Het voorspellen van de kniklast van een balk door analyse van de eigenfrequentie bij toenemende axiale belasting

Citation for published version (APA):

Kraker, de, A. (1984). *Het voorspellen van de kniklast van een balk door analyse van de eigenfrequentie bij toenemende axiale belasting*. (DCT rapporten; Vol. 1984.020). Technische Hogeschool Eindhoven.

Document status and date:
Gepubliceerd: 01/01/1984

Document Version:
Uitgevers PDF, ook bekend als Version of Record

Please check the document version of this publication:

• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

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SUFFICIENT CONDITIONS FOR SYNCHRONIZATION IN AN ENSEMBLE OF HINDMARSH AND ROSE NEURONS: PASSIVITY-BASED APPROACH

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Abstract: In this paper we consider a system of globally, uniformly and linearly coupled Hindmarsh and Rose oscillators. This model is a reduction of the celebrated Hodgkin-Huxley equations, which are considered as the most physiologically realistic model of neural dynamics at the level of a single cell. Exploiting recently developed framework for analysis of synchronization phenomena - passivity-based approach (A. Pogromsky, H. Nijmeijer) - we derive sufficient conditions for global/local asymptotic synchronization in the system. Apart from simply showing a possibility of synchronization, we also try to estimate the least possible values for the coupling connections that are sufficient for convergence of the trajectories to the synchronization manifold.

Keywords: synchronization, passive systems, spiking neurons

1. INTRODUCTION

The Hindmarsh and Rose model (J.L. Hindmarsh, 1984) is a reduced version of the celebrated Hudgkin-Huxley equations for modelling spike initiation in the squid giant axon (A.L. Hodgkin, 1952). The model governs the dynamics of the current through the neuron depending on the membrane potential and internal currents in the cell. Despite that the membrane potential in the original model was described by PDE, in Hindmarsh and Rose model the equations were reduced to ODE under assumption that the axon is space-clamped.

Lacking the features of real neuron like dependence of the membrane potential on spacial distance from the soma along the axon, solutions of Hindmarsh and Rose equations capture such inherent property of the neuron like spiking in both periodic and bursting regimes depending on the external stimulation (see figure 1 for the illustration, where symbols x, y, z state for the membrane potential, recovery variable and adaptation current correspondingly). Furthermore, for a specific set of the parameters and input currents, the model can exhibit chaotic dynamics (Kaas-Petersen, 1987) which in turn is essential in the applications where human-like associative memory is required with the ability to retrieve more than one pattern simultaneously (A. Raffone, 2003).

It is suggested in (Malsburg, 1981; Malsburg, 1999) that a process of retrieval of the stored patterns is related to spontaneously occurring syn-
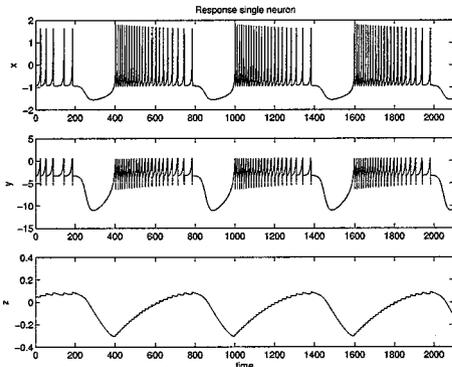
Fig. 1. Spiking dynamics of Hindmarsh and Rose
model

chrony in the arrays (or lattices) of the neurons.
For that reason investigation of the conditions
for synchronization in the ensembles of nonlin-
ear oscillators given by the model in (J.L. Hind-
marsh, 1984) is relevant for both theoretical and
experimental studies of human-like processing of
information.

Most of the published results in the field are
concentrated on numerical investigation of the
phenomenon (see for example (D. Hansel, 1992;
R. Huerta and Rabinovich, 1998)). According to
our knowledge, no successful attempts have been
made to attack the problem of synchronization
in arrays of Hindmarsh and Rose oscillators an-
alytically and especially from control-theoretic
prospective. There are a few publications that try
to apply control-theoretic analysis for the model
(A.E. Milne, 2001). However, applicability of these
and similar approaches is limited by assumptions
on availability of internal variables for direct mea-
surements and due to the requirements to apply
control efforts to every single equation in the sys-
tem. Therefore new theoretical framework is to be
provided to analyze the conditions of synchroniza-
tion in the system.

As a starting point for our theoretical analysis
a recently suggested technique of passivity-based
synchronization has been chosen (Pogromsky,
1998). Within this framework we aim to establish
an analytical proof for synchronization in an en-
semble of Hindmarsh and Rose models and derive
estimates of the coupling strengths for which the
synchronization is guaranteed.

The contribution of the paper is as follows: first
we derive sufficient conditions for synchronization
in a network of Hindmarsh and Rose oscillators.
These conditions should neither depend on the
bounds of the solutions nor should they result
in the growing of the coupling parameter when
the number of oscillators is increasing. Once the

bound for the coupling parameter is defined, we
proceed with a local analysis and provide the con-
ditions for local stability of the synchronization
manifold.

2. NOMENCLATURE AND PRELIMINARIES

In this section we specify the mathematical model
of a Hindmarsh and Rose oscillator and introduce
necessary notations.

A single Hindmarsh and Rose oscillator is defined
by the following system:

$$\begin{aligned}
\dot{x} &= -ax^3 + bx^2 + I + y - z + u \\
\dot{y} &= c - dx^2 - y \\
\dot{z} &= \varepsilon(s(x + x_0) - z),
\end{aligned} \qquad (1)$$

where x is the membrane potential, y – recovery
variable and z – adaptation variable. External
stimulation is given by constant I and input u.
Variable x in (1) is usually considered as a natural
output of the cell. Parameters $a, b, c, d, s, x_0, \epsilon$
are all positive constants. The values of these
parameters are specified in Table 1. A network of

Table 1. Parameters of system (1)

$a = 1$	$b = 3$	$c = 1$	$d = 5$
$s = 4$	$I = 0$	$x_0 = 0.795$	$\varepsilon = 0.001$

oscillators (1) can be described by the following
system:

$$\begin{aligned}
\dot{x}_i &= -ax_i^3 + bx_i^2 + I + y_i - z_i + u_i \\
\dot{y}_i &= c - dx_i^2 - y_i \\
\dot{z}_i &= \varepsilon(s(x_i + x_0) - z_i)
\end{aligned} \qquad (2)$$

where index $i \in \{1, \ldots, n\}$ states for the number
of each oscillator in the network, and u_i is a
coupling function between the nodes.

Definition 1. Let coupling functions $u_i : \mathbb{R}^q \to \mathbb{R}$
be given. Coupling is said to be symmetric iff
$u_i(\nu e_j) = u_j(\nu e_i)$, where $\nu \in \mathbb{R}$, $e_i = (\delta_{ik})_{k=1}^n$,
δ_{ik} - Kronecker delta.

Definition 2. Coupling is said to be uniform iff it
is symmetric, $u_i(\nu e_j) = u_i(\nu e_k)$ and $k, j \neq i$.

Definition 3. Coupling is said to be preserving iff
it vanishes on the synchronization manifold.

We restrict ourselves to a class of linear coupling
functions:

$$\underline{u}(\underline{x}) = -\Gamma \underline{x}, \qquad (3)$$

where $\underline{u} = col(u_1, \ldots, u_2)$, $\underline{x} = col(x_1, \ldots, x_n)$,
$\Gamma = (\gamma_{ij})_{i,j=1}^n$ is an $n \times n$ matrix. The i-th
row of matrix Γ is denoted by symbol Γ_i. It
is clear that symmetric coupling corresponds to

symmetric matrices Γ. In case of uniform coupling it is convenient to factorize matrix Γ as follows:

$$\Gamma = \gamma(\alpha I_n + \Gamma_0), \quad \Gamma_0 = (1 - \delta_{ij}), \quad \gamma, \alpha \in \mathbb{R}, (4)$$

where I_n is the identity matrix of appropriate dimensions.

In our study we exploit *passivity-based* approach to synchronization. Therefore some additional notations are required for consistency. Consider the nonlinear time-invariant system:

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \quad (5)$$

where $x(t) \in \mathbb{R}^k$ - is the state vector, $f : \mathbb{R}^k \to \mathbb{R}^k$, $g : \mathbb{R}^k \to \mathbb{R}^{k \times m}$, $h : \mathbb{R}^k \to \mathbb{R}^l$, $f, g, h \in C^1$; $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^l$ are input and output vectors respectively.

Definition 4. (Pogromsky, 1998) System (5) is called C^r-*semipassive* if there exists a C^r-smooth, $r \geq 0$ nonnegative function $V : \mathbb{R}^k \to \mathbb{R}^+$ and a function $H : \mathbb{R}^k \to \mathbb{R}$ such that for any initial conditions $x(0)$ and any admissible input u the following *dissipation inequality* holds:

$$V(x(t)) - V(x(0)) \leq \int_0^t (y(s)^\top u(s) - H(x(s)))ds \quad (6)$$

for all $0 \leq t \leq T_{u,x_0}$, where the function H is nonnegative outside some ball:

$$\exists \rho > 0 \ |x| \geq \rho \Rightarrow H(x) \geq 0$$

The rest of the paper is organized as follows. In Section 3 we show that system (1) is semipassive with radially unbounded storage function. This fact implies boundedness of the solutions of interconnected system (2) for a class of the coupling functions. Relying on these properties we derive sufficient conditions for global/local asymptotic synchronization in system (2). These are formulated in Sections 4 and 5 respectively. Section 6 concludes the paper.

3. BOUNDEDNESS OF THE SOLUTIONS FOR THE COUPLED SYSTEMS

Proposition 5. System (1) is semipassive with radially unbounded storage function.

Proof of Proposition 5. Consider the following positive-definite function:

$$V(x,y,z) = \tfrac{1}{2}(c_1 x^2 + c_2 y^2 + c_3 z^2) \quad (7)$$

According to Definition 4 the proof is completed if we find nonnegative numbers c_1, c_2, c_3 such that inequality (6) holds for some nonnegative (outside a ball in $\mathbb{R}^3$) function $H(\cdot)$. It is easy to verify

that the following domain of parameters c_i suffices these requirements: $c_1 = 1$; $c_2 < c_1 \frac{4a\lambda_2(1-\lambda_1)}{d^2}$, $0 < \lambda_i < 1$, $i \in 1,2$; $c_3 = \frac{c_1}{\varepsilon s}$. In particular inequality (6) is satisfied for $c_1 = 1, c_2 = 0.01, c_3 = 125$ with

$$H(x,y,z) = 0.455\Big(41403 - (-1.65 + x)^4 -$$
$$(8.98 + x)^2 - 0.1(-642.2 + x^2)^2 - 1.1(x^2 + 0.05y)^2 -$$
$$0.00825(-1.33 + y)^2 - 0.011(-100x + y)^2 -$$
$$0.00275(-159 + z)^2 - 0.272(2.02x + z)^2\Big). \quad (8)$$

The proposition is proven.

Proposition 5 allows us to show boundedness of solutions for the whole ensemble and a class of matrices Γ.

Proposition 6. Let system (2) be given. Let, in addition, coupling function be given by (3) with positive semi-definite Γ. Then solutions of (2) are bounded for any initial conditions.

Proof of Proposition 6. According to Proposition 5 each i-th subsystem in (5) is semi-passive with radially unbounded storage function $V(x_i, y_i, z_i)$. The dissipation inequality for the i-th system in the ensemble can be written as

$$V(x_i(t), y_i(t), z_i(t)) - V(x_i(0), y_i(0), z_i(0)) \leq$$
$$\int_0^t (x_i(s)u_i(s) - H(x_i(s), y_i(s), z_i(s)))ds.$$

Denoting $W(x,y,z) = \sum_{i=1}^n V(x_i, y_i, z_i)$ we obtain

$$W(x(t), y(t), z(t)) - W(x(0), y(0), z(0)) \leq$$
$$\sum_{i=1}^n \int_0^t \big(-x_i(s)\Gamma_i \underline{x}(s) - H(x_i(s), y_i(s), z_i(s))\big)ds$$
$$= \int_0^t \big(-\underline{x}(s)^\top \Gamma \underline{x}(s) - H(x(s), y(s), z(s))\big)ds$$

where $H(x,y,z)$ is nonnegative outside a ball in the extended state space. The rest of the proof is straightforward. *The proposition is proven.*

4. GLOBAL UPPER BOUND GAIN

In this section we provide analytically calculated bounds for the coupling parameter which guarantees asymptotic synchronization of an ensemble of linear, uniform and preserving coupled Hindmarsh and Rose oscillators. The results are formulated in Proposition 7:

Proposition 7. Let system (2) be given with linear, uniform and preserving with respect to the manifold $x_1 = x_2 = \cdots = x_n$ coupling with

$$\gamma > \frac{0.5d^2 + b^2}{n}.$$

Then all solutions of the system are bounded and

$$\lim_{t\to\infty} x_i(t) - x_j(t) = 0,$$
$$\lim_{t\to\infty} y_i(t) - y_j(t) = 0,$$
$$\lim_{t\to\infty} z_i(t) - z_j(t) = 0$$

for any $i, j \in \{1, \ldots, n\}$.

Proof of Proposition 7. According to the conditions of the proposition, coupling function is linear, uniform and preserving with respect to the manifold $x_1 = x_2 = \cdots = x_n$. Then $col(1, \ldots, 1) \in \text{Ker}(\Gamma)$. The last automatically implies that $\alpha = -(n-1)$ in decomposition (4). Hence according to Gershgorin's circle theorem, matrix Γ is positive semi-definite. Therefore, it follows from Proposition 6 that solutions of system (2) are bounded. Let us derive synchronization conditions for (2). Consider the following nonnegative function:

$$V = 0.5 \sum_{i=1}^{n-1} \Big(C_x(x_i - x_{i+1})^2 + C_y(y_i - y_{i+1})^2$$
$$+ C_z(z_i - z_{i+1})^2 \Big), \tag{9}$$

where C_x, $C_y > 0$ are to be defined and $C_z = C_x/(s\varepsilon)$. Its time-derivative can be expressed as follows:

$$\dot{V} = \sum_{i=1}^{n-1} -C_x(x_i - x_{i+1})^2 \Big(\frac{x_i^2}{2} + \frac{x_{i+1}^2}{2} +$$
$$\frac{(x_i + x_{i+1})^2}{2} - b(x_i + x_{i+1}) + \gamma n \Big) +$$
$$C_x(y_i - y_{i+1})(x_i - x_{i+1}) - C_y d(x_i - x_{i+1}) \times$$
$$(x_i + x_{i+1})(y_i - y_{i+1}) - C_y(y_i - y_{i+1})^2$$
$$- C_z \varepsilon(z_i - z_{i+1})^2 \tag{10}$$

Consider the following term in (10):

$$C_x(y_i - y_{i+1})(x_i - x_{i+1}) - C_y d(x_i - x_{i+1}) \times$$
$$(x_i + x_{i+1})(y_i - y_{i+1}) - C_y(y_i - y_{i+1})^2.$$

It can be written as follows:

$$\frac{C_x^2}{4C_y\Delta}(x_i - x_{i+1})^2 - \Big(\frac{C_x^2}{4C_y\Delta} \Big)^{0.5}(x_i - x_{i+1}) -$$
$$\Delta C_y^{0.5}(y_i - y_{i+1}) \Big)^2 + \frac{C_y d^2}{4(1-\Delta)}(x_i - x_{i+1})^2 \times$$
$$(x_i + x_{i+1})^2 - C_y \Big(\frac{d}{2(1-\Delta)^{0.5}}(x_i^2 - x_{i+1}^2) +$$
$$(1-\Delta)^{0.5}(y_i - y_{i+1}) \Big)^2, \quad \Delta \in (0,1)$$

Taking this into account one can rewrite (10) as:

$$\dot{V} \le \sum_{i=1}^{n-1} -C_x(x_i - x_{i+1})^2 \Big(\frac{x_i^2}{2} + \frac{x_{i+1}^2}{2} + \frac{(x_i + x_{i+1})^2}{2}$$
$$- \frac{C_y d^2}{C_x 4(1-\Delta)}(x_i + x_{i+1})^2 - b(x_i + x_{i+1}) + \gamma n$$
$$- \frac{C_x}{4C_y\Delta} \Big) - C_z\varepsilon(z_i - z_{i+1})^2 \tag{11}$$

Let

$$\frac{C_y}{C_x} = \frac{2(1-\Delta)}{d^2}.$$

Then

$$\dot{V} \le \sum_{i=1}^{n-1} -C_x(x_i - x_{i+1})^2 \Big(\frac{1}{2}(x_i - b)^2 + \frac{1}{2}(x_{i+1} - b)^2$$
$$+ \gamma n - \frac{d^2}{8(1-\Delta)\Delta} - b^2 \Big) - C_z\varepsilon(z_i - z_{i+1})^2. \tag{12}$$

Let $\gamma > \big(\frac{d^2}{8(1-\Delta)\Delta} + b^2 \big)/n$. Then we have that

$$\lim_{t\to\infty} \int_0^t (x_i(\tau) - x_{i+1}(\tau))^2 d\tau < \infty$$
$$\lim_{t\to\infty} \int_0^t (z_i(\tau) - z_{i+1}(\tau))^2 d\tau < \infty$$

Furthermore, the system trajectories are bounded and the system right-hand side is continuous. Hence according to Barbalat's lemma we can conclude that

$$\lim_{t\to\infty} (x_i(t) - x_j(t)) = 0, \quad \lim_{t\to\infty} (z_i(t) - z_j(t)) = 0$$

To show that differences $y_i(t) - y_j(t)$ tend to zero as $t \to 0$ it is sufficient to notice that

$$\frac{d}{dt} \big(y_i(t) - y_j(t) \big) = - \big(y_i(t) - y_j(t) \big) + d(x_j^2 - x_i^2),$$

where $d(x_j^2(t) - x_i^2(t)) \to 0$ as $t \to \infty$. The lowest admissible bound $\bar{\gamma}$ for $\gamma(d, b, n, \Delta)$ with respect to Δ can be defined by $\bar{\gamma} = 0.5d^2 + b^2/n$. The proposition is proven.

The proposition provides bounds for γ which are *independent* of the initial conditions, the excitation parameter I in the model and the parameter c which regulates the dynamics of the spikes. Furthermore, it is necessary to point out that the value for $\bar{\gamma}$ is decreasing with the rate of $O(1/n)$ if the number of interconnected oscillators is increasing. This observation is similar to the results in (Pogromsky, 1998) except, however, the fact that the bound for γ in our case is defined explicitly (and only) by the parameters of the model itself.

One question, however, is still open: whether the bound for γ can be lowered? In order to answer this question we should notice that the results formulated in Proposition 7 are global and are independent of the initial conditions. Therefore it is natural to expect that there is room for further

improvements if we assume that only initial conditions in a neighborhood of the synchronization manifold are allowed. The analysis for this case is given in the next section.

5. LOCAL UPPER BOUND

The main idea behind our approach is first to define a neighborhood of the synchronization manifold and then design a Lyapunov candidate with non-positive derivative in the same domain of the system state space. The estimates of the coupling parameter are expected to depend on the size of the domain of admissible initial conditions. The results of this local analysis are formulated in Proposition 8.

Proposition 8. Let system (2) be given, coupling function $\underline{u}(\underline{x})$ be linear, uniform and preserving with respect to the manifold $x_1 = x_2 = \cdots = x_n$. Let, in addition,

$$(x_i(0) - x_j(0))^2 + \frac{3}{d^2 + 2|b - \frac{1}{2}d||d|}(y_i(0) - y_j(0))^2$$

$$+ \frac{1}{s\varepsilon}(z_i(0) - z_j(0))^2 < \delta^2$$

for some $\delta > 0$. Furthermore let $\gamma > \bar{\gamma}_l$, where

$$\bar{\gamma}_l = \frac{1}{n}\left(\left(\frac{|b - \frac{1}{2}d|}{2|d|} + \frac{1}{4}\right)\frac{d^2 + 2|b - \frac{1}{2}d||d|}{3} + \left|1 - \frac{3d^2}{4d^2 + 2|d||b - \frac{1}{2}d|}\right|\delta^2 + \left(|b - \frac{1}{2}d| + 3\left|1 - \frac{d^2}{d^2 + 2|d||b - \frac{1}{2}d|}\right|\right)\mathcal{B}_x\delta\right) \quad (13)$$

Then

$$\lim_{t\to\infty} x_i(t) - x_j(t) = 0$$
$$\lim_{t\to\infty} y_i(t) - y_j(t) = 0$$
$$\lim_{t\to\infty} z_i(t) - z_j(t) = 0$$

Proof of Proposition 8. To prove the proposition consider the Lyapunov candidate given by equation (9). Its time-derivative is defined by (10). Rewrite it as follows:

$$\dot{V} = \sum_{i=1}^{n-1} -C_x(x_i - x_{i+1})^2\left(x_i^2 + x_{i+1}^2 + x_i x_{i+1}\right.$$

$$-b(x_i + x_{i+1}) - \left(\frac{C_x}{C_y} - d(x_i + x_{i+1})\right)^2\frac{C_y}{4C_x} +$$

$$\left.\gamma n\right) - C_y\left((y_i - y_{i+1}) - \left(\frac{C_x}{C_y} - d(x_i + x_{i+1})\right) \times\right.$$

$$\left.\frac{(x_i - x_{i+1})}{2}\right)^2 - C_z(z_i - z_{i+1})^2$$

Moreover, assume that $|x_i - x_{i+1}| \leq \delta$. This automatically implies that $x_{i+1}(t) = x_i(t) + \mu(t)$, where $|\mu(t)| \leq \delta$. Denote $\alpha = \frac{C_x}{C_y}$, then:

$$\dot{V} \leq \sum_{i=1}^{n-1} -C_x(x_i - x_{i+1})^2\left(\left(3 - \frac{d^2}{\alpha}\right)x_i^2 - 2(b -\right.$$

$$\frac{1}{2}d)x_i - \frac{\alpha}{4} + \gamma n\right) - C_x(x_i - x_{i+1})^2\left(\left(1 - \frac{d^2}{4\alpha}\right)\mu^2(t)\right.$$

$$-\left(b - \frac{1}{2}d\right)\mu(t) + \left(3 - \frac{d^2}{\alpha}\right)x_i\mu(t)\right) - C_y\left((y_i - y_{i+1})\right.$$

$$-\left(\frac{C_x}{C_y} - d(x_i + x_{i+1})\right)\frac{(x_i - x_{i+1})}{2}\right)^2$$

$$-C_z(z_i - z_{i+1})^2 \quad (14)$$

Consider term

$$\left(1 - \frac{d^2}{4\alpha}\right)\mu^2(t) - \left(b - \frac{1}{2}d\right)\mu(t) + \left(3 - \frac{d^2}{\alpha}\right)x_i\mu(t)$$

in (14). Given that the solutions of the system are bounded ($|x_i| < \mathcal{B}_x$) and $|\mu(t)| < \delta$, we can estimate it as follows:

$$\left|\left(1 - \frac{d^2}{4\alpha}\right)\mu^2(t) - \left(b - \frac{1}{2}d\right)\mu(t) + \left(3 - \frac{d^2}{\alpha}\right) \times\right.$$

$$\left.x_i\mu(t)\right| \leq \left|1 - \frac{d^2}{4\alpha}\right|\delta^2 + \left(\left|b - \frac{1}{2}d\right| + \left|3 - \frac{d^2}{\alpha}\right|\right)\mathcal{B}_x\delta$$

Taking this estimate and inequality (14) into account we can derive the following:

$$\dot{V} \leq \sum_{i=1}^{n-1} -C_x(x_i - x_{i+1})^2\left(\left(3 - \frac{d^2}{\alpha}\right)x_i^2 -\right.$$

$$2(b - \frac{1}{2}d)x_i - \frac{\alpha}{4} + \gamma n -$$

$$\left|1 - \frac{d^2}{4\alpha}\right|\delta^2 - \left(\left|b - \frac{1}{2}d\right| + \left|3 - \frac{d^2}{\alpha}\right|\right)\mathcal{B}_x\delta\right)$$

$$-C_y\left((y_i - y_{i+1}) - \left(\frac{C_x}{C_y} - d(x_i + x_{i+1})\right) \times\right.$$

$$\left.\frac{(x_i - x_{i+1})}{2}\right)^2 - C_z(z_i - z_{i+1})^2$$

It is clear that $\dot{V}$ will be nonpositive as long as

$$\left(3 - \frac{d^2}{\alpha}\right)x_i^2 - 2(b - \frac{1}{2}d)x_i - \frac{\alpha}{4} + \gamma n - \left|1 - \frac{d^2}{4\alpha}\right|\delta^2$$

$$-\left(\left|b - \frac{1}{2}d\right| + \left|3 - \frac{d^2}{\alpha}\right|\right)\mathcal{B}_x\delta \quad (15)$$

is strictly positive. Consider

$$\left(3 - \frac{d^2}{\alpha}\right)x_i^2 - 2(b - \frac{1}{2}d)x_i - \frac{\alpha}{4}$$

in (15) and let $3 - \frac{d^2}{\alpha} > 0$. Then this term can be written as

$$\left(\left(3 - \frac{d^2}{\alpha}\right)^{0.5}x_i - (b - \frac{1}{2}d)\left(\frac{\alpha}{3\alpha - d^2}\right)^{0.5}\right)^2 -$$

$$\frac{(b - \frac{1}{2}d)^2\alpha}{3\alpha - d^2} - \frac{\alpha}{4}$$

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Notice that the minimal value of

$$\frac{(b - \frac{1}{2}d)^2 \alpha}{3\alpha - d^2} + \frac{\alpha}{4}$$

for $\alpha \in (d^2/3, \infty)$ is equal to

$$\left(\frac{|b - \frac{1}{2}d|}{2|d|} + \frac{1}{4} \right) \frac{d^2 + 2|b - \frac{1}{2}d||d|}{3}$$

for

$$\alpha = \frac{d^2 + 2|b - \frac{1}{2}d||d|}{3}$$

Hence for any

$$\gamma > \frac{1}{n} \left(\left(\frac{|b - \frac{1}{2}d|}{2|d|} + \frac{1}{4} \right) \frac{d^2 + 2|b - \frac{1}{2}d||d|}{3} + \left| 1 - \right.$$

$$\frac{3d^2}{4d^2 + 2|d||b - \frac{1}{2}d|} \left| \delta^2 + \left(\left| b - \frac{1}{2}d \right| + 3 \right| 1 - \right.$$

$$\left. \frac{d^2}{d^2 + 2|d||b - \frac{1}{2}d|} \right| \right) \mathcal{B}_x \delta \right)$$

we get that $\dot{V} \leq 0$. This fact in turn implies that function $V(\cdot)$ is not increasing as soon as $|x_i - x_j| < \delta$. The last inequality can be satisfied by the choice of initial conditions as follows:

$$C_x(x_i(0) - x_j(0))^2 + C_y(y_i(0) - y_j(0))^2 +$$
$$C_z(z_i(0) - z_j(0))^2 < C_x \delta^2. \qquad (16)$$

Taking into account that $C_y/C_x = 1/\alpha$ and that $C_z/C_x = 1/(s\varepsilon)$ we can rewrite inequality (16) as

$$(x_i(0) - x_j(0))^2 + \frac{3}{d^2 + 2|d||b - \frac{1}{2}d|}(y_i(0) - y_j(0))^2$$

$$+ \frac{1}{s\varepsilon}(z_i(0) - z_j(0))^2 < \delta^2$$

The rest of the proof is analogous to that of Proposition 7 and follows explicitly from Barbalat's lemma. *The proposition is proven.*

It is desirable to notice that the estimate (13) for $b = 3$, $d = 5$ and $n = 2$ results in the limit $\delta \to 0$, in the following inequality: $\gamma > 1.5$. This estimate is much closer to the bounds for γ obtained in our computer experiments.

6. CONCLUSION

In this paper we formulated sufficient conditions for asymptotic synchronization in the ensembles of global, linear and uniformly coupled Hindmarsh and Rose oscillators. We have shown that local stability conditions result in significantly smaller value for the coupling parameter γ in comparison to that derived for the arbitrary initial conditions. One of the explicit applications of this result is in defining the domain for the values of the coupling parameters for which the *on-off intermittency* (N. Platt, 1993) effects are more likely to

appear given the specific connections and set of parameters.

We have also shown that sufficient conditions for asymptotic synchronization of linear, preserving and uniformly coupled nodes can be derived as a function of the system parameters which is not explicitly dependent on the bounds of the system solutions. On the other hand, the coupling gain ensuring asymptotic synchronization is decreasing at least as $O(1/n)$ when the number of interconnected systems is growing.

These results, however, are restricted to very specific classes of coupling functions. The more realistic cases would be *diffusive* and nonlinear couplings between the elements of the network. These are topics for our future study.

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