

Asymptotic behavior of nonexpansive semigroups in normed spaces

Citation for published version (APA):

Liu, G. Z. (1985). *Asymptotic behavior of nonexpansive semigroups in normed spaces*. (Eindhoven University of Technology : Dept of Mathematics : memorandum; Vol. 8515). Technische Hogeschool Eindhoven.

Document status and date:

Published: 01/01/1985

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy

If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

702618

EINDHOVEN UNIVERSITY OF TECHNOLOGY

Department of Mathematics and Computing Science

Memorandum 85 - 15

December 1985

ASYMPTOTIC BEHAVIOR OF NONEXPANSIVE SEMIGROUPS IN NORMED SPACES

by

Guizhong Liu

University of Technology
Dept. of Mathematics & Computing Science
Den Dolech 2, P.O. box 513
5600 MB EINDHOVEN
The Netherlands

by

Guizhong Liu

1. Introduction and statement of the results.

A family of operators $S = \{S(t): C \rightarrow C \mid t \geq 0\}$ defined on an arbitrary convex set C in a normed linear space X is said to be a nonexpansive semigroup if $S(0) = I$ (the identity on C), $S(t+s) = S(t)S(s)$, $\forall t, s \geq 0$, $\|S(t)x - S(t)y\| \leq \|x - y\|$, $\forall t \geq 0, \forall x, y \in C$, and $S(t)x: [0, \infty) \rightarrow C$ is continuous for each $x \in C$.

The main result of this paper is:

1.1. Theorem Let C be a convex set in a normed linear space X and $S = \{S(t): C \rightarrow C \mid t \geq 0\}$ be a nonexpansive semigroup of operators defined on it. Then, there exists an $f \in S(X^*) = \{f \in X^* \mid \|f\| = 1\}$ such that for every $x \in C$,

$$\lim_{t \rightarrow \infty} f \left(\frac{S(t)x}{t} \right) = \lim_{t \rightarrow \infty} \left\| \frac{S(t)x}{t} \right\| = \alpha$$

where

$$\alpha = \inf_{x \in C, \varepsilon > 0} \frac{1}{\varepsilon} \|x - S(\varepsilon)x\|.$$

Two immediate consequences are

1.2. Corollary $\frac{S(t)x}{t}$ converges for all $x \in C$, if X has the following property:

(#) every function $x: [0, \infty) \rightarrow X$ satisfying $\|x(t)\| = 1$ for each $t \in [0, \infty)$ and $f(x(t)) \rightarrow 1$ for some $f \in S(X^*)$ must converge.

1.3. Corollary $\frac{S(t)x}{t}$ converges weakly for all $x \in C$ if the following property holds:

every function $x: [0, \infty) \rightarrow X$ satisfying $\|x(t)\| = 1$ for each $t \in [0, \infty)$ (##) and $f(x(t)) \rightarrow 1$ for some $f \in S(X^*)$ must converge weakly.

We note that (#) holds if and only if X is a Banach space whose dual has Fréchet differential norm and (##) holds if and only if X is a strictly convex and reflexive Banach space. (See [1] and [2] .)

It was Kohlberg and Neyman whose work ([2]) on the asymptotic behavior of nonexpansive operators inspired the present work on nonexpansive semigroups. We even deliberately follow the composition of [2]. We also point out that the main result in [2] is an immediate consequence of ours ((1.1.)). It suffices to note that for any given nonexpansive operator $T: C \rightarrow C$ (C is convex) we have the relation between $\{T^n\}$ and the semigroup $\{S(t)\}$ generated by $A = I - T$ that

$$(*) \quad \|S(n)x - T^n x\| \leq \sqrt{n} \|x - Tx\| .$$

Corollaries 1.2 and 1.3 generalize the results in [3] , [4] and [5] , where only semigroups generated by accretive operators are considered. Noting the construction of a nonexpansive T in [2] and the relation (*) between T and the nonexpansive semigroup S generated by $A = I - T$, we see that the converses of corollaries 1.2 and 1.3 are also valid.

Thus we have:

1.4. Theorem The following conditions on a Banach space X are equivalent: :

- (i) X^* has Fréchet differentiable norm.
- (ii) If C is any closed convex subset of X and $S(t): C \rightarrow C$ ($t \geq 0$) is any nonexpansive semigroup on C , then $\frac{S(t)x}{t}$ converges for every $x \in C$ and

1.5. Theorem The following conditions on a Banach space X are equivalent:

- (i) X is strictly convex and reflexive.

(ii) If C is a closed convex subset of X and $S(t): C \rightarrow C$ ($t \geq 0$) is a nonexpansive semigroup, then $\frac{S(t)x}{t}$ converges weakly for every $x \in C$.

2. Proof of the results.

Let the assumptions in Theorem 1.1 be satisfied. Fix $x \in C$ and $\varepsilon > 0$ and suppose $t \geq \varepsilon$. Expressing $t = n\varepsilon + \delta$, where the integer $n \geq 1$ and $\delta \in [0, \varepsilon)$ are uniquely determined, we have

$$\begin{aligned} \|S(t)x - x\| &\leq \|S(t)x - S(\delta)x\| + \|S(\delta)x - x\| \\ &\leq \|S(n\varepsilon)x - x\| + \|S(\delta)x - x\| \\ &\leq \sum_{k=0}^{n-1} \|S((k+1)\varepsilon)x - S(k\varepsilon)x\| + \|S(\delta)x - x\| \\ &\leq nM_\varepsilon(x) + \|S(\delta)x - x\| \end{aligned}$$

where $M_\varepsilon(x) = \|x - S(\varepsilon)x\|$. Thus

$$\|(S(t)x - x) / t\| \leq \frac{n}{t} M_\varepsilon(x) + \max_{0 \leq \delta \leq \varepsilon} \|S(\delta)x - x\| / t$$

from which it follows that

$$(2.1) \quad \limsup_{t \rightarrow \infty} \|S(t)x/t\| \leq \frac{1}{\varepsilon} M_\varepsilon(x).$$

Since $\|S(t)x/t - S(t)y/t\| \leq \|x - y\| / t \rightarrow 0$ ($t \rightarrow \infty$), $\limsup_{t \rightarrow \infty} \|S(t)x/t\| = \limsup_{t \rightarrow \infty} \|S(t)y/t\|$. So, by (2.1) we have

$$(2.2) \quad \limsup_{t \rightarrow \infty} f(S(t)x/t) \leq \limsup_{t \rightarrow \infty} \|S(t)x\| \leq \alpha$$

for any $f \in S(X^*)$, where $\alpha = \inf_{x \in C, \varepsilon > 0} \frac{1}{\varepsilon} M_\varepsilon(x)$.

Thus, to prove Theorem 1.1 it is sufficient to show that there exists an $f \in S(X^*)$ such that, for some $y \in C$, $\liminf_{t \rightarrow \infty} f(S(t)y/t) \geq \alpha$. Assuming, without loss of generality, that $0 \in C$, it is therefore sufficient to show that

$$(2.3) \quad \liminf_{t \rightarrow \infty} f(S(t)0/t) \geq \alpha.$$

Each mapping $S(t): C \rightarrow C$ ($t > 0$) has an obvious extension to a nonexpansive mapping on a closed convex subset of the completion of X . There is no loss of generality in assuming that X is a Banach space and that C is closed. Fix $t_1 > 0$. Since $0 \in C$ and C is closed, if $r > 0$ then $S(t_1)/1+r$ is a contraction mapping that maps C into C , and therefore has a unique fixed point, $x(r)$, satisfying $S(t_1)x(r) = (1+r)x(r)$.

Clearly $\|rx(r)\| = \|S(t_1)x(r) - x(r)\| \geq t_1\alpha$.

For $\alpha = 0$, the theorem follows trivially from (2.2). Now we assume $\alpha > 0$.

For $x \in C$ and $r > 0$ we have

$$\begin{aligned} \|S(t_1)x - x(r)\| &= (1+r) \|S(t_1)x - x(r)\| - r \|S(t_1)x - x(r)\| \\ &\leq \|S(t_1)x - (1+r)x(r)\| - \|rx(r)\| + 2r \|S(t_1)x\| \\ &\leq \|x - x(r)\| - t_1\alpha + 2r \|S(t_1)x\|. \end{aligned}$$

Let $t \geq t_1$. Expressing $t = nt_1 + \delta$, where the integer n and $\delta \in [0, t_1)$ are uniquely determined, and using the above inequality n times we obtain that

$$\begin{aligned} \|S(t)0 - x(r)\| &= \|S^n(t_1)S(\delta)0 - x(r)\| \\ &\leq \|S(\delta)0 - x(r)\| - nt_1\alpha + 2r \sum_{k=1}^n \|S^k(t_1)S(\delta)0\| \\ &\leq \|x(r)\| + \|S(\delta)0\| - nt_1\alpha + 2r \sum_{k=1}^n \|S^k(t_1)S(\delta)0\|. \end{aligned}$$

In what follows, for $x \neq 0$, f_x denotes a functional of norm 1 satisfying

$f_x(x) = \|x\|$. Then, the above inequality implies that

$$\begin{aligned} f_{x(r)}(S(t)0) &= f_{x(r)}(x(r)) + f_{x(r)}(S(t)0 - x(r)) \\ &\geq \|x(r)\| - \|S(t)0 - x(r)\| \\ &\geq nt_1\alpha - \|S(\delta)0\| + 0(r) \quad (r \rightarrow 0). \end{aligned}$$

According to Banach - Alaoglu theorem, there exists a sequence $\{r_i\} \rightarrow 0$

such that $\{f_{x(r_i)}\}$ converges $*$ -weakly to some $f \in X^*$, while $\|f\| \leq 1$.

Thus, it follows from the above inequality that

$$f(S(t)0) \geq nt_1\alpha - \|S(\delta)0\|, \quad \forall t \geq t_1.$$

Dividing both sides by t and letting $t \rightarrow \infty$, we have

$$f\left(\frac{S(t)0}{t}\right) \geq \alpha, \quad \forall t \geq t_1$$

and (2.3) is satisfied with f replaced by $f/\|f\|$. Thus the proof of

Theorem 1.1 is completed.

REFERENCES

- [1] J. Diestel, *Geometry of Banach Spaces - Selected Topics*, Springer-Verlag, 1975.

- [2] E. Kohlberg and A. Neyman, Asymptotic behavior of nonexpansive mappings in normed linear spaces, *Israel J. Math.*, 38 (1981), 269 - 275.

- [3] I. Miyadera, On the infinitesimal generator and the asymptotic behavior of nonlinear contraction semigroups, *Proc. Japan Acad.*, 58, Ser. A (1982), 1 - 4.

- [4] A. Fazy, Asymptotic behavior of contractions in Hilbert space, *Israel J. Math.* 9 (1971), 235 - 240.

- [5] S. Reich, On the asymptotic behavior of nonlinear semigroups and the range of accretive operators II. *J. Math. Anal. Appl.* 87, 134 - 146 (1982).

Department of Mathematics & Computing Science
Eindhoven University of Technology
Den Dolech 2, P.O. box 513
5600 MB EINDHOVEN
The Netherlands (current address)

and

Department of Mathematics
Xi'an Jiaotong University
Xi'an, Shaanxi Province
China

Abstract.

It is proved that if $S = \{S(t): C \rightarrow C \mid t \geq 0\}$ is a nonexpansive semigroup on a convex subset C of a normed linear space X , then there exists an $f \in S(X^*) = \{f \in X^* \mid \|f\| = 1\}$ such that for every $x \in C$,

$$\lim_{t \rightarrow \infty} f \left(\frac{S(t)x}{t} \right) = \lim_{t \rightarrow \infty} \left\| \frac{S(t)x}{t} \right\| = \alpha$$

where $\alpha = \inf_{x \in C, \epsilon > 0} \frac{1}{\epsilon} \|x - S(\epsilon)x\|$. In particular, if X is strictly convex and reflexive, $\lim_{t \rightarrow \infty} \frac{S(t)x}{t}$ converges weakly for every $x \in C$; and if X satisfies the stronger condition that X^* is Fréchet differentiable, then the convergence is strong. We point out that the converses of these statements also hold true.