Asymptotics of the delay differential equation $x%5E{-\alpha} f(x) = f(x-1)-f(x)$ en $\frac{1}{4}

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Memorandum 1979-01

March 1979

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by

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ABSTRACT

By means of Fourier methods we shall prove that $\lim_{x\to\infty} f(x)$ exists if f is a solution of the equation in the title.

1. INTRODUCTION

In this note we consider the delay differential equation

$$(1.1) \qquad x^{-\alpha}f'(x) = f(x-1) - f(x), \quad x \geq 1 ,$$

where α is a real number. We say that f is a solution of (1.1) if f is a real valued function, continuous on $[0,\infty)$, differentiable on $[1,\infty)$, satisfying (1.1) on $[1,\infty)$. Every continuous function on $[0,1]$ can be extended to a solution of (1.1) by solving the differential equation $x^{-\alpha}f'(x) + f(x) = f(x-1)$ successively on $[n-1,n]$, $n = 1,2,\ldots$, each time with initial values that quarantee continuity.

The asymptotic behaviour of solutions of (1.1) is known: in 1949, 1950 N.G. de Bruijn ([1], [2]) treated the cases $\alpha \leq 0$ and $\alpha > \frac{1}{2}$ exhaustively; in 1972 J.J.A.M. Brands ([3]) solved the question of the asymptotic behavior for the cases $0 < \alpha \leq \frac{1}{2}$. Actually both authors dealt with generalizations of (1.1).

The main result of [3] (as far as (1.1) is concerned) can be expressed by the following

THEOREM 1.1. *Let* α *satisfy* $0 \leq \alpha \leq \frac{1}{2}$ *and let* f *be a solution of* (1.1). *Then* $\lim_{x\to\infty} f(x)$ *exists and* $\lim_{x\to\infty} f^{k}(x) = 0$, $k = 1,2,\ldots$.

In this note we present a totally different proof. Unfortunately it only covers the cases $\frac{1}{4} < \alpha \leq \frac{1}{2}$. It is an open question whether the ideas of this proof can be extended to the remaining cases $0 \leq \alpha \leq \frac{1}{4}$.

2. PRELIMINARIES

We state several properties of solutions f of (1.1). Proofs can be found in [1].

LEMMA 2.1. *Let* f *be a solution of* (1.1). *Then*

i) f *is bounded,*

ii) $f^{(n)}$ *is continuous and bounded on* $[n,\infty)$, $(n = 1,2,\ldots)$,

iii) $|f^{(n)}(x) - f^{(n)}(x-1)| = \mathcal{O}(x^{-\alpha})$, $(x \geq n+1)$, $n = 1,2,\ldots$,

iv) $x^{-\alpha}f^{(n+1)}(x) = f^{(n)}(x-1) - f^{(n)}(x) + \mathcal{O}(x^{-\alpha-1})$ $(x \geq n+1)$, $n=1,2,\ldots$.

3. PROOF OF THEOREM 1.1 IN CASE $\frac{1}{4} < \alpha \leq \frac{1}{2}$.

We shall use notations like $f(x) = \mathcal{O}(g(x))$ in order to indicate what happens when $x \to \infty$. So $f(x) = \mathcal{O}(g(x))$ means that positive numbers A and B exist such that $|f(x)| \leq A|g(x)|$ for all $x \geq B$.

Using a two-term Taylor expansion and lemma 2.1 we obtain

$$f^{(k)}(x + x^{-\alpha}) - f^{(k)}(x-1) = \mathcal{O}(x^{-2\alpha}), \quad k = 0,1,2,\ldots .$$

This relation suggests the transformation

$$g(y) = f(x), \qquad y + (1-\alpha)^{-1}y^{1-\alpha} = x(y) , \tag{3.1}$$

which was used already in [1].

LEMMA 3.1. *We have*

$$y^{-2\alpha}[A(y)g'(y) + B(y)g''(y) + C(y)g'''(y)] = g(y) - g(y-1) + \mathcal{O}(y^{-\beta}), \tag{3.2}$$

where

$$A(y) = \alpha(1-\alpha)^{-1} - \tfrac{1}{2}\alpha(3-\alpha)(1-\alpha)^{-2}y^{-\alpha} ,$$

$$B(y) = \tfrac{1}{2} - (1-\alpha)y^{-\alpha},\ C(y) = -\frac{1}{3}y^{-\alpha},\ \beta = \min\{4\alpha,\ 1+\alpha\} .$$

PROOF OF LEMMA 3.1. Applying lemma 2.1 we easily obtain that g and its derivatives are bounded (on their domain of existence), and furthermore that

$$g^{(k)}(y) - g^{(k)}(y-1) = \mathcal{O}(y^{-2\alpha}), \quad k = 0,1,2,\ldots .$$

Let

$$\Delta := x(y-1) + 1 - x(y),\ R := f(x(y-1)+1) - f(x(y)) .$$

Then

$$g(y-1) - g(y) = f(x(y-1)) - f(x(y-1)+1) + R =$$

$$= (x(y-1)+1)^{-\alpha}f'(x(y-1)+1) + R ,$$

which leads to (3.2) by substituting of

$$f'(x(y-1)+1) = f'(x(y)) + \Delta f''(x(y)) + \tfrac{1}{2}\Delta^2 f'''(x(y)) + \mathcal{O}(y^{-3\alpha}) ,$$

$$R = \Delta f'(x(y)) + \tfrac{1}{2}\Delta^2 f''(x(y)) + \frac{1}{6}\Delta^3 f'''(x(y)) + \mathcal{O}(y^{-4\alpha}) ,$$

$$\Delta = -y^{-\alpha} + \mathcal{O}(y^{-1-\alpha}) ,$$

$$f'(x(y)) = (1 + y^{-\alpha})g'(y) ,$$

$$f''(x(y)) = (1 + y^{-\alpha})^2 g''(y) + \mathcal{O}(y^{-1-\alpha}) ,$$

$$f'''(x(y)) = (1 + y^{-\alpha})^3 g'''(y) + \mathcal{O}(y^{-1-\alpha}) ,$$

$$(x(y-1)+1)^{-\alpha} = y^{-\alpha} - \alpha(1-\alpha)^{-1} y^{-2\alpha} + \tfrac{1}{2}\alpha(1+\alpha)(1-\alpha)^{-2} y^{-3\alpha} + \mathcal{O}(y^{-\beta}).$$

It is possible to improve on (3.2), replacing it by something with a smaller error term. We shall not do this, however. The reason will be given after the proof of lemma 3.2.

Proceeding with the proof of theorem (1.1) we derive a recurrence relation for the coefficients $c(n,m)$ of the Fourier series $\sum_{m=-\infty}^{\infty} c(n,m)e^{2\pi imy}$ of g on $[n,n+1]$, $n \geq 1$.

LEMMA 3.2.

(3.3) $$[1 + a(m)n^{-2\alpha} + b(m)n^{-3\alpha}]c(n,m) = c(n-1,m) + (1+m^2)\mathcal{O}(n^{-\beta}) ,$$

where $\mathcal{O}(n^{-\beta})$ *is uniform in* m, *and*

$$a(m) = 2\pi^2 m^2 - 2\pi i\alpha(1-\alpha)^{-1} m ,$$

$$b(m) = -\frac{8}{3}\pi^3 im^3 - 4\pi^2(1-\alpha)^{-1} m^2 + \pi i\alpha(3-\alpha)(1-\alpha)^{-2} m .$$

PROOF OF LEMMA 3.2. It is clear that $y^{-2\alpha} - n^{-2\alpha} = \mathcal{O}(n^{-\beta})$, $y^{-3\alpha} - n^{-3\alpha} = \mathcal{O}(n^{-\beta})$, uniformly in $y \in [n,n+1]$. Therefore we can rewrite (3.2) as follows:

(3.4) $$n^{-2\alpha}[A(n)g'(y) + B(n)g''(y) + C(n)g'''(y)] = g(y) - g(y-1) + \mathcal{O}(n^{-\beta})$$

uniformly in $y \in [n,n+1]$.

Integrating by parts, and using the fact that $g^{(k)}(n) - g^{(k)}(n-1) = \mathcal{O}(n^{-2\alpha})$, $k = 0,1,2,\ldots$, we obtain the following formulas:

$$\int_n^{n+1} e^{-2\pi imy} g'(y)dy = 2\pi im\, c(n,m) + \mathcal{O}(n^{-2\alpha}) ,$$

$$\int_n^{n+1} e^{-2\pi imy} g''(y)dy = -4\pi^2 m^2 c(n,m) + (1+|m|)\mathcal{O}(n^{-2\alpha}) ,$$

$$\int_n^{n+1} e^{-2\pi imy} g'''(y)dy = -8\pi^3 im\, c(n,m) + (1+m^2)\mathcal{O}(n^{-2\alpha}) ,$$

where the symbols $\mathcal{O}(n^{-2\alpha})$ are uniform in m. Multiplying both sides of (3.4) by $e^{-2\pi imy}$, integrating from n to n+1 and substituting the above results we obtain (3.3).

Now it becomes clear why we did not develop beyond $y^{-\beta}$ in (3.2). The $\mathcal{O}(n^{-2\alpha})$ in the estimates for the integrals give rise to a term $\mathcal{O}(n^{-4\alpha})$ in (3.2). It is not clear how these estimates can be sharpened.

Taking m = 0 in (3.3) we obtain

$$c(n,0) = c(n-1,0) + \mathcal{O}(n^{-\beta}) .$$

It follows that $\lim_{n\to\infty} c(n,0)$ exists. Denoting this limit by $\bar{g}$ we have

(3.5) $$c(n,0) = \bar{g} + \mathcal{O}(n^{1-\beta}) .$$

From lemma 3.2 we infer that there is a positive number K, independent of m, such that for n sufficiently large, say $n \geq n_1$, with n_1 independent of m, we have

(3.6) $$[1 + 2\pi^2 m^2 n^{-2\alpha}(1 - 2(1-\alpha)^{-1} n^{-\alpha})]\,|c(n,m)| \leq |c(n-1,m)| + K(1+m^2)n^{-\beta}.$$

Suppose that $m \neq 0$. We choose a positive number $\delta < \beta - 1$. Then for $n \geq 64$ the following inequality holds:

$$n^{\delta}[1 + 2\pi^2 m^2 n^{-2\alpha}(1 - 2(1-\alpha)^{-1} n^{-\alpha})] \geq (n+1)^{\delta} .$$

From (3.6) we infer that for $n \geq n_2 := \max\{n_1, 64\}$

$$(n+1)^{\delta}|c(n,m)| \leq n^{\delta}|c(n-1,m)| + K(1+m^2)n^{-\beta+\delta} .$$

It follows that

$$(n+1)^{\delta}|c(n,m)| \leq K(1+m^2)\sum_{k=n_2}^{n} k^{-\beta+\delta} + n_2^{\delta}|c(n_2-1,m)| .$$

Observing that $|c(n,m)| \leq \int_n^{n+1} |g(y)|dy \leq M$ independent of m and n we can conclude that there is a positive number A, independent of n and $m \neq 0$, such that

(3.7) $$|c(n,m)| \leq Am^2 n^{-\delta}, \quad n \geq n_2,\ m \neq 0 .$$

Moreover, we have

(3.8) $$|c(n,m)| \leq B|m|^{-1}, \quad n \geq 1,\ m \neq 0 ,$$

where B is a positive number independent of n and $m \neq 0$. The inequality (3.8) follows from the following calculation, ~~using~~ where the boundedness of g and g' is used

$$|c(n,m)| \leq \left| \int_n^{n+1} g(y) e^{-2\pi i m y} dy \right| \leq |g(n+1) - g(n)| \, |2\pi m|^{-1} +$$

$$|2\pi m|^{-1} \int_n^{n+1} |g'(y)| dy \; .$$

Applying (3.5), (3.7) and (3.8) in Parseval's formula we obtain

$$\int_n^{n+1} (g(y) - \bar{g})^2 dy = \sum_{|m|>0} |c(n,m)|^2 + |c(n,0) - \bar{g}|^2 =$$

$$\sum_{0<|m|\leq n^{\delta/3}} + \sum_{|m|>n^{\delta/3}} + \mathcal{O}(n^{2(1-\beta)}) \leq$$

$$A^2 n^{-2\delta} \sum_{0<|m|\leq n^{\delta/3}} m^4 + B^2 \sum_{|m|>n^{\delta/3}} m^{-2} + \mathcal{O}(n^{2(1-\beta)}) \leq$$

$$2A^2 n^{-\delta/3} + 2B^2 (n^{\delta/3} - 1)^{-1} + \mathcal{O}(n^{2(1-\beta)}) = \mathcal{O}(n^{-\delta/3}) \; .$$

Hence $\lim_{n\to\infty} \int_n^{n+1} (g(y) - \bar{g})^2 dy = 0$, which together with the boundedness of g' implies that $\lim_{y\to\infty} g(y) = \bar{g}$. From the definition (3.1) we infer that $\lim_{x\to\infty} f(x)$ exists. From the boundedness of the derivatives of f it follows that $\lim_{x\to\infty} f^{(k)}(x) = 0$, $k = 1,2,\ldots$, a consequence of a theorem of H.D. Kloosterman ([4], theorem 3). A simplified form of that theorem says: If $\lim_{x\to\infty} f(x)$ exists and if the n-th derivative ($n \geq 2$) is bounded, then $\lim_{x\to\infty} f^{(k)}(x) = 0$, $k = 1,2,\ldots,n-1$.

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