On the zeros of a polynomial and of its derivative

Mathematics. — *On the zeros of a polynomial and of its derivative.* By N. G. DE BRUIJN. (Natuurkundig Laboratorium der N.V. Philips' Gloeilampenfabrieken, Eindhoven-Nederland.) (Communicated by Prof. W. VAN DER WOUDE.)

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1. Considering polynomials $f(z)$ with real coefficients, we observe that, very roughly speaking, there exists a tendency for the zeros of its derivative $f'(z)$ to lie closer to the real axis, than those of $f(z)$. This is illustrated by the following well-known facts:

A. The number of imaginary [1]) zeros of $f'(z)$ does not exceed that of $f(z)$. This is a consequence of ROLLE's theorem.

B. If the zeros of $f(z)$ lie in a strip $|\operatorname{Im} z| \leq a$, the same can be said concerning the zeros of $f'(z)$. This is a special case of a famous theorem of GAUSS, expressing that the zeros of $f'(z)$ all lie in any convex domain, containing all the zeros of $f(z)$.

C. JENSEN's *Circles Theorem.* [2]) If $a_\nu \pm i\, b_\nu$ denote the imaginary zeros of $f(z)$, then any imaginary zero of $f'(z)$ lies inside at least one of the circles $|z - a_\nu| \leq |b_\nu|$.

In this paper we prove another theorem, illustrating the same tendency.

Theorem 1. *Let the real* [3]) *polynomial $f(z)$ of degree n ($n > 1$) have the zeros* [4]) $\alpha_1, \ldots, \alpha_n$, *and let* $\beta_1, \ldots, \beta_{n-1}$ *be those of $f'(z)$. Then we have*

$$\frac{1}{n-1} \sum_{\nu=1}^{n-1} |\operatorname{Im} \beta_\nu| \leq \frac{1}{n} \sum_{\nu=1}^{n} |\operatorname{Im} \alpha_\nu| . \quad \quad (1)$$

There is equality if, and only if, all the zeros of $f(z)$ are real.

In section **2**, a generalization of Theorem 1 is stated and proved. Section **3** contains some applications of Theorems 1 and 2.

We do not know, whether Theorem 1 remains valid if the condition about the reality of the coefficients of $f(z)$ is omitted. The cases $n = 2, 3$ are easy to deal with, but the general case seems to be difficult. We are, however, able to prove it if all the zeros of $f(z)$ are purely imaginary, but not necessarily conjugated by pairs. This will be carried out in section **4**, which is independent of sections **1**, **2**, **3**. As a corrolary, we will obtain

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[1]) We call a complex number α ($\alpha = \operatorname{Re} \alpha + i \operatorname{Im} \alpha$) imaginary, if $\operatorname{Im} \alpha \neq 0$; and purely imaginary, if $\operatorname{Re} \alpha = 0$.

[2]) Acta Math., **36**, 181—195 (1913).

[3]) Throughout the paper, a polynomial or rational function of z is called a real function, if it is real for real z.

[4]) A double zero is counted twice, etc.

a simple and more general inequality, containing an arbitrary convex function (Theorem 7).

2. The results concerning the zeros of $f'(z)$ and $f(z)$ can also be expressed in terms of zeros and poles of the function $f'(z)/f(z)$. We extend our considerations to rational functions of the more general type

$$\left.\begin{array}{l} \varphi(z) = az + b + \sum_{j=1}^{k} \frac{s_j}{z-a_j} + \sum_{j=1}^{l} \left(\frac{t_j}{z-\varrho_j} + \frac{t_j}{z-\bar{\varrho}_j} \right), \\ a \leqslant 0,\ b,\ a_1, \dots, a_k \text{ real};\ s_1 \geqslant 0, \dots, s_k \geqslant 0;\ t_1 \geqslant 0, \dots, t_l \geqslant 0,\ \varrho_1, \dots, \varrho_l \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{imaginary} \end{array}\right\} . \quad (2)$$

Henceforth, we consider the point $z = \infty$ as a possible pole or zero of rational functions. For instance, if $a \neq 0$, $z = \infty$ represents a simple pole of (2); we call $-a$ its residue [5]). If $a = b = 0$, $z = \infty$ is a zero of $\varphi(z)$. Moreover, $z = \infty$ is considered as a point on the real axis; thus we always take Im z to be zero, if z represents the point at infinity.

We can now describe a function of the type (2) as a real rational function, all of whose poles are simple, with positive residues. In analogy to the behaviour of $f'(z)/f(z)$, referred to above, we observe the tendency for the zeros of $\varphi(z)$ to lie closer to the real axis than the poles of $\varphi(z)$. Properties analogous to A, B, C hold true, and also their proofs are analogous [6]).

Our generalization of theorem 1 reads:

Theorem 2. *Let $\varphi(z)$ be of the type (2), and let $\alpha_1, \dots, \alpha_n$ denote the poles of $\varphi(z)$, and $\beta_1, \dots, \beta_n$ its zeros. Then we have*

$$\sum_{\nu=1}^{n} |\operatorname{Im} \beta_\nu| \leqslant \sum_{\nu=1}^{n} |\operatorname{Im} \alpha_\nu|.$$

If, moreover, $a = b = 0$, we have

$$\sum_{\nu=1}^{n} |\operatorname{Im} \beta_\nu| \leqslant \sum_{\nu=1}^{n} |\operatorname{Im} \alpha_\nu| - 2 \left(\sum_1^k s_j + 2 \sum_1^l t_j\right)^{-1} \sum_1^l t_j \, |\operatorname{Im} \varrho_j|.$$

In both cases, there is equality if, and only if, the poles of $\varphi(z)$ all lie on the real axis.

Theorem 1 follows immediately from the second part of Theorem 2. For, on taking $\varphi(z) = f'(z)/f(z)$, we obtain $a = b = 0$, $t_j = 1$, $\Sigma s_j + 2 \Sigma t_j = n$.

Our proof of Theorem 2 is relatively simple when $\varphi(z)$ has neither real poles, nor real zeros. In the general case, however, we need an auxiliary function $\psi(z)$, to be constructed in Lemma 2.

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[5]) With this (rather unusual) convention, the sign of the residue of a pole on the real axis remains invariant with respect to transformations $z = (az' + b)/(cz' + d)$ $(ad - bc > 0;\ a,\ b,\ c,\ d$ real).

[6]) The analogue of JENSEN's Circles Theorem was proved by J. V. SZ. NAGY, Jhrsber. D. Math. Ver. **31**, 238—251 (1922).

Lemma 1. *If*

$$f(z) = \prod_{\nu=1}^{n} (z-\alpha_\nu), \quad g(z) = \prod_{\nu=1}^{n} (z-\beta_\nu),$$

then

$$\text{V. P.} \int_{-\infty}^{\infty} \log \left| \frac{g(z)}{f(z)} \right| dz = \pi \left\{ \sum_{1}^{n} |\operatorname{Im} \beta_\nu| - \sum_{1}^{n} |\operatorname{Im} \alpha_\nu| \right\},$$

where V.P. *means 'valeur principale': V.P.* $\int_{-\infty}^{\infty} = \lim_{A \to +\infty} \int_{-A}^{A}$.

Proof. It is easily verified, that, for α arbitrary and $A \to +\infty$

$$\int_{-A}^{A} \log |z-\alpha| \, dz = 2 (A \log A - A) + \pi |\operatorname{Im} \alpha| + O(A^{-1}).$$

Now our lemma is evident.

Lemma 2. *Let $\varphi(z)$ be a real rational function (not identically zero) of the type (2). Then we can construct a real rational function $\psi(z)$, all of whose poles are simple and real, with negative residues, such that*

	a) $0 \leq \varphi(z)\, \psi(z) < \infty$	*for real z,*
and	*b)* $\varphi(z)\, \psi(z) = 1$	*for* $z = \infty$.

Proof. A point α on the real axis is called a *positive change of sign* for $\varphi(z)$, if the behaviour of the sign of $\varphi(z)$ in the neighbourhood of α is the same, as it is for a real function, which has a simple pole at $z = \alpha$ with positive residue. So if α is finite, this means $(z-\alpha)\,\varphi(z) > 0$ for $0 < |z-\alpha| < \varepsilon$; if $\alpha = \infty$ it means $z\,\varphi(z) < 0$ for $0 < |z^{-1}| < \varepsilon$. Negative changes of sign are defined accordingly. Any change of sign represents a pole or zero of odd order of $\varphi(z)$.

Now construct a real rational function $\psi(z)$, which has its (simple) zeros in the positive changes of sign for $\varphi(z)$, and its (simple) poles in the negative ones. Moreover, we take care, that $\varphi(z)\ \psi(z) \geq 0$ for all real values of z. It follows, that the poles of $\psi(z)$ are negative changes of sign for $\psi(z)$; hence all the residues of $\psi(z)$ are negative.

Any pole or zero of $\psi(z)$ is a pole or zero of $\varphi(z)$. The residues of $\varphi(z)$ being positive, any real pole of $\varphi(z)$ is a positive change of sign for $\varphi(z)$, and hence it is a zero of $\psi(z)$. So if α is a pole of one of the functions $\varphi(z)$ or $\psi(z)$, it is a zero of the other one. $\varphi(z)$ and $\psi(z)$ having simple zeros only, we now conclude, that the product $\varphi(z)\ \psi(z)$ *has no poles on the real axis.*

We shall prove, that $\varphi(z)\ \psi(z)$ has no zero at $z = \infty$. If $z = \infty$ is a pole of $\varphi(z)$, it will be a zero of $\psi(z)$. Both pole and zero are simple, and hence $0 \neq \varphi(\infty)\ \psi(\infty) \neq \infty$. If, in the second place, $0 \neq \varphi(\infty) \neq \infty$, we have also $0 \neq \psi(\infty) \neq \infty$, whence it follows $0 \neq \varphi(\infty)\ \psi(\infty) \neq \infty$. If, lastly, $\varphi(\infty) = 0$, we represent $\varphi(z)$ in the form (2), with $a = b = 0$.

From $s_j \geq 0$, $t_j \geq 0$, we infer, that $z = \infty$ is a simple zero, and moreover, that it is a negative change of sign for φ. Hence $z = \infty$ is a pole of $\psi(z)$, and again $0 \neq \varphi(\infty)\,\psi(\infty) \neq \infty$.

Now having obtained $\varphi(\infty)\,\psi(\infty) = c > 0$, we take $\psi_1(z) = c^{-1}\,\psi(z)$, which is easily seen to satisfy the conditions of our lemma.

Proof of Theorem 2. Let $\varphi(z)$ be given by (2), and let $\psi(z)$ satisfy the conditions of Lemma 2. Then we have, by Lemma 1:

$$\text{V. P.} \int_{-\infty}^{\infty} \log |\varphi(z)\,\psi(z)|\,dz = \pi \left\{ \sum_1^n |\operatorname{Im} \beta_\nu| - \sum_1^n |\operatorname{Im} \alpha_\nu| \right\}, \quad (3)$$

where $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are the poles and zeros of $\varphi(z)$, respectively. For, the zeros and poles of $\psi(z)$ are real, and do not contribute to the right-hand side of (3).

It follows from the inequality

$$\log u \leqslant u - 1 \qquad (u > 0), \quad (4)$$

that, for z real,

$$\log |\varphi(z)\,\psi(z)| = \log \varphi(z)\,\psi(z) \leqslant \varphi(z)\,\psi(z) - 1. \quad (5)$$

In the upper half plane, the function $\varphi(z)\,\psi(z) - 1$ has the poles ϱ_j (we may, of course, suppose $\operatorname{Im} \varrho_j > 0 > \operatorname{Im} \bar{\varrho}_j$), with residues $t_j\,\psi(\varrho_j)$, At $z = \infty$ this function behaves like $c_1 z^{-1} + c_2 z^{-2} + \ldots$, with c_1 real. Now contour integration shows that

$$\text{V. P.} \int_{-\infty}^{\infty} \{\varphi(z)\,\psi(z) - 1\}\,dz = -2\pi \sum_{j=1}^{l} \operatorname{Im} \{t_j\,\psi(\varrho_j)\}. \quad (6)$$

By Lemma 2, $\psi(z)$ is a function of the type

$$\psi(z) = A\,z + B - \frac{R_1}{z - d_1} - \ldots - \frac{R_p}{z - d_p} \quad (7)$$

where $B, d_1, \ldots, d_p$ are real, and $A \geq 0$, $R_1 \geq 0, \ldots, R_p \geq 0$. This implies $\operatorname{Im} \psi(\varrho_j) \geq A \operatorname{Im} \varrho_j$, hence

$$-\operatorname{Im} \{t_j\,\psi(\varrho_j)\} \leqslant 0, \quad (8)$$

and even

$$-\operatorname{Im} \{t_j\,\psi(\varrho_j)\} \leqslant -A\,t_j \operatorname{Im} \varrho_j \quad (9)$$

The first part of Theorem 2 follows from (3), (5), (6) and (8). Now take $a = b = 0$. In this case, we have

$$\lim_{z \to \infty} z\,\varphi(z) = s_1 + \ldots + s_k + 2\,(t_1 + \ldots + t_l).$$

According to $\varphi(\infty)\,\psi(\infty) = 1$, we infer from (7), that

$$A = [s_1 + \ldots + s_k + 2\,(t_1 + \ldots + t_l)]^{-1}.$$

Now using (9) instead of (8), the second part of Theorem 2 is readily proved.

In (4), the sign $=$ only holds for $u = 1$. Hence, in Theorem 2, equality occurs only, if $\varphi(z)\,\psi(z) = 1$ identically in z. This means, that $\varphi(z)$ has no imaginary poles.

3. An interesting specialization of Theorem 1 is obtained by taking all the zeros of $f(z)$ to lie on the imaginary axis. This leads to:

Theorem 3. *Let $F(y)$ be a real polynomial, whose zeros $\gamma_1, \ldots, \gamma_n$ $(n > 0)$ are all ≥ 0, but not all $= 0$, and let $\delta_1, \ldots, \delta_{n-1}$ be the zeros of $F'(y)$. Then we have the inequality*

$$\sum_1^{n-1} \delta_\nu^{\frac{1}{2}} < \frac{n - \frac{1}{2}}{n} \sum_1^n \gamma_\nu^{\frac{1}{2}}. \qquad (10)$$

Proof. The real polynomial $f(z) = F(-z^2)$ has the 2n zeros $\pm i\gamma_1^{\frac{1}{2}}, \ldots, \pm i\gamma_n^{\frac{1}{2}}$. Its derivative $f'(z) = -2zF'(-z^2)$ having the zeros $0, \pm i\delta_1^{\frac{1}{2}}, \ldots, \pm i\delta_{n-1}^{\frac{1}{2}}$. Theorem 3 follows by applying Theorem 1 to $f(z)$.

An inequality in the opposite direction was given by KAKEYA [7]) for general exponents. He proved, under the assumptions of Theorem 3, putting

$$D_p = \frac{1}{n}\sum_1^n \gamma_\nu^p - \frac{1}{n-1}\sum_1^{n-1} \delta_\nu^p,$$

that $D_p \geq 0$ for $p \geq 1$ or $p \leq 0$, and $D_p \leq 0$ for $0 \leq p \leq 1$, with equality only in three cases: $p = 0$, $p = 1$, and $\gamma_1 = \ldots = \gamma_n$ (p arbitrary). For $p = \frac{1}{2}$, KAKEYA's result reads

$$\sum_1^{n-1} \delta_\nu^{\frac{1}{2}} \geqslant \frac{n-1}{n} \sum_1^n \gamma_\nu^{\frac{1}{2}}.$$

In section 4, KAKEYA's inequality for D_p will appear to be a special case of Theorem 7.

We give two more applications of Theorem 2.

Theorem 4. *Let the real polynomial $f(z) = a_0 + a_1 z + \ldots + a_n z^n$ have the zeros $\alpha_1, \ldots, \alpha_n$, and let $P(y)$ be a polynomial, all of whose zeros are ≤ 0. Furthermore, let $\beta_1, \ldots, \beta_n$ be the zeros of*

$$g(z) = a_0\,P(0) + a_1\,P(1)\,z + \ldots + a_n\,P(n)\,z^n.$$

We then have

$$\sum_{\nu=1}^n |\operatorname{Im}\beta_\nu| \leqslant \frac{P(n-1)}{P(n)} \sum_{\nu=1}^n |\operatorname{Im}\alpha_\nu|.$$

Theorem 1 follows from this one by taking $P(y) = y$.

Proof. It is sufficient to prove the case $P(y) = y + a$ $(a \geq 0)$, the

7) Proc. Phys. Math. Soc. Japan (3) **15**, 149—154 (1933). The case, that p is a natural number, was considered before by H. E. BRAY, Am. J. Math. **53**, 864—872 (1931). KAKEYA uses BRAY's result.

general case being obtained by iteration. Now $g(z) = zf'(z) + af(z)$; hence $\beta_1, \ldots, \beta_n$ and ∞ are the zeros of the function

$$\varphi(z) = \frac{f'(z)}{f(z)} + \frac{a}{z},$$

whose poles are $\alpha_1, \ldots, \alpha_n, 0$. Applying the second part of Theorem 2, we obtain $\Sigma s_j + 2\Sigma t_j = n + a$, $t_2 = \ldots = t_k = 1$, and, consequently

$$\sum_1^n |\operatorname{Im} \beta_\nu| \leqslant \left(1 - \frac{1}{n+a}\right) \sum_1^n |\operatorname{Im} \alpha_\nu| = \frac{P(n-1)}{P(n)} \sum_1^n |\operatorname{Im} \alpha_\nu|.$$

The following theorem is obtained by a similar iteration process.

Theorem 5. *Let $\alpha_1, \ldots, \alpha_n$ be the zeros of the real polynomial $f(z)$, and suppose that $g(y) = b_0 + b_1 y + \ldots + b_m y^m$ has real zeros only. If, furthermore, $\beta_1, \ldots, \beta_n$ are the zeros of $b_0 f(z) + b_1 f'(z) + \ldots + b_m f^{(m)}(z)$, then we have*

$$\sum_1^n |\operatorname{Im} \beta_\nu| \leqslant \sum_1^n |\operatorname{Im} \alpha_\nu|. \quad \ldots \ldots \ldots \quad (11)$$

Proof. It is sufficient to consider the linear function $g(y) = b_0 + y$. Taking $\varphi(z) = b_0 + f'(z)/f(z)$ and applying the first part of Theorem 2, we immediately obtain (11).

4. We do not know whether the inequality (1) holds true for polynomials with complex coefficients. We can however prove it, if all the zeros of $f(z)$ are assumed to lie on the imaginary axis. Introducing a rotation $z = ix$, our result reads

Theorem 6. *Let the polynomial $f(x)$ of degree $n > 1$ have the zeros $\alpha_1, \ldots, \alpha_n$, and let $\beta_1, \ldots, \beta_{n-1}$ denote the roots of $f'(x) = 0$. Supposing $\alpha_1, \ldots, \alpha_n$ to be real (which implies the reality of $\beta_1, \ldots, \beta_{n-1}$), we have*

$$\frac{1}{n-1} \sum_1^{n-1} |\beta_\nu| \leqslant \frac{1}{n} \sum_1^n |\alpha_\nu|,$$

with equality only if

$$a) \quad \alpha_1 \leqslant 0, \alpha_2 \leqslant 0, \ldots, \alpha_n \leqslant 0,$$

or if

$$b) \quad \alpha_1 \geqslant 0, \alpha_2 \geqslant 0, \ldots, \alpha_n \geqslant 0.$$

Proof. Since we have

$$\frac{1}{n-1} \sum_1^{n-1} \beta_\nu = \frac{1}{n} \sum_1^n \alpha_\nu, \quad \ldots \ldots \ldots \quad (12)$$

(both sides equalling $-a_{n-1}/na_n$, if $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots$) it is sufficient to prove

$$D(f) = \frac{1}{n} \sum_1^n \varphi(\alpha_\nu) - \frac{1}{n-1} \sum_1^{n-1} \varphi(\beta_\nu) \geqslant 0, \quad \ldots \ldots \quad (13)$$

where

$$\varphi(x) = \tfrac{1}{2}(x + |x|) = \begin{cases} x \text{ if } x \geqslant 0, \\ 0 \text{ if } x \leqslant 0. \end{cases}$$

In case a) we have also $\beta_1 \leq 0, \ldots, \beta_{n-1} \leq 0$, and consequently $D = 0$. The same holds true in case b). Now suppose

$$0 < k < n,\ \alpha_1 \geq 0, \ldots, \alpha_k \geq 0,\ \alpha_{k+1} < 0, \ldots, \alpha_n < 0.$$

Suppose furthermore, that at least one β_i is positive, and not equal to a multiple root of $f(x) = 0$, for otherwise (13) is trivial. Put

$$f_\varrho(x) = \prod_1^k (x - \alpha_\nu) \prod_{k+1}^n (x - \varrho\, \alpha_\nu), \qquad f(x) = f_1(x).$$

The zeros $\beta_1(\varrho), \ldots, \beta_{n-1}(\varrho)$ of $f'_\varrho(x)$ are continuous functions of ϱ for $0 \leq \varrho \leq 1$, and differentiable at least for $0 < \varrho \leq 1$. Let $\beta_1, \ldots, \beta_j$ $(j \geq 1)$ be the positive zeros of $f'(x)$, and suppose $h \geq 1$, $f(\beta_1) \neq 0, \ldots, f(\beta_h) \neq 0$, $f(\beta_{h+1}) = \ldots = f(\beta_j) = 0$. The zeros $\beta_1(\varrho), \ldots, \beta_h(\varrho)$ are increasing, if ϱ decreases from 1 to 0. For, by differentiation of the relations

$$\sum_{\nu=1}^{k} \frac{1}{\beta_i - \alpha_\nu} + \sum_{\nu=k+1}^{n} \frac{1}{\beta_i - \varrho\, \alpha_\nu} = 0$$

with respect to ϱ, we obtain

$$\frac{d\beta_i}{d\varrho} \left(\sum_{\nu=1}^{k} \frac{1}{(\beta_i - \alpha_\nu)^2} + \sum_{\nu=k+1}^{n} \frac{1}{(\beta_i - \varrho\, \alpha_\nu)^2} \right) = \sum_{\nu=k+1}^{n} \frac{\alpha_\nu}{(\beta_i - \varrho\, \alpha_\nu)^2},$$

whence it follows $\dfrac{d\beta_i}{d\varrho} < 0$ $(i = 1, \ldots, h;\ 0 < \varrho \leq 1)$. Furthermore

$$\beta_{h+1}(\varrho), \ldots, \beta_j(\varrho)$$

are constant. Hence

$$\sum_{i=1}^{j} \beta_i(1) < \sum_{i=1}^{j} \beta_i(0) \leqslant \sum_{\nu=1}^{n-1} \varphi(\beta_\nu(0)). \qquad (14)$$

Now considering the expressions $D\{f_1(x)\}$ and $D\{f_0(x)\}$, we observe the contributions of the α's to be equal for both. So by (14), we have

$$D\{f(x)\} > D\{f_0(x)\}. \qquad (15)$$

The polynomial $f_0(x)$ belonging to case b), we have $D\{f_0(x)\} = 0$. Now (15) proves our theorem.

Theorem 7. *Let the polynomial $f(x)$ of degree $n > 1$ have the n real zeros $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$, and let $\beta_1, \ldots, \beta_{n-1}$ denote the zeros of $f'(x)$. Let the function $\psi(x)$ be convex in the interval $\alpha_1 \leq x \leq \alpha_n$. Putting*

$$D(\psi;\ f) = \frac{1}{n} \sum_1^n \psi(\alpha_\nu) - \frac{1}{n-1} \sum_1^{n-1} \psi(\beta_\nu),$$

we have $D(\psi;\, f) \geq 0$. $D(\psi;\, f) = 0$ *holds only if* $\psi(x)$ *is linear for* $\alpha_1 \leq x \leq \alpha_n$ [8]).

Proof. Let $\gamma_1 < \gamma_2 < \ldots < \gamma_m$ denote the set $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_{n-1}$, arranged in ascending order (each number of this set is taken only once). It follows that $\gamma_1 = \alpha_1$, $\gamma_m = \alpha_n$. We can evidently construct a convex function $\psi^*(x)$, satisfying $\psi^*(\gamma_i) = \psi(\gamma_i)$ $(i = 1, \ldots, m)$, which is linear and continuous upon the intervals

$$x \leqslant \gamma_2,\, \gamma_2 \leqslant x \leqslant \gamma_3, \ldots, \gamma_{m-2} \leqslant x \leqslant \gamma_{m-1},\, \gamma_{m-1} \leqslant x.$$

This function can be represented as

$$\psi^*(x) = \sum_{i=2}^{m-1} \lambda_i \,|x - \gamma_i| + C\,x, \qquad \lambda_i \geqslant 0. \quad . \quad . \quad . \quad . \quad (16)$$

According to Theorem 6, we have

$$D\{|x-\gamma_i|;\, f(x)\} = D\{|x|;\, f(x+\gamma_i)\} \geqslant 0,$$

and by (12) $D(x;\, f) = 0$. Hence

$$D(\psi;\, f) = D(\psi^*,\, f) = \sum_{2}^{m-1} \lambda_i\, D\{|x-\gamma_i|;\, f\} + C\, D(x;\, f) \geqslant 0 \ . \quad (17)$$

Now suppose $\alpha_1 < \alpha_n$, and suppose that $\psi(x)$ is not linear upon the interval $\alpha_1 \leq x \leq \alpha_n$. The open interval $\alpha_1 < x < \alpha_n$ containing at least one root of $f'(x) = 0$, we observe that $m > 2$, and that $\psi^*(x)$ is not linear upon $\alpha_1 \leq x \leq \alpha_n$. Hence at least one term on the right-hand side of (16), say the term with $i = k$, is non-linear, and consequently $\lambda_k > 0$. By theorem 6 it follows, that $\lambda_k D(|\,x - \gamma_k\,|;\, f) > 0$, the roots of $f(x)$ occurring on both sides of γ_k. Now (17) yields $D(\psi;\, f) > 0$, which proves our theorem.

KAKEYA's result, mentioned in section 3, is contained in Theorem 6. This follows from the convexity of y^p for $y > 0$, $p \geq 1$ or $p \leq 0$, and of $-y^p$ for $y > 0$, $0 \leq p \leq 1$.

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[8]) This includes the case $\alpha_1 = \alpha_2 = \ldots = \alpha_n$.

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