The logarithm of a matrix

EINDHOVEN UNIVERSITY OF TECHNOLOGY

Department of Mathematics

Memorandum 1977-02

January 1977

The logarithm of a matrix

by

M.L.J. Hautus

University of Technology
Department of Mathematics
PO Box 513, Eindhoven
The Netherlands

THE LOGARITHM OF A MATRIX

M.L.J. Hautus

Introduction. In the theory of systems of linear differential equations with periodic coefficients, Floquet's theorem plays a central role [3 , § 2.5]. Its proof depends crucially on the following matrix theoretic result:

THEOREM 1. *If* A *is a nonsingular (complex)* $n \times n$ *matrix, there exists a matrix* P *such that* $e^P = A$.

This theorem is easily proved once a suitable operational calculus for matrix functions has been set up [2, § V.1.]. In most textbooks, a proof depending on the Jordan canonical form is given. For undergraduate cources a simpler and more elementary proof is desirable. One such proof was given in [1, § 1.15]. In this note we propose an alternative proof, which is more closely related to the theory of linear differential equations.

Proof of Theorem 1. We need a preliminary result.

LEMMA

$$\exp\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \begin{bmatrix} e^A & \int_0^1 e^{(1-\tau)A} C e^{\tau B} d\tau \\ 0 & e^B \end{bmatrix}$$

PROOF. Consider the system of differential equations

$$(1) \qquad \begin{aligned} \dot{x}(t) &= Ax(t) + Cy(t) \\ \dot{y}(t) &= \qquad\qquad By(t) \end{aligned} .$$

Its fundamental solution $\Phi(t)$ with initial value $\Phi(0) = I$, equals

$$\Phi(t) = \exp\begin{bmatrix} tA & tC \\ 0 & tB \end{bmatrix} .$$

On the other hand, (1) can be solved by observing that $y(t) = e^{tB}y(0)$ and by applying the variation of constants formula to the first equation of (1), which yields:

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Ce^{B\tau}d\tau \; y(0)$$

$$y(t) = e^{tB}y(0)$$

Hence

$$\Phi(t) = \begin{bmatrix} e^{tA} & \int_0^t e^{(t-\tau)A}Ce^{B\tau}d\tau \\ 0 & e^{tB} \end{bmatrix}$$ □

In the proof of Theorem 1 we assume without loss of generality that A is an upper triangular (abbreviated UT) matrix. (Compare [1, § 1.15]. Consequently, Theorem 1 follows from

THEOREM 2. *Let* A *be a nonsingular* n × n *UT matrix. Then there exists a unique* n × n *UT matrix* P *such* $e^P = A$ *and* $p_{ii} = \log a_{ii}$ (Here log denotes the principal value).

PROOF. We proceed by induction with respect to n. The result is obvious for n = 1. Now, let A be a nonsingular n × n UT matrix. We decompose A as follows:

$$A = \begin{bmatrix} B & c \\ 0 & \delta \end{bmatrix},$$

where B is a nonsingular (n - 1) × (n - 1) UT matrix, c is an (n - 1) column vector and δ is a nonzero number. We split the sought matrix P analogously

$$P = \begin{bmatrix} Q & r \\ 0 & \sigma \end{bmatrix}.$$

According to the Lemma, the equation $e^P = A$ is equivalent to

(2) $e^Q = B$,

(3) $e^\sigma = \delta$,

(4) $\int_0^1 e^{(1-\tau)Q} r e^{\sigma\tau} d\tau = c$.

By induction (2) has a unique solution satisfying the requirements. Also, σ is uniquely determined by $\sigma = \log \delta$ (principal value). Finally, (4) can be rewritten as

$$e^Q M r = c \ ,$$

where

$$M := \int_0^1 e^{(\sigma I - Q)\tau} d\tau \ .$$

Consequently, it suffices to show that M is nonsingular. Since M is also UT, this is the case iff

$$m_{ii} = \int_0^1 e^{(\sigma - q_{ii})\tau} d\tau \neq 0$$

for $i = 1,\dots,n-1$. If $q_{ii} = \sigma$, then $m_{ii} = 1$ and if $q_{ii} \neq \sigma$, then

$$m_{ii} = (e^{\sigma - q_{ii}} - 1) \ / \ (\sigma - q_{ii}) \neq 0,$$

since $|\operatorname{Im}(\sigma - q_{ii})| < 2\pi$. (Recall that δ and q_{ii} are principal values of logarithms).

References

[1] R. BELLMAN, *Stability theory of differential equations*. Dover, New York, 1969.

[2] J.L. DALECKII and M.G. KREIN. *Stability of solutions of differential equations in Banach space*. Transl. of Math. Mon. 43, Am. Math. Soc., Providence, Rhode Island 1974.

[3] M. ROSEAU. *Vibrations non linéaires et théorie de la stabilité*. Springer tracts in natural philosophy 8, Springer-Verlag Berlin 1966.