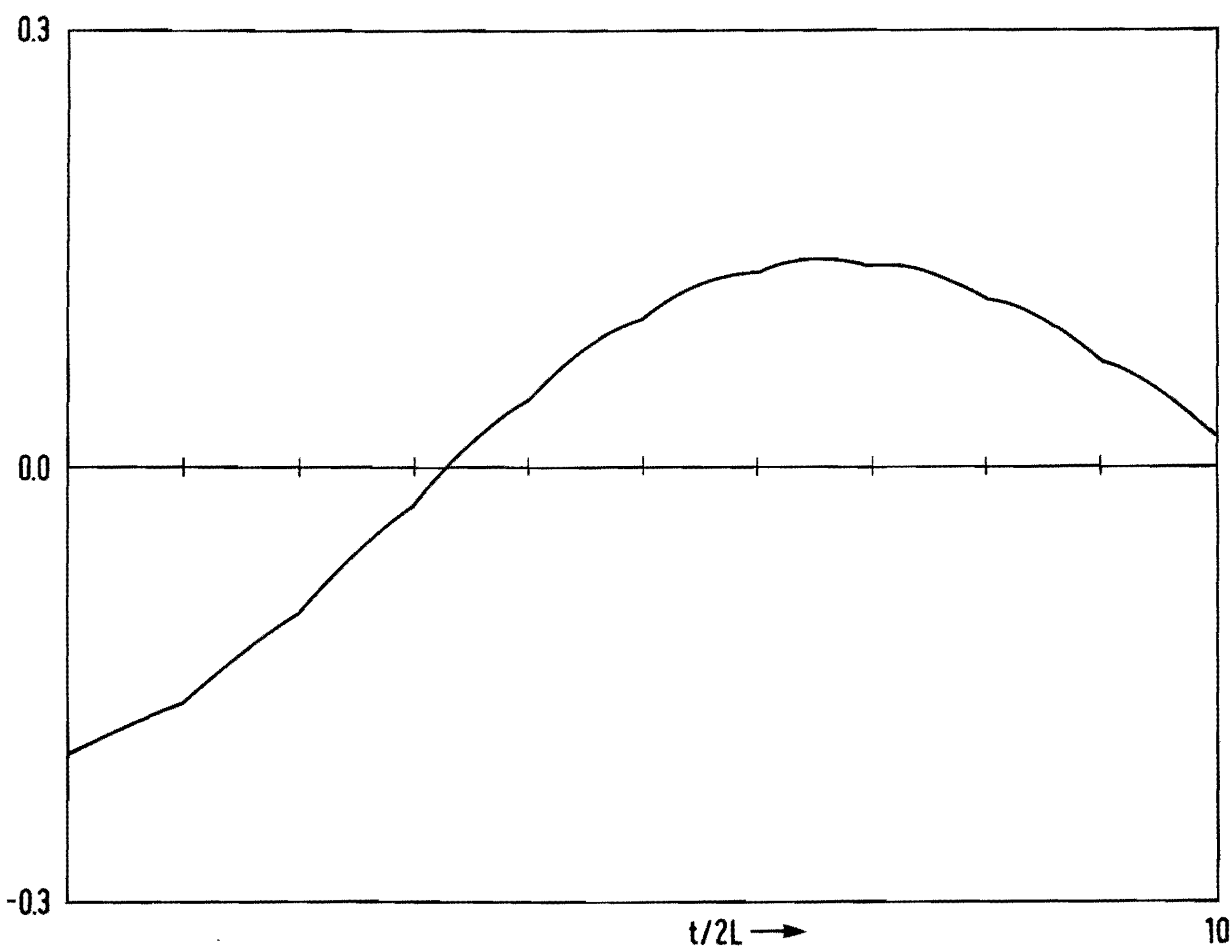

Figure 6. g(t) with R=0, m/2L=10

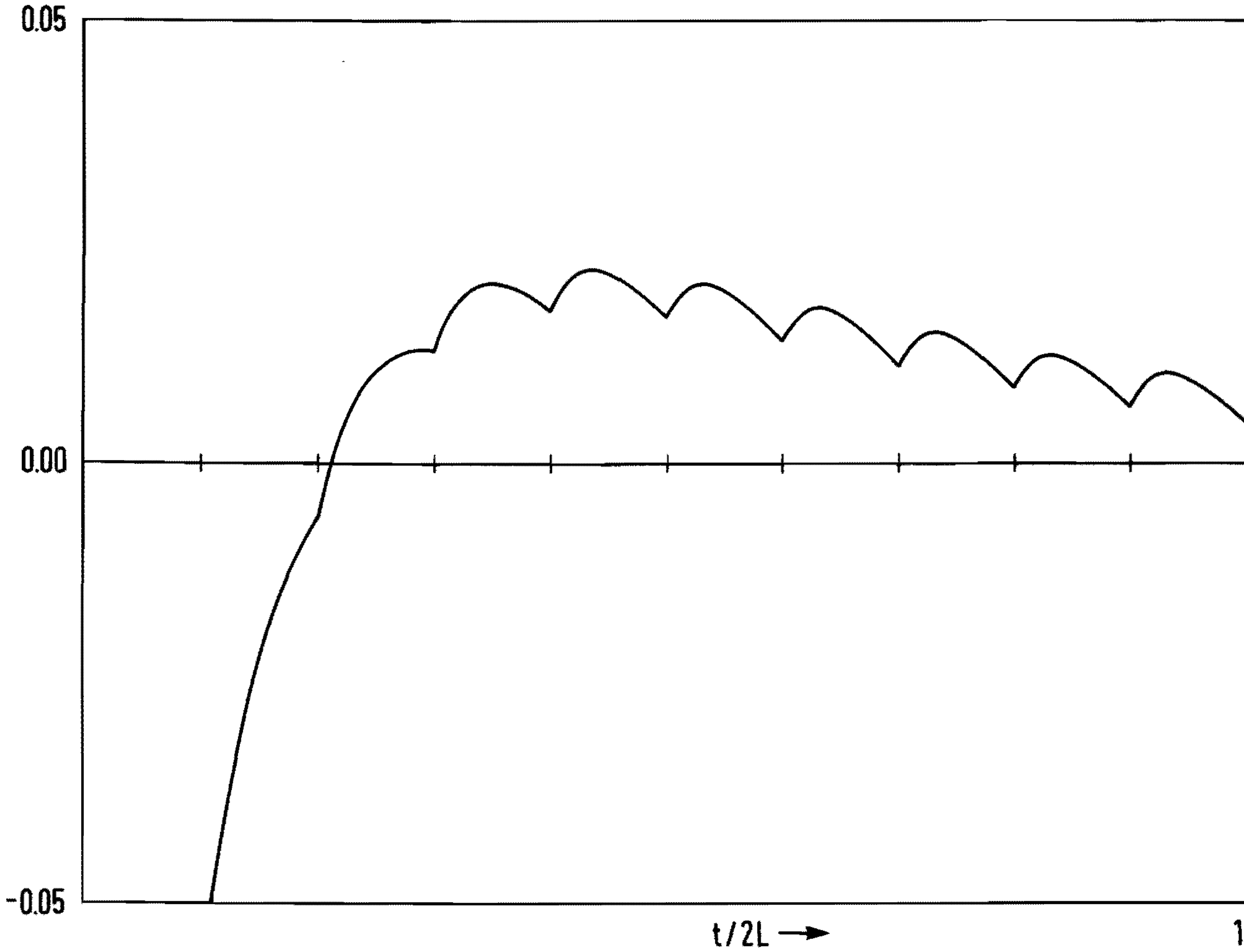

Figure 7. g(t) with R=10, m/2L=10

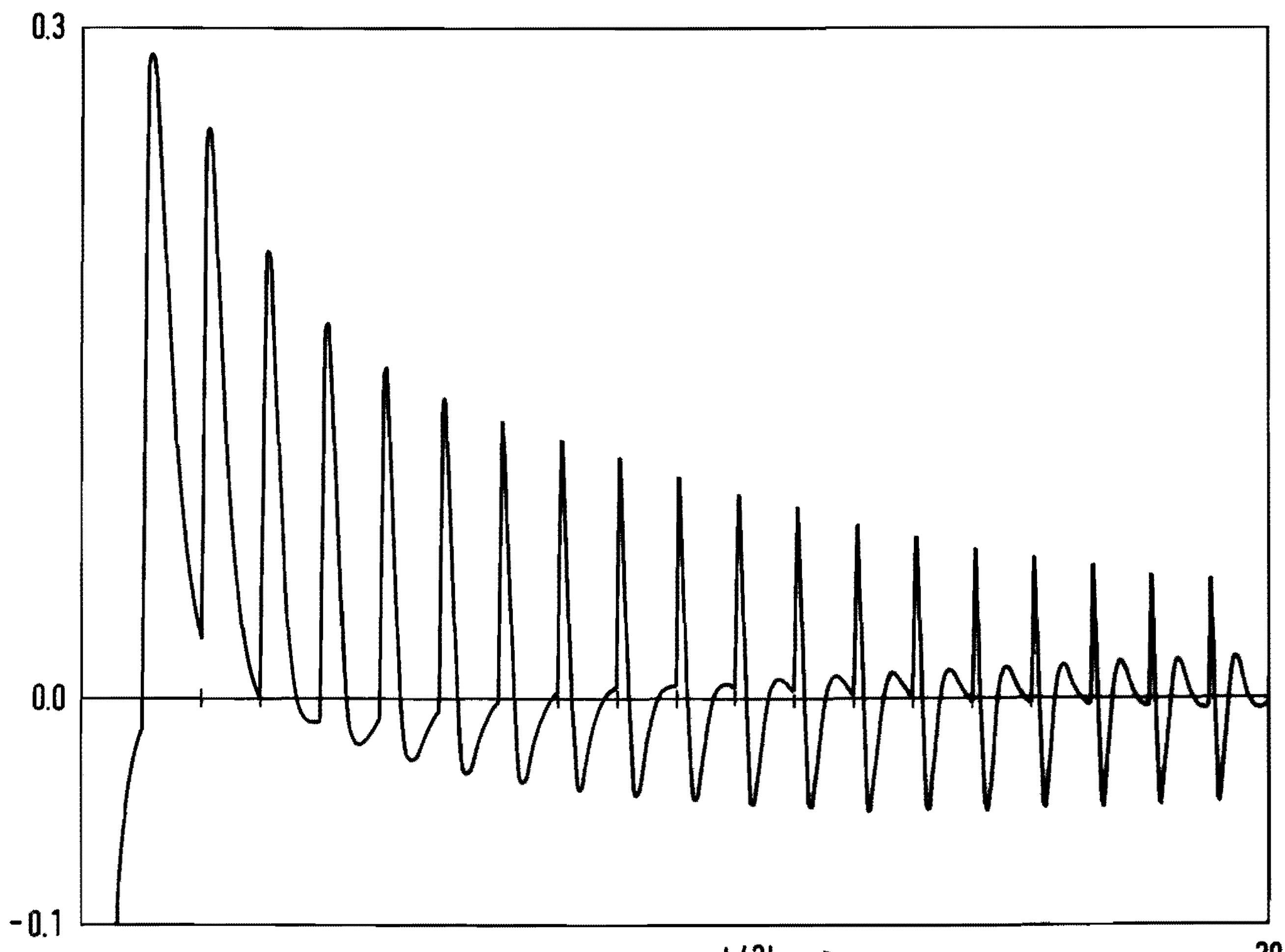

Figure 8. g(t) with R=3, m/2L=1

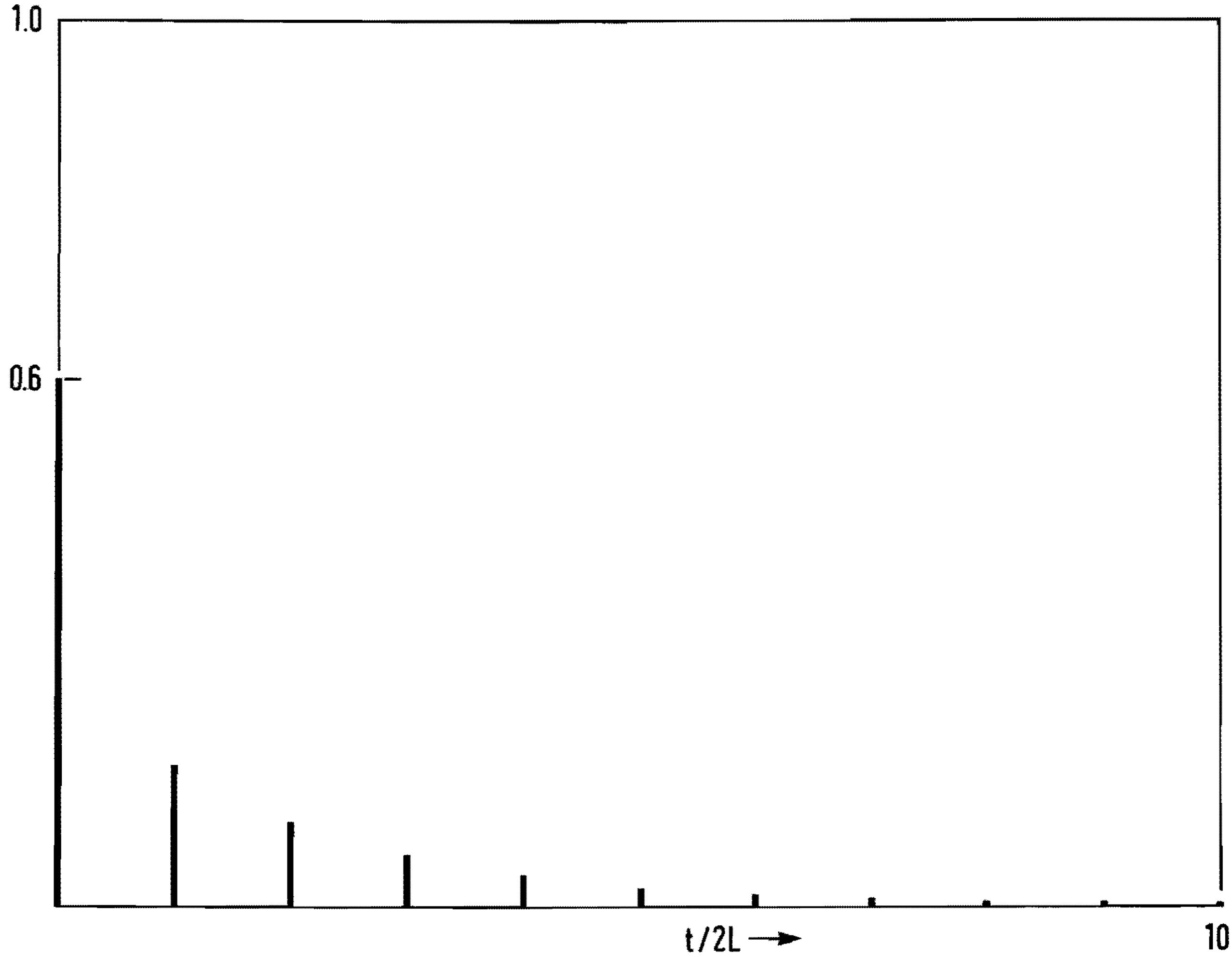

Figure 9. Amplitudes of g(t), with R=3, m/2L=0

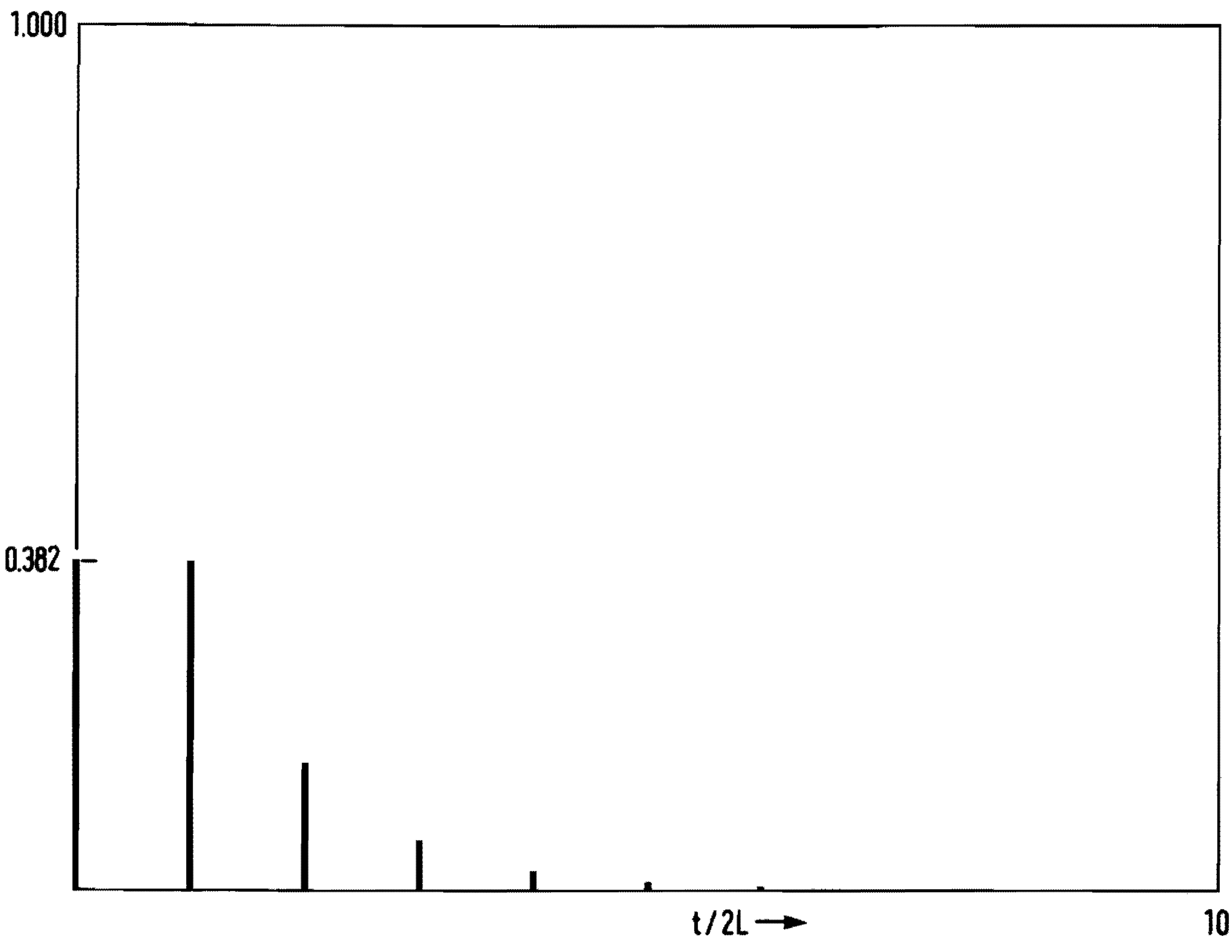

Figure 10. Amplitudes of g(t), with R = 1.236, m/2L=0

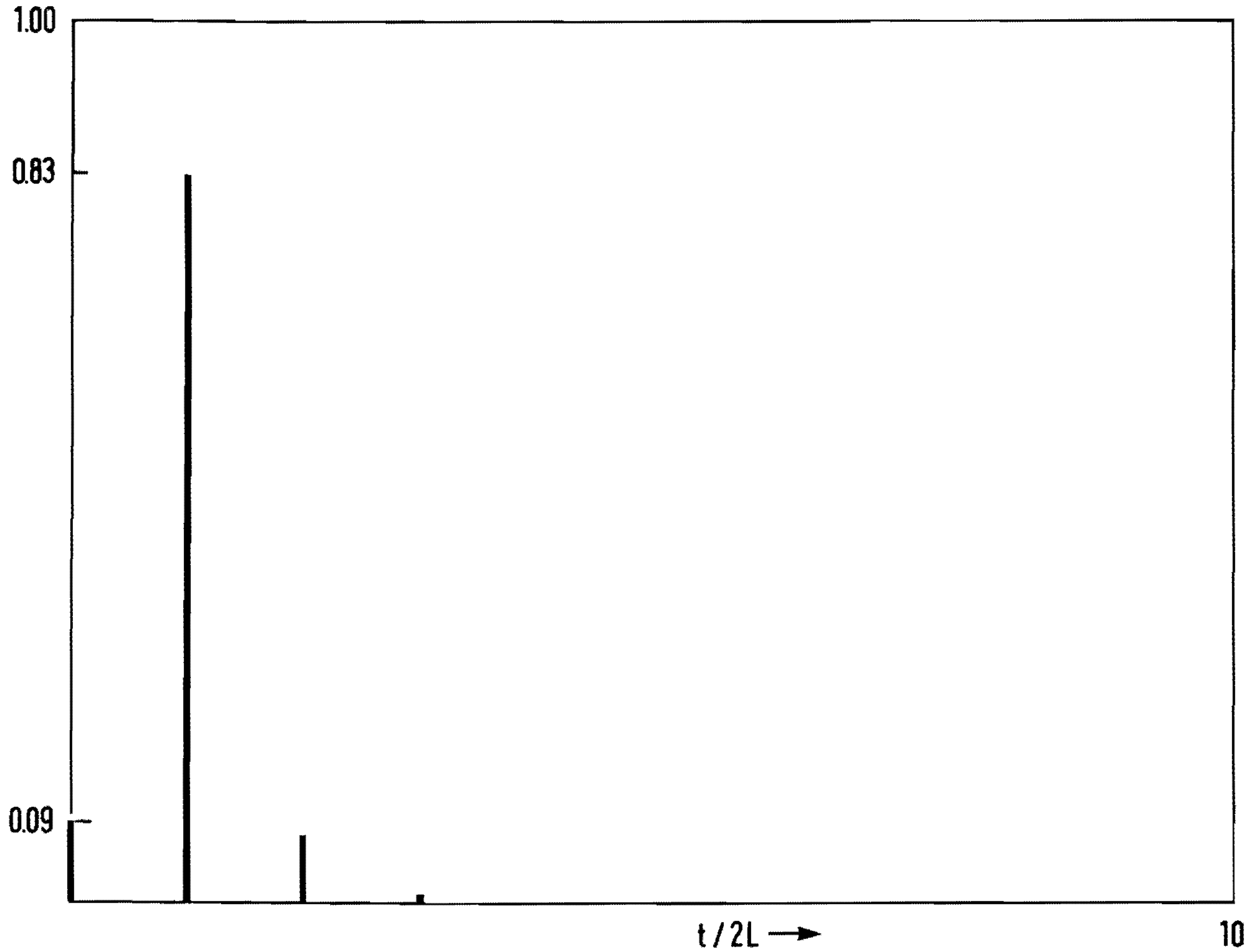

Figure 11. Amplitudes of g(t), with R = 0.2, m / 2L = 0