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1. Introduction and statement of the results.

A family of operators $S = \{S(t): C \to C \mid t \geqq 0\}$ defined on an arbitrary convex set C in a normed linear space X is said to be a nonexpansive semigroup if $S(0) = I$ (the identity on C), $S(t+s) = S(t)S(s)$, $\forall t,s \geqq 0$, $\|S(t)x - S(t)y\| \leq \|x - y\|$, $\forall t \geqq 0$, $\forall x,y \in X$, and $S(t)x: [0,\infty) \to X$ is continuous for each $x \in C$.

The main result of this paper is:

1.1. <u>Theorem</u> Let C be an convex set in a normed linear space X and $S = \{S(t): C \to C \mid t \geqq 0\}$ be a nonexpansive semigroup of operators defined on it. Then, there exists an $f \in S(X^*) = \{f \in X^* \mid \|f\| = 1\}$ such that for every $x \in C$,

$$\lim_{t\to\infty} f\left(\frac{S(t)x}{t}\right) = \lim_{t\to\infty} \left\|\frac{S(t)x}{t}\right\| = \alpha$$

where

$$\alpha = \inf_{x\in C,\varepsilon>0} \frac{1}{\varepsilon}\|x - S(\varepsilon)x\|.$$

Two immediate consequences are

1.2. <u>Corollary</u> $\frac{S(t)x}{t}$ converges for all $x \in C$, if X has the following property:

(#) every function $x: [0,\infty) \to X$ satisfying $\|x(t)\| = 1$ for each $t \in [0,\infty)$ and $f(x(t)) \to 1$ for some $f \in S(X^*)$ must converge.

1.3. <u>Corollary</u> $\frac{S(t)x}{t}$ converges weakly for all $x \in C$ if the following property holds:

every function $x: [0,\infty) \to X$ satisfying $\|x(t)\| = 1$ for each $t \in [0,\infty)$
(##)
and $f(x(t)) \to 1$ for some $f \in S(X^*)$ must converge weakly.

We note that (#) holds if and only if X is a Banach space whose dual has Fréchet differential norm and (##) holds if and only if X is a strictly convex and reflexive Banach space. (See [1] and [2] .)

It was Kohlberg and Neyman whose work ([2]) on the asymptotic behavior of nonexpansive operators inspired the present work on nonexpansive semigroups. We even deliberately follow the composition of [2]. We also point out that the main result in [2] is an immediate consequence of ours ((1.1.)). It suffices to note that for any given nonexpansive operator $T: C \to C$ (C is convex) we have the relation between $\{T^n\}$ and the semigroup $\{S(t)\}$ generated by $A = I - T$ that

(*) $$\| S(n)x - T^n x \| \leq \sqrt{n} \, \| x - Tx \| .$$

Corollaries 1.2 and 1.3 generalize the results in [3] , [4] and [5] , where only semigroups generated by accretive operators are considered. Noting the construction of a nonexpansive T in [2] and the relation (*) between T and the nonexpansive semigroup S generated by $A = I - T$, we see that the converses of corollaries 1.2 and 1.3 are also valid. Thus we have:

1.4. <u>Theorem</u> The following conditions on a Banach space X are equivalent:

(i) X^* has Fréchet differentiable norm.

(ii) If C is any closed convex subset of X and $S(t): C \to C$ $(t \geq 0)$ is any nonexpansive semigroup on C, then $\frac{S(t)x}{t}$ converges for every $x \in C$

and

1.5. <u>Theorem</u> The following conditions on a Banach space X are equivalent:

(i) X is strictly convex and reflexive.

(ii) If C is a closed convex subset of X and $S(t): C \to C$ $(t \geqq 0)$ is a nonexpansive semigroup, then $\frac{S(t)x}{t}$ converges weakly for every $x \in C$.

2. Proof of the results.

Let the assumptions in Theorem 1.1 be satisfied. Fix $x \in C$ and $\varepsilon > 0$ and suppose $t \geqq \varepsilon$. Expressing $t = n\varepsilon + \delta$, where the integer $n \geqq 1$ and $\delta \in [0,\varepsilon)$ are uniquely determined, we have

$$\begin{aligned}\|S(t)x - x\| &\leq \|S(t)x - s(\delta)x\| + \|S(\delta)x - x\| \\ &\leq \|S(n\varepsilon)x - x\| + \|S(\delta)x - x\| \\ &\leq \sum_{k=0}^{n-1} \|S((k+1)\varepsilon)x - S(k\varepsilon)x\| + \|S(\delta)x - x\| \\ &\leq nM_\varepsilon(x) + \|S(\delta)x - x\|\end{aligned}$$

where $M_\varepsilon(x) = \|x - S(\varepsilon)x\|$. Thus

$$\|(S(t)x - x)/t\| \leq \frac{n}{t} M_\varepsilon(x) + \max_{0 \leq \delta \leq \varepsilon} \|S(\delta)x - x\|/t$$

from which it follows that

$$\limsup_{t \to \infty} \|S(t)x/t\| \leq \frac{1}{\varepsilon} M_\varepsilon(x). \tag{2.1}$$

Since $\|S(t)x/t - S(t)y/t\| \leq \|x - y\|/t \to 0$ $(t \to \infty)$, $\limsup_{t \to \infty} \|S(t)x/t\| = \limsup_{t \to \infty} \|S(t)y/t\|$. So, by (2.1) we have

$$\limsup_{t \to \infty} f(S(t)x/t) \leq \limsup_{t \to \infty} \|S(t)x\| \leq \alpha \tag{2.2}$$

for any $f \in S(X^*)$, where $\alpha = \inf_{x \in C, \varepsilon > 0} \frac{1}{\varepsilon} M_\varepsilon(x)$.

Thus, to prove Theorem 1.1 it is sufficient to show that there exists an $f \in S(X^*)$ such that, for some $y \in C$, $\liminf_{t \to \infty} f(S(t)y/t) \geqq \alpha$. Assuming, without loss of generality, that $0 \in C$, it is therefore sufficient to show that

$$\liminf_{t \to \infty} f(S(t)0/t \geqq \alpha. \tag{2.3}$$

Each mapping $S(t): C \to C$ $(t > 0)$ has an obvious extension to a nonexpansive mapping on a closed convex subset of the completion of X. There is no loss of generality in assuming that X is a Banach space and that C is closed.

Fix $t_1 > 0$. Since $0 \in C$ and C is closed, if $r > 0$ then $S(t_1)/1 + r$ is a contraction mapping that maps C into C, and therefore has a unique fixed point, $x(r)$, satisfying $S(t_1)x(r) = (1 + r)x(r)$.

Clearly $\| rx(r) \| = \| S(t_1)x(r) - x(r) \| \geq t_1\alpha$.

For $\alpha = 0$, the theorem follows trivially from (2.2). Now we assume $\alpha > 0$.

For $x \in C$ and $r > 0$ we have

$$\| S(t_1)x - x(r) \| = (1+r) \| S(t_1)x - x(r) \| - r \| S(t_1)x - x(r) \|$$
$$\leq \| S(t_1)x - (1+r)x(r) \| - \| rx(r) \| + 2r \|S(t_1)x\|$$
$$\leq \| x - x(r) \| - t_1\alpha + 2r \| S(t_1)x \| .$$

Let $t \geq t_1$. Expressing $t = nt_1 + \delta$, where the integer n and $\delta \in [0,t_1)$ are uniquely determined, and using the above inequality n times we obtain that

$$\| S(t)0 - x(r) \|$$
$$= \| S^n(t_1)S(\delta)0 - x(r) \|$$
$$\leq \| S(\delta)0 - x(r) \| - nt_1\alpha + 2r \sum_{k=1}^{n} \| S^k(t_1)S(\delta)0 \|$$
$$\leq \| x(r) \| + \| S(\delta)0 \| - nt_1\alpha + 2r \sum_{k=1}^{n} \| S^k(t_1)S(\delta)0\|.$$

In what follows, for $x \neq 0$, f_x denotes a functional of norm 1 satisfying $f_x(x) = \| x \|$. Then, the above inequality implies that

$$f_{x(r)} (S(t)0) = f_{x(r)}(x(r) + f_{x(r)}(S(t)0 - x(r)$$
$$\geq \| x(r) \| - \| S(t)0 - x(r) \|$$
$$\geq nt_1\alpha - \| S(\delta)0 \| + 0(r) \quad (r \to 0).$$

According to Banach - Alaoglu theorem, there exists a sequence $\{r_i\} \to 0$ such that $\{f_{x(r_i)}\}$ converges $*$-weakly to some $f \in X^*$, while $\| f \| \leq 1$.

Thus, it follows from the above inequality that

$$f(S(t)0 \geq nt_1\alpha - \| S(\delta)0 \| , \forall t \geq t_1.$$

Dividing both sides by t and letting $t \to \infty$, we have

$$f\left(\frac{S(t)0}{t}\right) \geq \alpha, \forall t \geq t_1$$

and (2.3) is satisfied with f replaced by $f/\|f\|$. Thus the proof of Theorem 1.1 is completed.

REFERENCES

[1] J. Diestel, Geometry of Banach Spaces - Selected Topics, Spring-Verlag, 1975.

[2] E. Kohlberg and A. Neyman, Asymptotic behavior of nonexpansive mappings in normed linear spaces, Israel J. Math., 38 (1981), 269 - 275.

[3] I. Miyadera, On the infinitesimal generator and the asymptotic behavior of nonlinear contraction semigroups, Proc. Japan Acad., 58, Ser. A (1982), 1 - 4.

[4] A. Pazy, Asymptotic behavior of contractions in Hilbert space, Israel J. Math. 9 (1971), 235 - 240.

[5] S. Reich, On the asymptotic behavior of nonlinear semigroups and the arange of accretive operators II. J. Math. Anal. Appl. 87, 134 - 146 (1982).

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Abstract.

It is proved that if $S = \{S(t):C \to C \mid t \geqq 0\}$ is a nonexpansive semigroup on a convex subset C of a normed linear space X, then there exists an $f \in S(X^*) = \{f \in X^* \mid \|f\| = 1\}$ such that for every $x \in C$,

$$\lim_{t \to \infty} f\left(\frac{S(t)x}{t}\right) = \lim_{t \to \infty} \left\|\frac{S(t)x}{t}\right\| = \alpha$$

where $\alpha = \inf_{x \in C, \varepsilon > 0} \frac{1}{\varepsilon} \|x - S(\varepsilon)x\|$. In particular, if X is strictly convex and reflexive, $\lim_{t \to \infty} \frac{S(t)x}{t}$ converges weakly for every $x \in C$; and if X satisfies the stronger condition that X^* is Fréchet differentiable, then the convergence is strong. We point out that the converses of these statements also hold true.