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ESTIMATION OF PARIS-ERDOGAN LAW PARAMETERS AND THE INFLUENCE OF ENVIRONMENTAL FACTORS ON CRACK GROWTH

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Abstract

Following earlier stochastic models of crack growth a simple function of the crack length follows a normal distribution. From this observation the parameters of the Paris-Erdogan model are readily evaluated without the need to estimate the crack growth rate. Moreover, the approach lends itself to the analysis of properly designed experiments to determine the effect of environmental factors on the parameters of the Paris-Erdogan equation through the medium of accelerated failure time models borrowed from reliability theory.

Introduction

There is now an extensive literature on the subject of the statistical nature of crack growth[1-6]. Most of the literature is concerned with model building and the agreement between the general features of the model and the observed behaviour of the crack. However, little use has been made of the statistical nature of the models to analyse experimental results. The basis of most of the models is the Paris-Erdogan equation relating the rate of growth of crack length, a, after N cycles to the stress intensity, ΔK,

$$\frac{da}{dN} = A(\Delta K)^m$$

where A and m are constants. If the stress intensity is taken to be proportional to $a^{1/2}$ the Paris-Erdogan equation becomes

$$\frac{da}{dN} = \alpha a^{1/2 m} = \alpha a^q, \text{ with } m = 2q \tag{1}$$

which, as is well known, integrates easily to give the length as

$$a = \{a_0^{1-q} - \alpha(q-1)(N-N_0)\}^{1/(1-q)}, \quad \text{for } m>2, \tag{2}$$

and

$$a = a_0 \exp\{\alpha(N-N_0)\}, \quad \text{for } m=2, \tag{3}$$

where a_0 and N_0 are the initial values.

The basis of most data analyses seems to be to take logarithms in (1) and estimate m and α by least squares in the equation

$$\ell n\left[\frac{da}{dN}\right] = \ell n(\alpha) + q\ell n(a) \quad . \tag{4}$$

Unfortunately to use this equation estimates of $\frac{da}{dN}$ are required. Estimates of derivatives are notoriously unreliable. If several repetitions of an experiment under the same conditions are done it is not always clear how to combine the results. Moreover, as a regression model the properties of the estimates of the coefficients in (4) are not the same as those of estimates of the coefficients in (2). Thus it is sensible to ask why the estimation does not proceed directly from the data on crack length and cycles through equation (2) or equation (3). It is interesting to note that if q were known α could be estimated from a straight line

$$a_0^{1-q} - a^{1-q} = \alpha(q-1)(N - N_0).$$

and indeed such a plot for a few values of q is indicative of the nature of the Paris–Erdogan equation in a particular case.

Stochastic models

When the stochastic models are considered the probability density of the crack length is exhibited as a solution of the Fokker–Planck equation[7]. A stationary solution is given by Sobczyk[7] in the form

$$\mathrm{f}(a,N\,|\,a_0,N_0) = a^{-q}\,\frac{1}{\beta\sqrt{2\pi(N-N_0)}}\exp\left\{-\frac{1}{2}\left[\frac{a_0^{1-q} - a^{1-q} - (q-1)\alpha(N-N_0)}{(q-1)\beta\sqrt{N-N_0}}\right]^2\right\} \tag{5}$$

and in non-stationary cases $(N-N_0)$ is replaced by, for example, a measure of accumulated loading[7]. This density function (5) gives the basis for direct analyses based on the crack data. The density says that the function $a_0^{1-q} - a^{1-q}$ is normally distributed with mean $(q-1)\alpha(N-N_0)$ and standard deviation $(q-1)\beta\sqrt{N-N_0}$. Thus using the standard normal distribution function, Φ, we can write the cumulative distribution

$$\mathrm{F}(a,N\,|\,a_0,N_0) = \Phi\left\{ \frac{a_0^{1-q} - a^{1-q} - (q-1)\alpha(N-N_0)}{(q-1)\beta\sqrt{N-N_0}} \right\} \tag{6}$$

although as has been noted elsewhere[5,6] the distribution is defective because there is a finite escape time.

Data analysis for a single crack

The regression model corresponding to this normal distributions states that

$$a_0^{1-q} - a^{1-q} = \alpha(q-1)(N - N_0) + u_c, \tag{7}$$

where u_c is normally distributed with standard deviation $(q-1)\beta\sqrt{N-N_0}$. Because the variance is non-constant (7) is a non-standard model, however, on dividing by $\sqrt{N-N_0}$ the model becomes

$$\frac{a_0^{1-q} - a^{1-q}}{\sqrt{N-N_0}} = \alpha(q-1)\sqrt{N - N_0} + \varepsilon\ . \tag{8}$$

where ε is normally distributed with mean zero and standard deviation $(q-1)\beta$ independent of N. Thus if q is known the estimator of $(q-1)\alpha$ is just the least squares estimator of the coefficient in equation (8) and the estimate of $(q-1)\beta$ is just the estimate of the variance of the regression. It remains to determine what to do about q.

Given the data describing a single crack, say a sequence $\{(a_i,N_i)\}_{i=0}^{n}$, it is easy to construct a log-likelihood[8] using density (5) and estimate the parameters q, α and β by maximum likelihood. The log-likelihood is

$$\mathcal{L}(q,\alpha,\beta\,|\,\{(a_i,N_i)\} \;=\; -q\sum \ell n(a) \;-\; \mathrm{n}\ell n(\beta) \;-\; \frac{1}{2}\sum\left[\frac{a_0^{1-q} - a^{1-q} - (q-1)\alpha(N-N_0)}{(q-1)\beta\sqrt{N-N_0}}\right]^2$$

where for ease of reading the subscript i has been dropped on the right hand side of the equation. Inspection shows that this differs from the standard least squares equation only in the term $-q\sum \ell n(a)$. The likelihood estimators are obtained by solving the equations

$$\frac{\partial\mathcal{L}}{\partial q} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\alpha} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\beta} = 0$$

From the invariance property of likelihood estimators[8] the estimator $\hat{q}$ of q gives the estimator for m directly as $\hat{m} = 2\hat{q}$. In this case the equations have no closed solution. However, it is easy to see that the estimators for α and β given m are the usual least squares estimators for the coefficients in (8) conditioned on q,

$$\hat{\alpha}(q) \;=\; \frac{1}{(q-1)}\left[\,\mathrm{n}a_0^{1-q} - \sum_1^{\mathrm{n}} a^{1-q}\right]\left[\,\sum_1^{\mathrm{n}} (N - N_0)\,\right]^{-1} \tag{9}$$

$$\hat{\beta}^2(q) \;=\; \frac{1}{\mathrm{n}(q-1)^2}\sum \frac{\left[\,a_0^{1-q} - a^{1-q} - \hat{\alpha}(q)(q-1)(N-N_0)\right]^2}{(N - N_0)}$$

and on substituting these back in the log-likelihood gives a function of q alone,

$$\mathcal{L}^*(q) \;=\; -q\sum \ell n(\mathrm{a}) \;-\; \mathrm{n}\ell n\left[\,\hat{\beta}(q)\,\right] \;-\; \frac{\mathrm{n}}{2}\,.$$

Thus the technique is to search for the value of q which maximises $\mathcal{L}^*$ by estimating α and β as functions of q and substituting in $\mathcal{L}$. In this study a simple golden–section search[9] worked very effectively.

Pooling data

When several experiments have been performed it is possible to combine the log–likelihoods from each experiment to give estimators of the parameters of interest. Suppose that several experiments have been performed. Each experiment is labelled with i, i runs from 1 to n, and yields s_i observations. The data is then a set of sequences $\{(a_{ij},N_{ij})\}$, with i=1, .., n, j=1, ..., s_i. The log–likelihood for the whole set of experiments is simply the sum of the log–likelihoods[9] for the individual cracks, writing $\mathcal{L}_i(N_i,\alpha_i,\beta_i)$ for the log–likelihood for the i–th crack gives

$$\mathcal{L}_i(N_i,\alpha_i,\beta_i \,|\, \{(a_{ij},N_{ij})\}) = \ldots$$

$$-q_i\sum_j \ell n(a_{ij}) - s_i \ell n(\beta_i) - \frac{1}{2}\sum_j \left[\frac{a_{i0}^{1-q_i} - a_{ij}^{1-q_i} - (q_i-1)\alpha_i(N_{ij}-N_{i0})}{(q_i-1)\beta_i\sqrt{N_{ij}-N_{i0}}} \right]^2$$

and

$$\mathcal{L} = \sum_i \mathcal{L}_i(q_i,\alpha_i,\beta_i) \quad .$$

The global log–likelihood can be used to investigate explicit parametric models for the parameters, or simply as a way to pool data. In the illustrative example data from 9 cracks are available, and the log–likelihood is used to obtain an estimate of a value of the Paris–Erdogan parameter, m, that is common to all the cracks, while the α's and β's are supposed to reflect the experimental conditions. Estimation by maximum likelihood proceeds exactly as above, the α's and β's are obtained as ordinary least squares estimators from equations like (9) and (10), one for each crack and substituted back into the log–likelihood to yield

$$\mathcal{L}^*(q_1, q_2, \ldots, q_n) = -\sum_i q_i \sum_j \ell n(a_{ij}) - \sum_i s_i \ell n[\ \hat{\beta}_i(q_i)\ ] - \frac{1}{2} \sum_i s_i \ .$$

Moreover, when the cracks are all assumed to be independent with distinct parameters the estimators from the joint log–likelihood are precisely those obtained by estimating from each separately as outlined above. In the simplest case when a common value of m (and thus q) is used and the α's and β's are assumed to absorb most of the experimental variability the joint log–likelihood reduces to, up to a constant,

$$\mathcal{L}^*(q) = -q \sum_i \sum_j \ell n(a_{ij}) - \sum_i s_i \ell n[\ \hat{\beta}_i(q)\ ]$$

which is the form used for the illustrative example.

Example

Data from nine crack growth experiments on two types of (A533B and A508) steel used in the manufacture of pressure vessels was made available to the author. The experiments were carried out in air and in water and the stress amplitude, temperature, and frequency, were also varied. The results are analysed using the approach outlined above to obtain estimators of the Paris–Erdogan parameter, m, and the constants α and β. The results of grouping the experiments are also reported. The calculations and graphics were done with the *PC–MATLAB* package. A FORTRAN program for estimation in the model with a fixed m and different α's and β's is available from the author. In Figure 1 are shown plots of the observed crack length against the estimated crack growth using parameters estimated by simple least squares in equation (4) and against the crack length estimated by the method of maximum likelihood applied to a single crack. It can be seen that the least squares estimates usually underestimate the crack length, and in Figure 1(i) the least squares method predicts an explosion after about 7.5×10^4 cycles. The results are summarised in Table 1.

Summary of experimental conditions

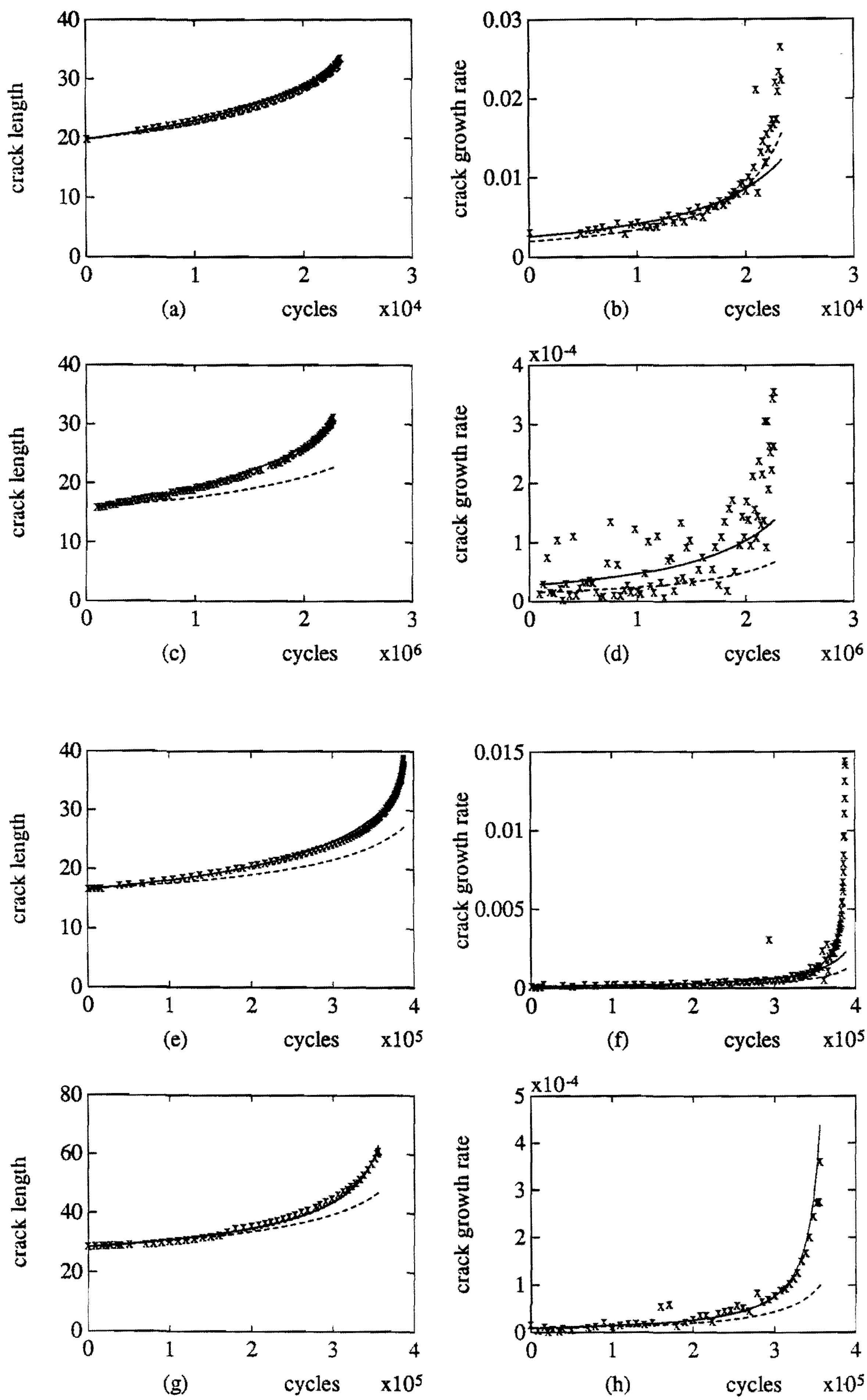
All the test specimens were subjected to a sinusoidal load.

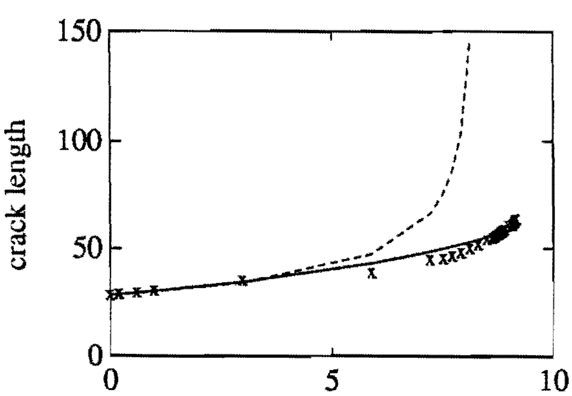
Experiment

A	Steel A533B;	air;	room temp.;	10Hz;	R=0.1;	P_{max}=35kN;	P_{min}=3.5kN
B	Steel A533B;	air;	room temp.;	10Hz;	R=0.1;	P_{max}=9kN;	P_{min}=0.9kN
C	Steel A533B;	air;	room temp.;	10Hz;	R=0.1;	P_{max}=16kN;	P_{min}=1.6kN
D	Steel A533B;	air;	room temp.;	0.5Hz;	R=0.1;	P_{max}=55kN;	P_{min}=5.5kN
E	Steel A533B;	water;	20°C;	0.1Hz;	R=0.1;	P_{max}=50kN;	P_{min}=5kN
F	Steel A533B;	water;	20°C;	0.1Hz;	R=0.1;	P_{max}=55kN;	P_{min}=5.5kN
G	Steel A508;	water;	20°C;	0.1Hz;	R=0.1;	P_{max}=55kN;	P_{min}=5.5kN
H	Steel A508;	water;	20°C;	0.1Hz;	R=0.1;	P_{max}=50kN;	P_{min}=5kN
I	Steel A508;	air;	room temp.;	0.5Hz;	R=0.1;	P_{max}=55kN;	P_{min}=5.5kN

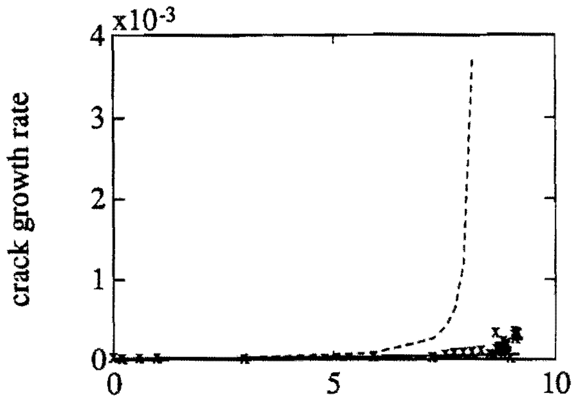
To illustrate the method of pooling the data are taken in four groups: steel A533B at low frequency (experiments A–C); steel A533B at high frequency (experiments D–F); steel A533B at both low and high frequency (experiments A–F); and steel A508 (experiments G–H). The results for A533B at high frequency are summarised in Table 2, and shown in Figure 2. The maximum-likelihood estimators again do better overall. The results for the 2nd to 4th groups are given in Tables 3–5 and Figures 3–5. Again it can be seen that the likelihood method gives closer fit for both crack length and crack growth rates. Clearly other subdivisions of the data could be used. The basic assumption was that the Paris-Erdogan parameter, m, is a fixed material property and that the parameters α and β reflect the effects of the factors (frequency, temperature, presence or not of water) on the rate of growth.

FIGURE 1: Cracks analysed individually

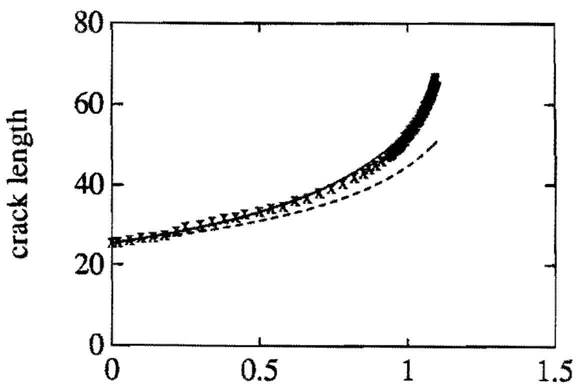




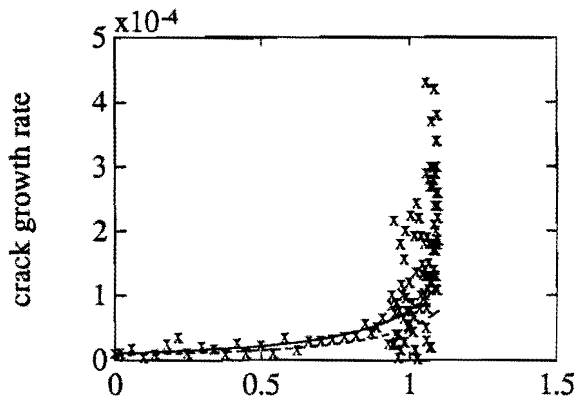
(i) cycles x10⁴



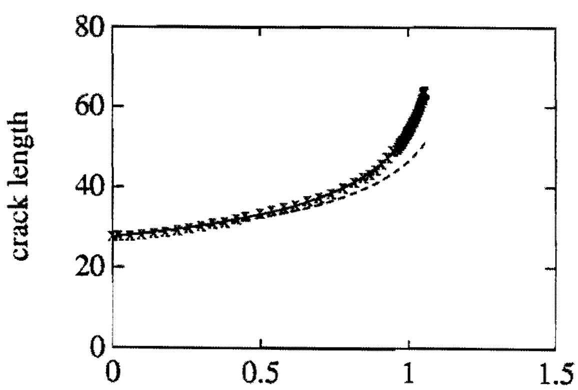
(j) cycles x10⁴



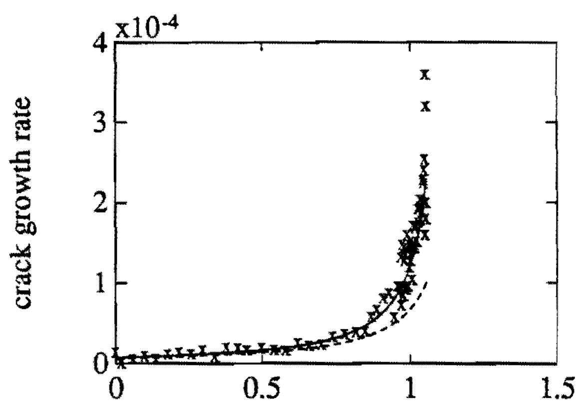
(k) cycles x10⁵



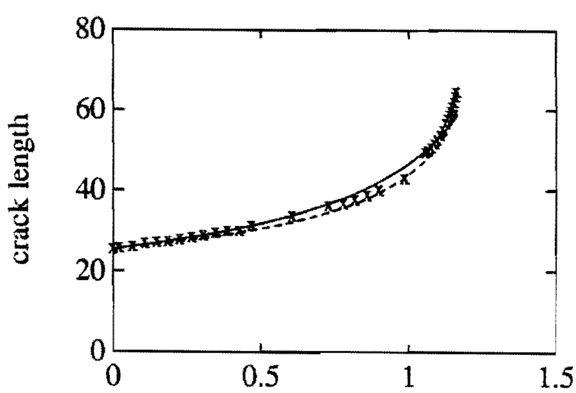
(l) cycles x10⁵



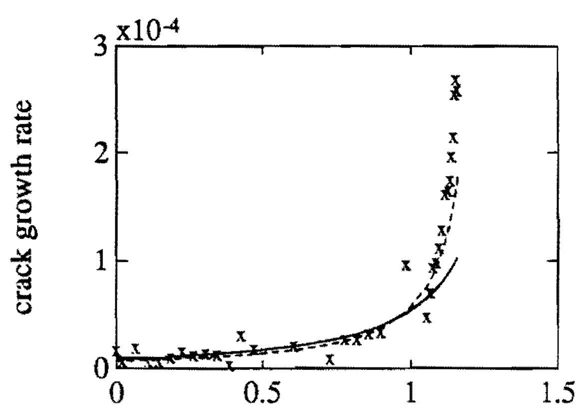
(m) cycles x10⁵



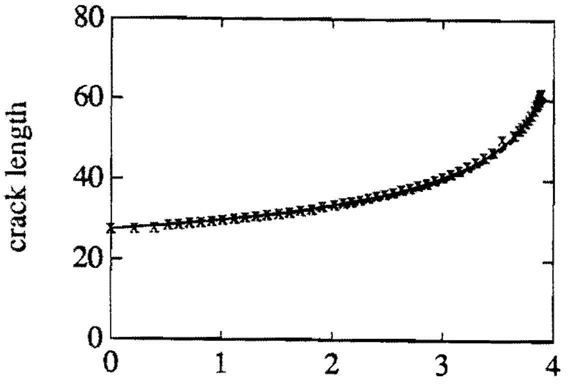
(n) cycles x10⁵



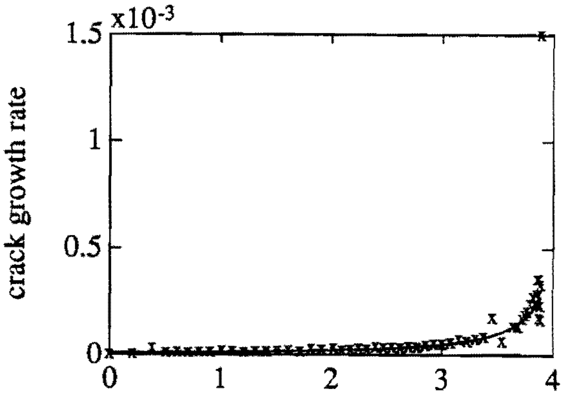
(o) cycles x10⁵



(p) cycles x10⁵



(q)



(r)

TABLE 1: each crack treated indiviually

experiment	likelihood estimators				least squares estimators	
	m	q	α	β	m	α
A	6.35	3.17	1.9749×10^{-8}	5.1699×10^{-8}	8.82	1.2952×10^{-9}
B	5.03	2.52	2.7784×10^{-9}	1.3430×10^{-7}	8.26	5.3604×10^{-11}
C	7.68	3.84	2.7612×10^{-10}	4.1223×10^{-9}	10.64	1.2321×10^{-11}
D	9.55	4.78	2.2686×10^{-12}	8.0426×10^{-11}	9.77	5.3553×10^{-12}
E	3.64	1.82	3.8762×10^{-7}	6.8368×10^{-6}	6.86	3.5318×10^{-9}
F	5.20	2.60	2.4419×10^{-8}	1.9751×10^{-7}	6.48	5.0924×10^{-9}
G	8.24	4.12	9.0141×10^{-11}	8.2903×10^{-10}	8.79	1.0737×10^{-10}
H	5.87	2.94	6.7198×10^{-9}	8.7244×10^{-8}	7.66	8.2238×10^{-10}
I	8.35	4.18	1.9897×10^{-11}	4.5322×10^{-10}	8.82	2.9787×10^{-11}

FIGURE 2: Steel A533B high frequency

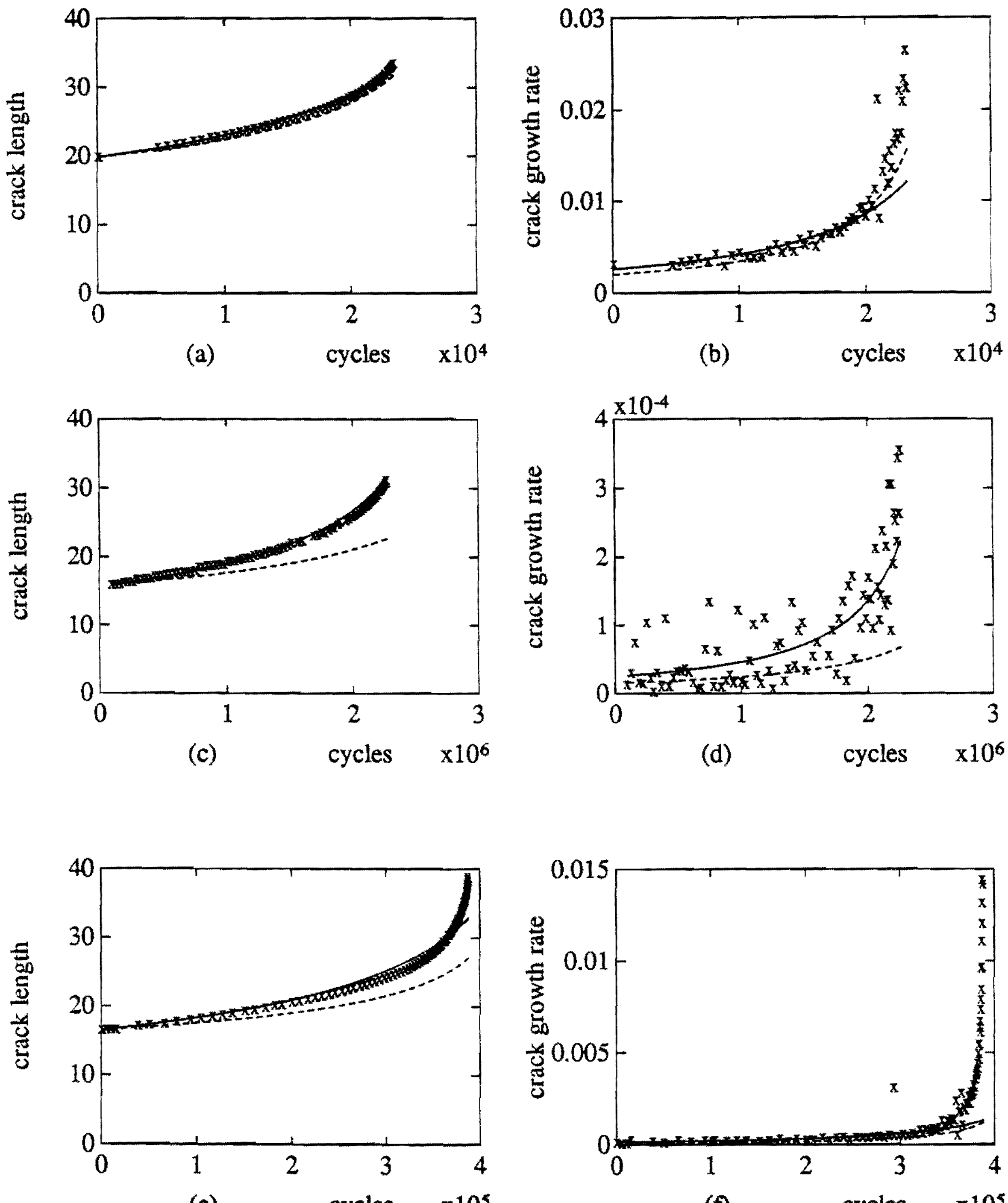


TABLE 2: steel A533B

m=6.30 q=3.15

Experiment	α	β
A	2.1370×10^{-8}	5.6343×10^{-8}
B	4.3325×10^{-10}	2.8941×10^{-8}
C	2.2312×10^{-9}	6.6027×10^{-8}

FIGURE 3: Steel A533B low frequency

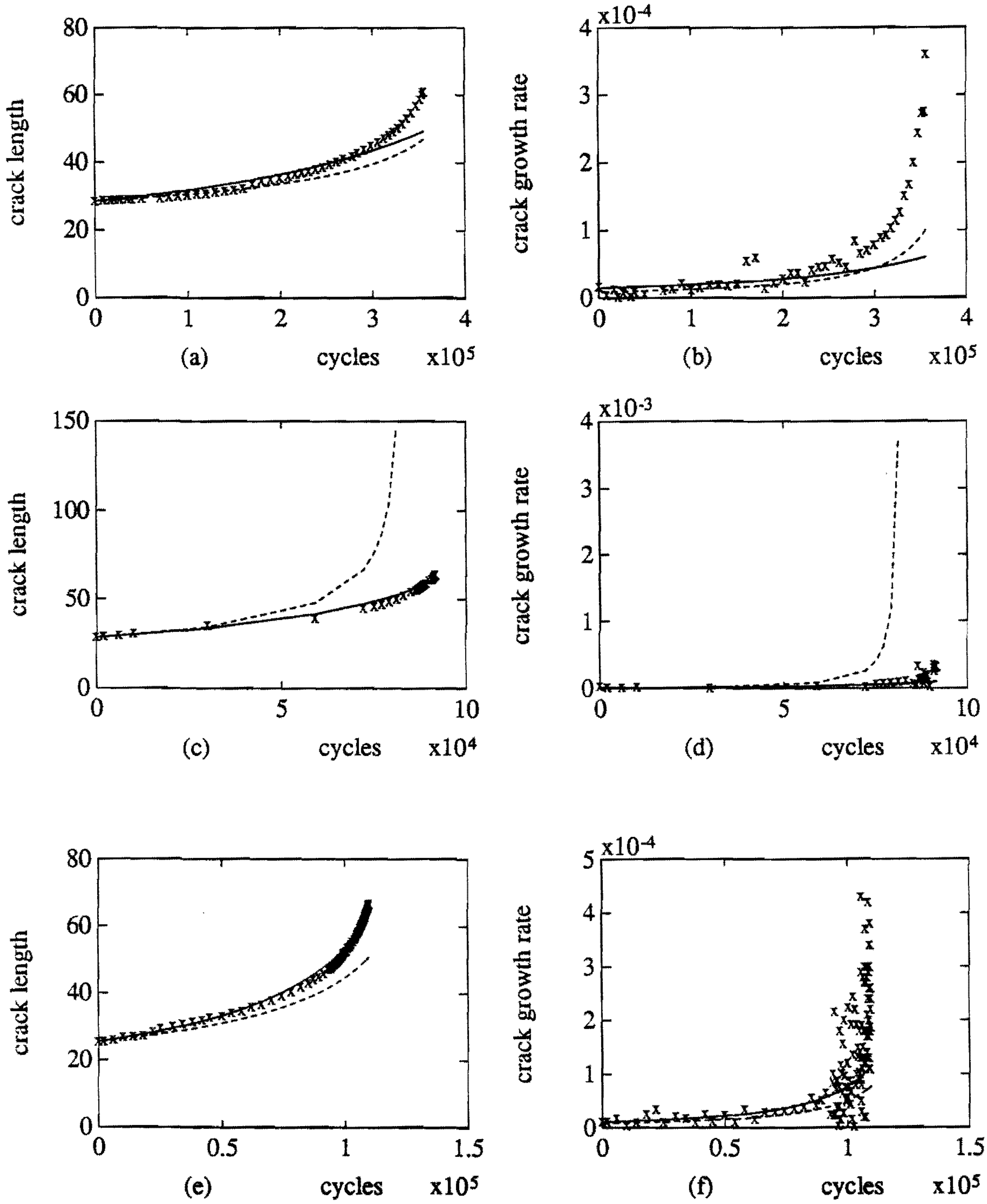
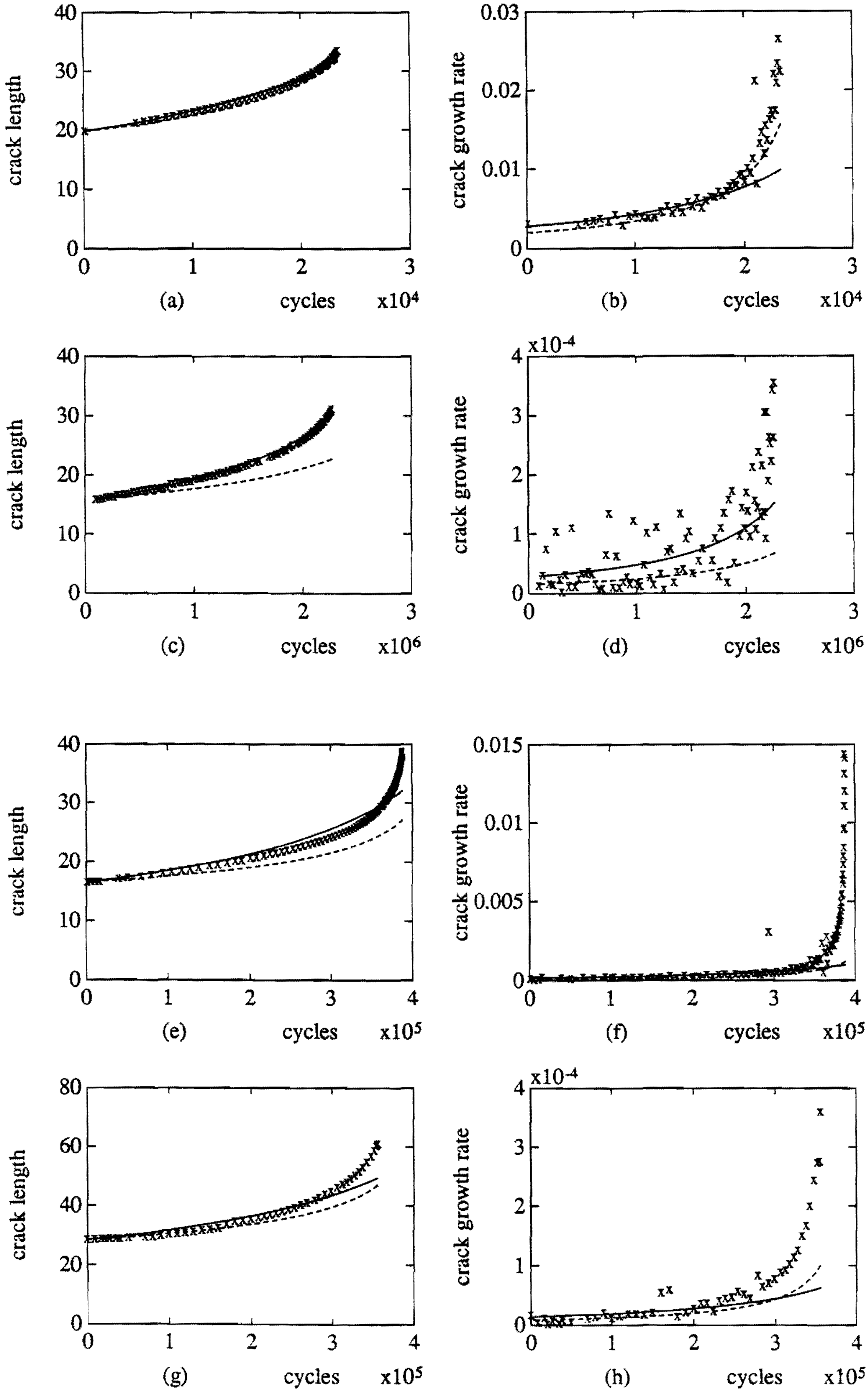


TABLE 3: Steel A533B

m=5.15 q=2.58

Experiment	α	β
D	5.2587×10^{-9}	4.3088×10^{-7}
E	2.4500×10^{-8}	3.9254×10^{-7}
F	2.6763×10^{-8}	2.1924×10^{-7}

FIGURE 4: Steel A533B high and low frequency



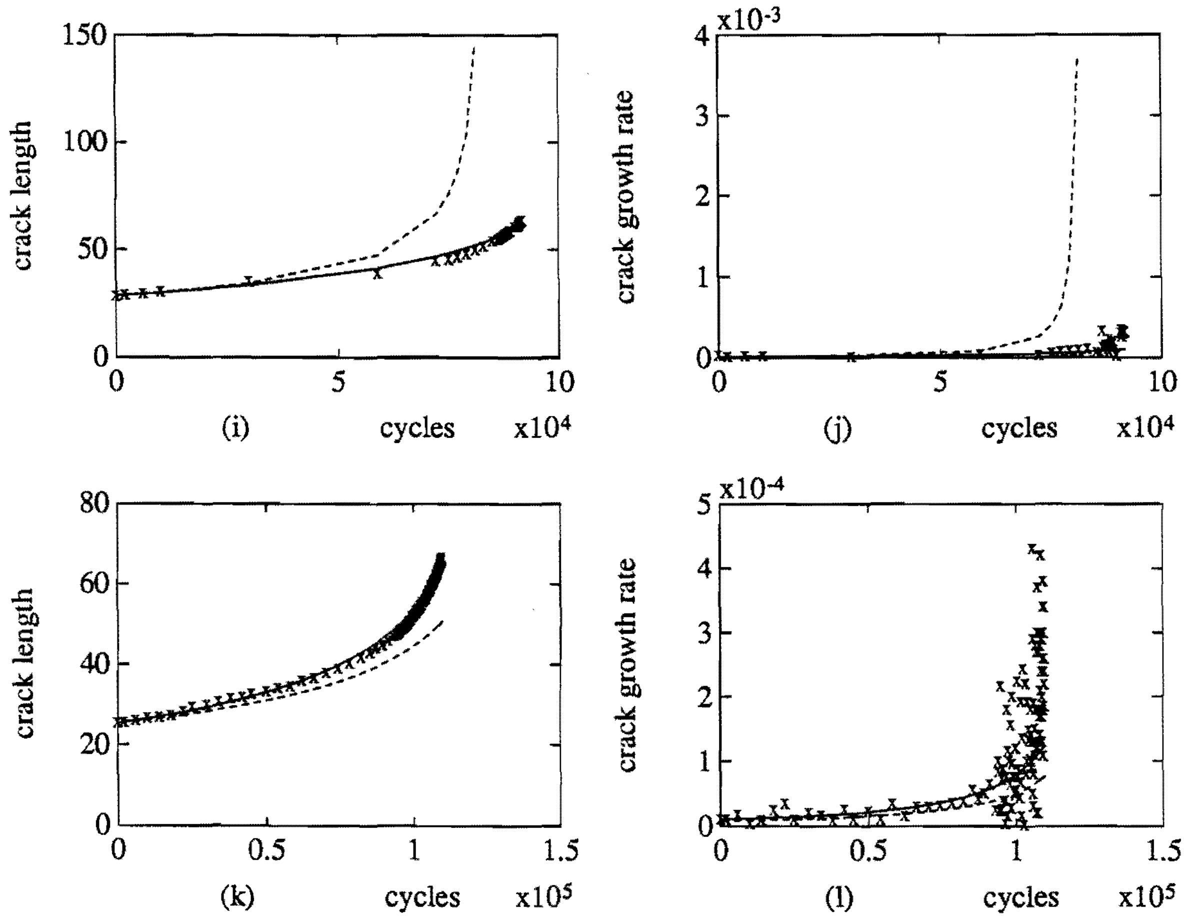


TABLE 4: STEEL A533B

$m=5.34 \quad q=2.67$

Experiment	α	β
A	9.5445×10^{-8}	4.0767×10^{-7}
B	1.7599×10^{-9}	8.5922×10^{-8}
C	9.4805×10^{-9}	4.3461×10^{-7}
D	3.7412×10^{-9}	2.9550×10^{-7}
E	1.7293×10^{-8}	2.8290×10^{-7}
F	1.9050×10^{-8}	1.5147×10^{-7}

FIGURE 5: Steel A508

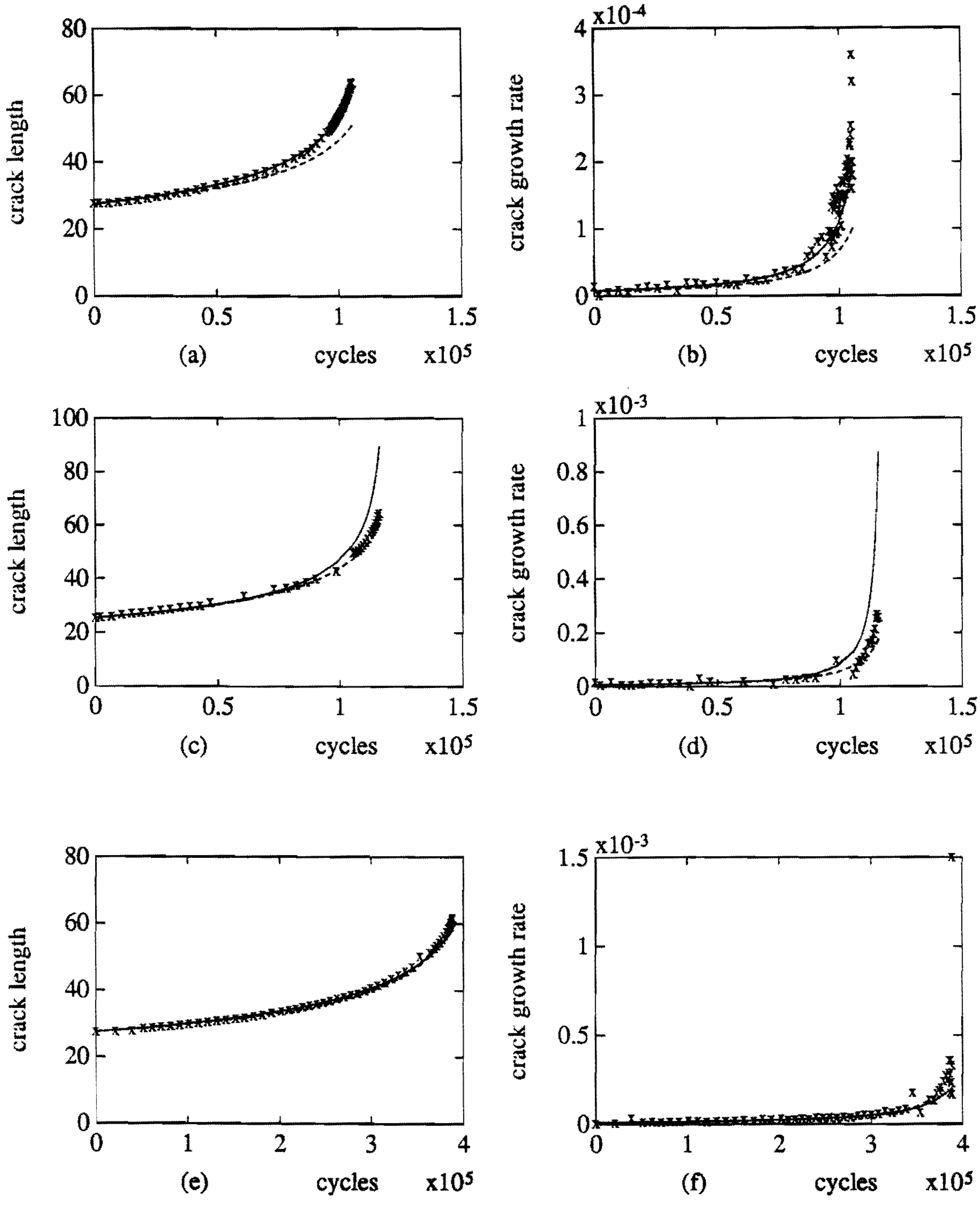


TABLE 5: Steel A508

m=7.97 q=3.98

Experiment	α	β
G	1.4450×10^{-10}	1.4226×10^{-9}
H	1.7748×10^{-10}	4.5093×10^{-9}
I	3.8895×10^{-11}	9.6733×10^{-10}

Extending the Model

The analysis above falls into a group of problems frequently considered by reliability analysts. In those cases the reliability analyst attempts to determine the effect of operating conditions on the failure behaviour of a component or a system. Two standard approaches are commonly used, the *accelerated failure time model*[10], and the *proportional hazards model*[11]. A widely used concept in reliability analysis is that of the hazard rate, h(t), a function of time which gives the chance of failure in the next instant of time given no earlier failure. The hazard rate is related to the distribution F(t) and density f(t) of the lifetimes of the system by

$$h(t) = -\frac{d}{dt}\ell n\{1-F(t)\} = \frac{f(t)}{1-F(t)} \quad .$$

The proportional hazards model assumes that the hazard rate for a system operating under conditions represented by a measurement z consists of two parts and may be written for a measurement z_i as

$$h_i(t) = \Psi(z_i)h_0(t)$$

where h_0 is called the baseline hazard rate and Ψ the relative risk function. When $\Psi(z_i)>1$, the risk of failure is increased and when $\Psi(z_i)<1$, the risk of failure is decreased. Proportional hazards is useful because the analysis of the effects of the conditions can be done in terms of the function Ψ alone without any assumptions about h_0.

The accelerated failure time model assumes that the life times are described by a probability distribution function $G(t|k,\vartheta)=F\left[\frac{t}{\vartheta}|k\right]$, where F is a one parameter distribution with parameter k, $\vartheta=1$ gives F in standard form. The effect of conditions on the system life time is assumed to act through changes in the scale parameter ϑ, that is $\vartheta=\vartheta(z)$. It is easy to see that both α and β play the rôle of a scale parameter in the distribution (5). Thus the accelerated failure time approach can be used to model the effects of experimental conditions on crack growth. The initial steps of the analysis are elementary, but there may well be computational problems. For simplicity of exposition suppose that the distribution (5) describes the length of a crack after N cycles, and that α is a function of the experimental conditions as measured by z but m and β are constant. Suppose further that α can be written as $\alpha=\alpha(z;\lambda,\mu)$ where λ and μ are parameters. Write the likelihood for the i-the experiment as $L(m,\alpha(z_i),\beta)$, then the overall likelihood is

$$\mathcal{L}(m,\underline{\alpha},\beta) = \sum_i L(m,\alpha(z_i),\beta)$$

and the likelihood equations are

$$\frac{\partial\mathcal{L}}{\partial m} = \sum_i \frac{\partial L}{\partial m} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\lambda} = \sum_i \frac{\partial L}{\partial\alpha}\frac{\partial\alpha}{\partial\lambda} = 0$$

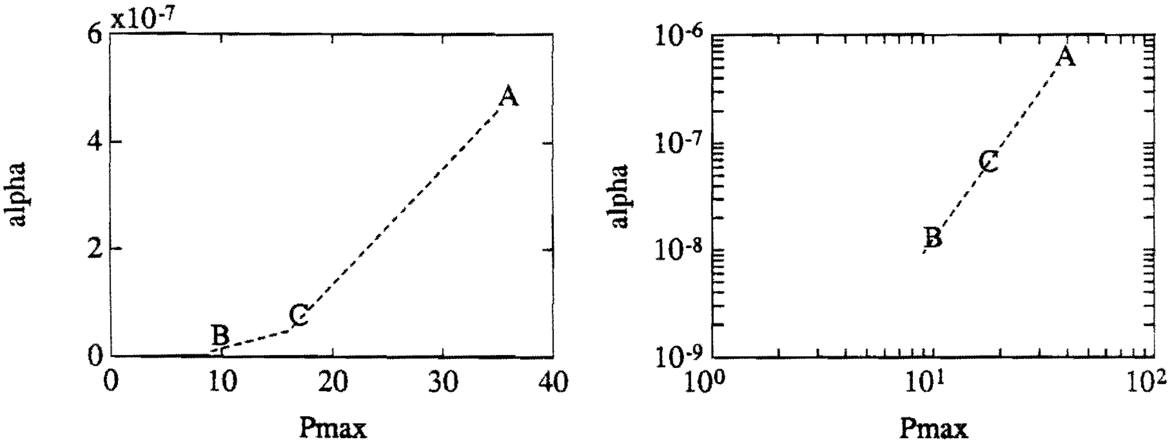
$$\frac{\partial\mathcal{L}}{\partial\mu} = \sum_i \frac{\partial L}{\partial\alpha}\frac{\partial\alpha}{\partial\mu} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\beta} = \sum_i \frac{\partial L}{\partial\beta} = 0$$

with a proper interpretation of the chain rule in the second and third equations. In this way it is easy to extend the approach of the above example to cover explicit parametric models of the effect of operating conditions on the material.

With the data available it is not possible to build models as described here, for example in considering the experiments A–C a two parameter form for α, say α is a function of P_{max}, requires 4 estimating equations, and there are only three observations on the values of P_{max}. However, an idea of the form of dependence may be obtained from simple graphical analyses. In Figure 6 the

FIGURE 6



α values reported in Table 3 are plotted against the P_{max}, on linear scales and on log–log scales, there is some indication of a linear or power relationship between α and P_{max}. It was not possible to conclude anything about the influence of the frequency of loading nor of the influence of the presence of water.

Experimental Design

The author came to this problem as do most statisticians by being asked to analyse data which had already been collected. Clearly there are a number of factors that may affect the rate of fatigue growth in a steel, the experiments used for the example had as variables frequency, stress range, temperature, and the presence or absence of water. However, the experiments were not carried out in a way which allows the best use to be made of the data. These comments also hold for the analysis of a similar problem given by Bhuyan, Swamidas and Vosikovsky[12], while the regression analysis gives no problems, the constants in the Paris–Erdogan equation are the responses and so a more explicit consideration of the choice of levels for the factors would have added to the confidence with which their results could be reported. Experimental design[8] begins by examining what needs to be done to best answer the question of interest, the result of a good design is an economical set of experiments that give the best chance of determining the effects of the variables of interest. In this report the response variables are m, α, and β and to determine the effects on them of the three factors which are varied in this set of experiments would need 30 experiments (three factors, one with two levels, one with three levels, and one with five levels) in a factorial design. The number of experiments can be significantly reduced by using orthogonal designs.

Discussion

The above description of a method of analysis and the example have been used to show how a better approach to the analysis of crack growth data can be developed. The important points are (i) the analysis proceeds directly from the data recorded without the need for intervening transformations and estimations and gives results as least as good as existing methods; (ii) it derives directly from the statistical properties of a now widely used model of crack growth; (iii) it is flexible, and given well designed experiments should be capable of demonstrating the effects of environmental or experimental conditions on the rate of crack propagation.

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