

## The logarithm of a matrix

***Citation for published version (APA):***

Hautus, M. L. J. (1977). *The logarithm of a matrix*. (Eindhoven University of Technology : Dept of Mathematics : memorandum; Vol. 7702). Technische Hogeschool Eindhoven.

***Document status and date:***

Published: 01/01/1977

***Document Version:***

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

***Please check the document version of this publication:***

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

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Memorandum 1977-02

January 1977

The logarithm of a matrix

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# THE LOGARITHM OF A MATRIX

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Introduction. In the theory of systems of linear differential equations with periodic coefficients, Floquet's theorem plays a central role [3, § 2.5]. Its proof depends crucially on the following matrix theoretic result:

**THEOREM 1.** *If A is a nonsingular (complex)  $n \times n$  matrix, there exists a matrix P such that  $e^P = A$ .*

This theorem is easily proved once a suitable operational calculus for matrix functions has been set up [2, §V.1.]. In most textbooks, a proof depending on the Jordan canonical form is given. For undergraduate courses a simpler and more elementary proof is desirable. One such proof was given in [1, § 1.15]. In this note we propose an alternative proof, which is more closely related to the theory of linear differential equations.

Proof of Theorem 1. We need a preliminary result.

LEMMA

$$\exp \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \begin{bmatrix} e^A & \int_0^1 e^{(1-\tau)A} C e^{\tau B} d\tau \\ 0 & e^B \end{bmatrix}$$

PROOF. Consider the system of differential equations

$$(1) \quad \begin{aligned} \dot{x}(t) &= Ax(t) + Cy(t) \\ \dot{y}(t) &= By(t) \end{aligned}$$

Its fundamental solution  $\Phi(t)$  with initial value  $\Phi(0) = I$ , equals

$$\Phi(t) = \exp \begin{bmatrix} tA & tC \\ 0 & tB \end{bmatrix}.$$

On the other hand, (1) can be solved by observing that  $y(t) = e^{tB}y(0)$  and by applying the variation of constants formula to the first equation of (1), which yields:

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A} C e^{B\tau} d\tau y(0)$$

$$y(t) = e^{tB}y(0)$$

Hence

$$\Phi(t) = \begin{bmatrix} e^{tA} & \int_0^t e^{(t-\tau)A} C e^{B\tau} d\tau \\ 0 & e^{tB} \end{bmatrix}$$

□

In the proof of Theorem 1 we assume without loss of generality that  $A$  is an upper triangular (abbreviated UT) matrix. (Compare [1, § 1.15]. Consequently, Theorem 1 follows from

**THEOREM 2.** *Let  $A$  be a nonsingular  $n \times n$  UT matrix. Then there exists a unique  $n \times n$  UT matrix  $P$  such  $e^P = A$  and  $p_{ii} = \log a_{ii}$  (Here  $\log$  denotes the principal value).*

**PROOF.** We proceed by induction with respect to  $n$ . The result is obvious for  $n = 1$ . Now, let  $A$  be a nonsingular  $n \times n$  UT matrix. We decompose  $A$  as follows:

$$A = \begin{bmatrix} B & c \\ 0 & \delta \end{bmatrix},$$

where  $B$  is a nonsingular  $(n-1) \times (n-1)$  UT matrix,  $c$  is an  $(n-1)$  column vector and  $\delta$  is a nonzero number. We split the sought matrix  $P$  analogously

$$P = \begin{bmatrix} Q & r \\ 0 & \sigma \end{bmatrix}.$$

According to the Lemma, the equation  $e^P = A$  is equivalent to

$$(2) \quad e^Q = B ,$$

$$(3) \quad e^\sigma = \delta ,$$

$$(4) \quad \int_0^1 e^{(1-\tau)Q} r e^{\sigma\tau} d\tau = c .$$

By induction (2) has a unique solution satisfying the requirements. Also,  $\sigma$  is uniquely determined by  $\sigma = \log \delta$  (principal value). Finally, (4) can be rewritten as

$$e^Q M r = c ,$$

where

$$M := \int_0^1 e^{(\sigma I - Q)\tau} d\tau .$$

Consequently, it suffices to show that  $M$  is nonsingular. Since  $M$  is also UT, this is the case iff

$$m_{ii} = \int_0^1 e^{(\sigma - q_{ii})\tau} d\tau \neq 0$$

for  $i = 1, \dots, n - 1$ . If  $q_{ii} = \sigma$ , then  $m_{ii} = 1$  and if  $q_{ii} \neq \sigma$ , then

$$m_{ii} = (e^{\sigma - q_{ii}} - 1) / (\sigma - q_{ii}) \neq 0,$$

since  $|\operatorname{Im}(\sigma - q_{ii})| < 2\pi$ . (Recall that  $\delta$  and  $q_{ii}$  are principal values of logarithms).

#### References

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- [3] M. ROSEAU. *Vibrations non linéaires et théorie de la stabilité*. Springer tracts in natural philosophy 8, Springer-Verlag Berlin 1966.