

## Asymptotics of the delay differential equation $x^{\alpha} f(x) = f(x-1) - f(x)$ en $\frac{1}{4}$

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Asymptotics of the delay differential equation

$$x^{-\alpha} f'(x) = f(x-1) - f(x) \text{ with } \frac{1}{4} < \alpha \leq \frac{1}{2}$$

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ASYMPTOTICS OF THE DELAY DIFFERENTIAL EQUATION

$$x^{-\alpha} f'(x) = f(x-1) - f(x) \text{ WITH } \frac{1}{4} < \alpha \leq \frac{1}{2}$$

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ABSTRACT

By means of Fourier methods we shall prove that  $\lim_{x \rightarrow \infty} f(x)$  exists if  $f$  is a solution of the equation in the title.

1. INTRODUCTION

In this note we consider the delay differential equation

$$(1.1) \quad x^{-\alpha} f'(x) = f(x-1) - f(x), \quad x \geq 1,$$

where  $\alpha$  is a real number. We say that  $f$  is a solution of (1.1) if  $f$  is a real valued function, continuous on  $[0, \infty)$ , differentiable on  $[1, \infty)$ , satisfying (1.1) on  $[1, \infty)$ . Every continuous function on  $[0, 1]$  can be extended to a solution of (1.1) by solving the differential equation  $x^{-\alpha} f'(x) + f(x) = f(x-1)$  successively on  $[n-1, n]$ ,  $n = 1, 2, \dots$ , each time with initial values that guarantee continuity.

The asymptotic behaviour of solutions of (1.1) is known: in 1949, 1950 N.G. de Bruijn ([1], [2]) treated the cases  $\alpha \leq 0$  and  $\alpha > \frac{1}{2}$  exhaustively; in 1972 J.J.A.M. Brands ([3]) solved the question of the asymptotic behavior for the cases  $0 < \alpha \leq \frac{1}{2}$ . Actually both authors dealt with generalizations of (1.1).

The main result of [3] (as far as (1.1) is concerned) can be expressed by the following

**THEOREM 1.1.** *Let  $\alpha$  satisfy  $0 \leq \alpha \leq \frac{1}{2}$  and let  $f$  be a solution of (1.1). Then  $\lim_{x \rightarrow \infty} f(x)$  exists and  $\lim_{x \rightarrow \infty} f^k(x) = 0$ ,  $k = 1, 2, \dots$ .*

In this note we present a totally different proof. Unfortunately it only covers the cases  $\frac{1}{4} < \alpha \leq \frac{1}{2}$ . It is an open question whether the ideas of this proof can be extended to the remaining cases  $0 \leq \alpha \leq \frac{1}{4}$ .

2. PRELIMINARIES

We state several properties of solutions  $f$  of (1.1). Proofs can be found in [1].

LEMMA 2.1. Let  $f$  be a solution of (1.1). Then

- i)  $f$  is bounded,
- ii)  $f^{(n)}$  is continuous and bounded on  $[n, \infty)$ , ( $n = 1, 2, \dots$ ),
- iii)  $|f^{(n)}(x) - f^{(n)}(x-1)| = O(x^{-\alpha})$ , ( $x \geq n+1$ ),  $n = 1, 2, \dots$ ,
- iv)  $x^{-\alpha} f^{(n+1)}(x) = f^{(n)}(x-1) - f^{(n)}(x) + O(x^{-\alpha-1})$  ( $x \geq n+1$ ),  $n = 1, 2, \dots$ .

3. PROOF OF THEOREM 1.1 IN CASE  $\frac{1}{4} < \alpha \leq \frac{1}{2}$ .

We shall use notations like  $f(x) = O(g(x))$  in order to indicate what happens when  $x \rightarrow \infty$ . So  $f(x) = O(g(x))$  means that positive numbers  $A$  and  $B$  exist such that  $|f(x)| \leq A|g(x)|$  for all  $x \geq B$ .

Using a two-term Taylor expansion and lemma 2.1 we obtain

$$f^{(k)}(x + x^{-\alpha}) - f^{(k)}(x - 1) = O(x^{-2\alpha}), \quad k = 0, 1, 2, \dots$$

This relation suggests the transformation

$$(3.1) \quad g(y) = f(x), \quad y + (1 - \alpha)^{-1} y^{1-\alpha} = x(y),$$

which was used already in [1].

LEMMA 3.1. We have

$$(3.2) \quad y^{-2\alpha} [A(y)g'(y) + B(y)g''(y) + C(y)g'''(y)] = g(y) - g(y-1) + O(y^{-\beta}),$$

where

$$A(y) = \alpha(1 - \alpha)^{-1} - \frac{1}{2}\alpha(3 - \alpha)(1 - \alpha)^{-2} y^{-\alpha},$$

$$B(y) = \frac{1}{2} - (1 - \alpha)y^{-\alpha}, \quad C(y) = -\frac{1}{3} y^{-\alpha}, \quad \beta = \min\{4\alpha, 1 + \alpha\}.$$

PROOF OF LEMMA 3.1. Applying lemma 2.1 we easily obtain that  $g$  and its derivatives are bounded (on their domain of existence), and furthermore that

$$g^{(k)}(y) - g^{(k)}(y-1) = O(y^{-2\alpha}), \quad k = 0, 1, 2, \dots$$

Let

$$\Delta := x(y-1) + 1 - x(y), \quad R := f(x(y-1) + 1) - f(x(y)).$$

Then

$$g(y-1) - g(y) = f(x(y-1)) - f(x(y-1) + 1) + R =$$

$$= (x(y-1) + 1)^{-\alpha} f'(x(y-1) + 1) + R,$$

which leads to (3.2) by substituting <sup>sp</sup> of

$$f'(x(y-1) + 1) = f'(x(y)) + \Delta f''(x(y)) + \frac{1}{2}\Delta^2 f'''(x(y)) + O(y^{-3\alpha}),$$

$$R = \Delta f'(x(y)) + \frac{1}{2}\Delta^2 f''(x(y)) + \frac{1}{6}\Delta^3 f'''(x(y)) + O(y^{-4\alpha}),$$

$$\Delta = -y^{-\alpha} + \mathcal{O}(y^{-1-\alpha}) ,$$

$$f'(x(y)) = (1 + y^{-\alpha})g'(y) ,$$

$$f''(x(y)) = (1 + y^{-\alpha})^2 g''(y) + \mathcal{O}(y^{-1-\alpha}) ,$$

$$f'''(x(y)) = (1 + y^{-\alpha})^3 g'''(y) + \mathcal{O}(y^{-1-\alpha}) ,$$

$$(x(y-1)+1)^{-\alpha} = y^{-\alpha} - \alpha(1-\alpha)^{-1} y^{-2\alpha} + \frac{1}{2}\alpha(1+\alpha)(1-\alpha)^{-2} y^{-3\alpha} + \mathcal{O}(y^{-\beta}) .$$

It is possible to improve on (3.2), replacing it by something with a smaller error term. We shall not do this, however. The reason will be given after the proof of lemma 3.2.

Proceeding with the proof of theorem (1.1) we derive a recurrence relation for the coefficients  $c(n,m)$  of the Fourier series  $\sum_{m=-\infty}^{\infty} c(n,m)e^{2\pi imy}$  of  $g$  on  $[n, n+1]$ ,  $n \geq 1$ .

LEMMA 3.2.

$$(3.3) \quad [1 + a(m)n^{-2\alpha} + b(m)n^{-3\alpha}]c(n,m) = c(n-1,m) + (1+m^2)\mathcal{O}(n^{-\beta}) ,$$

where  $\mathcal{O}(n^{-\beta})$  is uniform in  $m$ , and

$$a(m) = 2\pi^2 m^2 - 2\pi i \alpha (1-\alpha)^{-1} m ,$$

$$b(m) = -\frac{8}{3} \pi^3 i m^3 - 4\pi^2 (1-\alpha)^{-1} m^2 + \pi i \alpha (3-\alpha)(1-\alpha)^{-2} m .$$

PROOF OF LEMMA 3.2. It is clear that  $y^{-2\alpha} - n^{-2\alpha} = \mathcal{O}(n^{-\beta})$ ,  $y^{-3\alpha} - n^{-3\alpha} = \mathcal{O}(n^{-\beta})$ , uniformly in  $y \in [n, n+1]$ . Therefore we can rewrite (3.2) as follows:

$$(3.4) \quad n^{-2\alpha}[A(n)g'(y) + B(n)g''(y) + C(n)g'''(y)] = g(y) - g(y-1) + \mathcal{O}(n^{-\beta})$$

uniformly in  $y \in [n, n+1]$ .

Integrating by parts, and using the fact that  $g^{(k)}(n) - g^{(k)}(n-1) = \mathcal{O}(n^{-2\alpha})$ ,  $k = 0, 1, 2, \dots$ , we obtain the following formulas:

$$\int_n^{n+1} e^{-2\pi imy} g'(y) dy = 2\pi im c(n,m) + \mathcal{O}(n^{-2\alpha}) ,$$

$$\int_n^{n+1} e^{-2\pi imy} g''(y) dy = -4\pi^2 m^2 c(n,m) + (1+|m|)\mathcal{O}(n^{-2\alpha}) ,$$

$$\int_n^{n+1} e^{-2\pi imy} g'''(y) dy = -8\pi^3 im c(n,m) + (1+m^2)\mathcal{O}(n^{-2\alpha}) ,$$

where the symbols  $\mathcal{O}(n^{-2\alpha})$  are uniform in  $m$ . Multiplying both sides of (3.4)

by  $e^{-2\pi i m y}$ , integrating from  $n$  to  $n+1$  and substituting the above results we obtain (3.3).

Now it becomes clear why we did not develop beyond  $y^{-\beta}$  in (3.2). The  $\mathcal{O}(n^{-2\alpha})$  in the estimates for the integrals give rise to a term  $\mathcal{O}(n^{-4\alpha})$  in (3.2). It is not clear how these estimates can be sharpened.

Taking  $m = 0$  in (3.3) we obtain

$$c(n,0) = c(n-1,0) + \mathcal{O}(n^{-\beta}) .$$

It follows that  $\lim_{n \rightarrow \infty} c(n,0)$  exists. Denoting this limit by  $\bar{g}$  we have

$$(3.5) \quad c(n,0) = \bar{g} + \mathcal{O}(n^{1-\beta}) .$$

From lemma 3.2 we infer that there is a positive number  $K$ , independent of  $m$ , such that for  $n$  sufficiently large, say  $n \geq n_1$ , with  $n_1$  independent of  $m$ , we have

$$(3.6) \quad [1 + 2\pi^2 m^2 n^{-2\alpha} (1 - 2(1-\alpha)^{-1} n^{-\alpha})] |c(n,m)| \leq |c(n-1,m)| + K(1+m^2)n^{-\beta} .$$

Suppose that  $m \neq 0$ . We choose a positive number  $\delta < \beta - 1$ . Then for  $n \geq 64$  the following inequality holds:

$$n^\delta [1 + 2\pi^2 m^2 n^{-2\alpha} (1 - 2(1-\alpha)^{-1} n^{-\alpha})] \geq (n+1)^\delta .$$

From (3.6) we infer that for  $n \geq n_2 := \max\{n_1, 64\}$

$$(n+1)^\delta |c(n,m)| \leq n^\delta |c(n-1,m)| + K(1+m^2)n^{-\beta+\delta} .$$

It follows that

$$(n+1)^\delta |c(n,m)| \leq K(1+m^2) \sum_{k=n_2}^n k^{-\beta+\delta} + n_2^\delta |c(n_2-1,m)| .$$

Observing that  $|c(n,m)| \leq \int_n^{n+1} |g(y)| dy \leq M$  independent of  $m$  and  $n$  we can conclude that there is a positive number  $A$ , independent of  $n$  and  $m \neq 0$ , such that

$$(3.7) \quad |c(n,m)| \leq A m^2 n^{-\delta}, \quad n \geq n_2, \quad m \neq 0 .$$

Moreover, we have

$$(3.8) \quad |c(n,m)| \leq B |m|^{-1}, \quad n \geq 1, \quad m \neq 0 ,$$

where B is a positive number independent of n and  $m \neq 0$ . The inequality (3.8) follows from the following calculation, <sup>where</sup> using the boundedness of g and g' is used.

$$|c(n,m)| \leq \left| \int_n^{n+1} g(y) e^{-2\pi i m y} dy \right| \leq |g(n+1) - g(n)| |2\pi m|^{-1} + |2\pi m|^{-1} \int_n^{n+1} |g'(y)| dy .$$

Applying (3.5), (3.7) and (3.8) in Parseval's formula we obtain

$$\begin{aligned} \int_n^{n+1} (g(y) - \bar{g})^2 dy &= \sum_{|m|>0} |c(n,m)|^2 + |c(n,0) - \bar{g}|^2 = \\ &= \sum_{0 < |m| \leq n^{\delta/3}} + \sum_{|m| > n^{\delta/3}} + O(n^{2(1-\beta)}) \leq \\ &= A^2 n^{-2\delta} \sum_{0 < |m| \leq n^{\delta/3}} m^4 + B^2 \sum_{|m| > n^{\delta/3}} m^{-2} + O(n^{2(1-\beta)}) \leq \\ &= 2A^2 n^{-\delta/3} + 2B^2 (n^{\delta/3} - 1)^{-1} + O(n^{2(1-\beta)}) = O(n^{-\delta/3}) . \end{aligned}$$

Hence  $\lim_{n \rightarrow \infty} \int_n^{n+1} (g(y) - \bar{g})^2 dy = 0$ , which together with the boundedness of g' implies that  $\lim_{y \rightarrow \infty} g(y) = \bar{g}$ . From the definition (3.1) we infer that  $\lim_{x \rightarrow \infty} f(x)$  exists. From the boundedness of the derivatives of f it follows that  $\lim_{x \rightarrow \infty} f^{(k)}(x) = 0$ ,  $k = 1, 2, \dots$ , a consequence of a theorem of H.D. Kloosterman ([4], theorem 3). A simplified form of that theorem says: If  $\lim_{x \rightarrow \infty} f(x)$  exists and if the n-th derivative ( $n \geq 2$ ) is bounded, then  $\lim_{x \rightarrow \infty} f^{(k)}(x) = 0$ ,  $k = 1, 2, \dots, n-1$ .

REFERENCES

1. N.G. de Bruijn, The asymptotically periodic behaviour of the solutions of some linear functional equations, Amer. J. Math. 71 (1949), 313-330.
2. N.G. de Bruijn, On some linear functional equations, Publ. Math. Debrecen. 1 (1950), 129-134.
3. J.J.A.M. Brands, The asymptotic behaviour of the solutions of certain difference differential equations, Nederl. Akad. Wetensch. Proc. Ser A 75 = Indag. Math. 34 (1972), 73-81.
4. H.D. Kloosterman, On the convergence of series summable  $(c,r)$  and on the magnitude of the derivatives of a function of a real variable, J. London Math. Soc. 15 (1940), 91-96.