On fractional flow models and equivalent finite state processes

JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **133**, 395–403 (1988)

On Fractional Flow Models and Equivalent Finite State Processes

JAMES FLYNN

Department of Quantitative Business Analysis, Cleveland State University, Cleveland, Ohio 44115

AND

JACOB WIJNGAARD

Department of Industrial Endgineering Eindhoven University of Technology, Eindhoven, The Netherlands

Submitted by E. Stanley Lee

Received November 12, 1986

1. INTRODUCTION

This paper studies the relationships between a dynamic programming model of fractional flows in an n-sector system and certain finite state processes. The former, which is described in Section 2, is a nonstationary version of [4]. Under conditions given in Section 3, the flow model is "equivalent" to an n-state Markovian decision process, allowing for simpler computation. In certain cases where that "equivalence" fails, a partial solution to the flow model follows from the solution to an $(n+1)$-state semi-Markovian decision process. This result, which appears in Section 6, helps to explain why the algorithms of [4, 5] resemble ones for finite state processes. (It does not substantially reduce computation.)

2. THE FLOW MODEL

The system consists of various units, each of which is assigned, according to its condition, to one of the *sectors*, 1, ..., n. The population sizes of the respective sectors are represented by a $1 \times n$ *state* vector $s = (s_i)$. In period $m = 0, 1, \ldots$, the latter is subject to the constraint,

$$su^m = r \qquad \text{and} \qquad s_i \geqslant 0, \qquad 1 \leqslant i \leqslant n, \tag{2.1}$$

0022-247X/88 $3.00

where r and the components of the $n \times 1$ vector $u^m = (u_i^m)$ are positive. The state space S^m is the compact convex set of $1 \times n$ vectors s that satisfy (2.1). (Noninteger s_i's are allowed.) Such a constraint could arise from either a production requirement or a limit on population size (see [4]).

Periodically, one observes the current state and makes decisions that determine the flows into and out of the system and between the sectors. Denote by A_i^m the set of controls in period m on sector i units, $1 \leqslant i \leqslant n$. For each $a = (a_i) \in A^m \equiv \prod_{i=1}^n A_i^m$, there is given an $n \times n$ *flow matrix* $P^m(a) = (P_{ij}^m(a_i))$ and an $n \times 1$ *flow cost vector* $C^m(a) = (C_i^m(a_i))$, $1 \leqslant i, j \leqslant n$. One interpretation is that if s_i units are in sector i and receive control a_i for period m, then by the end of the period a cost $s_i C_i^m(a_i)$ is incurred, $s_i P_{ij}^m(a_i)$ of those units are in sector j, and $s_i(1 - \sum_{k=1}^n P_{ik}^m(a_i))$ of them exit the system, $1 \leqslant i, j \leqslant n$. Of course this assumes substochastic flow matrices, which we do not require (see [4, Section 9.3]).

Replacements enter the system to compensate for exiting units. For period m, denote the controls on the *entrants* by E^m, a set of nonnegative $1 \times n$ vectors $e = (e_i)$, and the costs of bringing entrants into the respective sectors by a fixed $n \times 1$ vector $c^m = (c_i^m)$. To simplify notation, define E^m so that $eu^{m+1} = 1$ $(e \in E^m)$. Each $e \in E^m$ corresponds to a possible "mix" of entrants, the number coming into sector i being proportional to e_i, $1 \leqslant i \leqslant n$. Of course the total number of entrants is determined by (2.1).

This means that if controls $a \in A^m$ and $e \in E^m$ are selected for state s in period m, then the resulting *entrant vector* equals $[r - sP^m(a)u^{m+1}]e \equiv s[u^m - P^m(a)u^{m+1}]e$ and costs $s[u^m - P^m(a)u^{m+1}]ec^m$, so the system incurs a *cost*, $C_{aes}^m \equiv s\{C^m(a) + [u^m - P^m(a)u^{m+1}]ec^m\}$, for period m and moves to a *new state*, $t_{aes}^m \equiv s\{P^m(a) + [u^m - P^m(a)u^{m+1}]e\}$, in period $m+1$. By (2.1), the set of $a \in A$ that are *feasible* at $s \in S$ is $A^m(s) \equiv \{a \in A^m : sP^m(a)u^{m+1} \leqslant r\}$.

Following [4], we require (A1)–(A3) below. Note that (A1) is automatic when E and A are nonempty and finite.

(A1) For $m \geqslant 0$ and $1 \leqslant i, j \leqslant n$, E^m and A_i^m are nonempty and compact, $P_{ij}^m : A_i^m \to R^1$ is continuous, and $C_i^m : A_i^m \to R^1$ is bounded and lower semicontinuous.

(A2) $P^m(a) \geqslant 0$ and $|\sigma(P^m(a))| \leqslant 1$, $(m \geqslant 0$ and $a \in A^m)$.

(A3) For $m \geqslant 0$, $A^m(s) \neq \varnothing$ $(s \in S^m)$ (or equivalently, $\{a \in A^m : P^m(a)u^{m+1} \leqslant u^m\} \neq \varnothing$).

EXAMPLE 2.1. If for some *discount factor* $\beta > 0$ and each $m \geqslant 0$, $u^m = u$, $A^m = A$, $E^m = E$, $P^m(a) \equiv P(a)$, $c^m = \beta^m c$, and $C^m(a) \equiv \beta^m C(a)$, then the flow model is said to be *stationary*.

The models of [4] are stationary. The normalization mapping of Section 4 transforms Example 2.2 below into a stationary model.

EXAMPLE 2.2. Suppose that some $\theta > 0$, some *discount factor* $\beta > 0$, and each $m \geqslant 0$, $u^m = \theta^{-m}u$, $A^m = A$, $E^m = E$, $P^m(a) \equiv P(a)$, $c^m = \beta^m c$, and $C^m(a) \equiv \beta^m C(a)$, there being "expansion," "no growth," or "contraction," respectively as $\theta > 1$, $\theta = 1$, or $0 < \theta < 1$.

In general, a *policy* is a rule for generating a sequence of feasible actions for each initial state $s \in S^0$. The goal is to find an optimal policy, i.e., one that minimizes in some sense the total costs over a specified horizon of length $N \leqslant \infty$. Most of our results require that N is finite or the flow model is stationary with $0 < \beta \leqslant 1$.

3. THE INVARIANT CASE

In Section 4, we indicate how to transform the flow model into an equivalent *normalized* flow model, whose state space is the unit simplex. Our main result, which appears in Section 5, is that under mild conditions, replacing (A3) by the following stronger assumption establishes an equivalence between the normalized flow model and an n-state Markovian decision process:

(A4) $P^m(a)u^{m+1} \leqslant u^m$ ($a \in A^m$ and $m \geqslant 0$).

By exploiting that equivalence, one can determine solutions to the flow model from ones to the n-state process. Under the resulting optimal policies for the flow model, the action depends only on the period $m \geqslant 0$ and not on the state (see Remark 3.3).

It can be shown that (A4) is equivalent to the condition that $A^m(s) = A^m$, for $s \in S^m$ and $m \geqslant 0$, i.e., that the set of available actions does not vary with the state. Following [4], refer to the case where (A4) holds as the *invariant case.*

The invariant case is restrictive; however, it does include instances of substantial interest. Assuming that the transition flow matrices are substochastic, (A4) is automatic whenever

$$u_j^{m+1} > u_i^m \text{ implies } P_{ij}^m(a_i) = 0, \qquad (m \geqslant 0,\ 1 \leqslant i, j \leqslant n,\ a_i \in A_i^m). \quad (3.1)$$

Remark 3.1. If (2.1) is due to a production constraint, then (3.1) holds provided that the "productivities of the units deteriorate from one period to the next" (see [4, Remark 9.1]). Of course (3.1) holds if (2.1) represents a constraint that the population size remains constant (see [4, Section 3]), which is true of some of the examples in [1, Chap. 4; 6, Chap. 5]. (Unfortunately, our model does not incorporate the extra state constraint in (5.4) of [6].)

Remark 3.2. There is a fixed transiton flow matrix P in [7], so the assumption that P be feasible at each state—which seems implicit—suffices for that model to satisfy (A4). The same is true of [10, Section 2(a)] in certain cases.

Remark 3.3. The existence of state-independent optimal policies is not surprising. Pliska [8] studies a model which reduces to an n-state Markovian decision process where that result holds. For stationary flow models that satisfy the invariant case, Flynn [4] proves that result for both the averaging and discounted cost criteria. Denardo and Rothblum [2] do the same thing for models having "affine structure." Stationary flow models that satisfy the invariant case also satisfy their assumptions when N is finite or $0<\beta<1$.

Remark 3.4. By (A3), $A_{\$}^{m} \equiv \{a \in A^{m} : P^{m}(a) u^{m+1} \leqslant u^{m}\} \neq \varnothing$, $(m \geqslant 0)$. Let (\$) denote the model where each A^m is replaced by $A_{\$}^{m}$. Evidently (\$) satisfies the invariant case. An optimal solution to (\$) is suboptimal for the original model, providing an upper bound on its objective function.

4. The Normalized Flow Model

A flow model is *normalized* if its state space always equals $\hat{S}$, the unit simplex in R^n, i.e., if $r=1$ and $u^m = \mathbf{1}$, $(m \geqslant 0)$. Here, we transform the flow model into an equivalent normalized one by using a mapping that is close to Veinott's [11] "similarity transformation."

Let the period $m \geqslant 0$ be arbitrary. Define $U_m = \text{Diag}\{u_1^m, ..., u_n^m\}$, $\sigma_m(s) = (1/r) s U_m$, $(s \in S^m)$, and $\eta_m(e) = e U_{m+1}$, $(e \in E^m)$. Let $\hat{r}=1$, $\hat{u}^m = \mathbf{1}$, $\hat{S}^m = \hat{S} \equiv \sigma_m(S^m)$, $\hat{A}^m = A^m$, $\hat{E}^m = \eta_m(E^m)$, $\hat{c}_m = r U_{m+1}^{-1} c_m$, $\hat{C}^m(a) = r U_m^{-1} C^m(a)$, $(a \in A^m)$, and $\hat{P}^m(a) = U_m^{-1} P^m(a) U_{m+1}$, $(a \in A^m)$. Refer to the model associated with the " ^ "-objects as the *normalized model.*

To see that the original and normlized models are equivalent, the reader should verify that if $a \in A^m$, $e \in E^m$, $s \in S^m$, $\hat{e} = \eta_m(e)$, and $\hat{s} = \sigma_m(s)$, then $C_{aes}^m = \hat{C}_{a\hat{e}\hat{s}}^m \equiv \hat{s}\{\hat{C}^m(a) + [\mathbf{1} - \hat{P}^m(a)\mathbf{1}]\hat{e}\hat{c}_m\}$, $\sigma_{m+1}(t_{aes}^m) = \hat{t}_{a\hat{e}\hat{s}}^m \equiv \hat{s}\{\hat{P}^m(a) + [\mathbf{1} - \hat{P}^m(a)\mathbf{1}]\hat{e}\}$, and $A^m(s) = \hat{A}^m(\hat{s}) \equiv \{a \in A^m : \hat{s}\hat{P}^m(a)\mathbf{1} \leqslant 1\}$.

It is easy to see that $\hat{P}^m(a)$ is transient (resp. substochastic) if and only if $P^m(a)$ is transient (resp. $P^m(a) u^{m+1} \leqslant u^m$), for $a \in A^m$. Thus, the invariant case holds if and only if $\hat{P}^m(a)$ is always substochastic.

Remark 4.1. The reader should verify that Example 2.2 above yields the same normalized model as does a stationary model with discount factor $\beta\theta$, where for each $m \geqslant 0$, $u^m = u$, $A^m = A$, $E^m = E$, $P^m(a) \equiv (1/\theta) P(a)$, $c^m = (\beta\theta)^m c$, and $C^m(a) \equiv (\beta\theta)^m C(a)$.

5. A Markovian Decision Process

Unless stated otherwise, we assume throughout this section that (A1)–(A4) hold and that the flow model is normalized. By Section 4, the latter entails no loss of generality. Theorem 5.1 establishes that under mild additional conditions, the flow model is equivalent to an n-state Markovian decision process. Note that without (A4) that equivalence breaks down (see Remark 5.1).

For $m \geqslant 0$, $a \in A^m$, and $e \in E^m$, define $Q^m(a, e) = (Q^m_{ij}(a_i, e)) = P^m(a) + [I - P^m(a)]\mathbf{1}e$ and $C^m(a, e) = (C^m_i(a_i, e)) = C^m(a) + [I - P^m(a)]\mathbf{1}ec^m$. The next lemma ensures that under this section's assumptions, $Q^m(a, e)$ is always stochastic.

Lemma 5.1. (a) *If the flow model is normalized, then* $C^m_{aes} = sC^m(a, e)$ *and* $t^m_{aes} = sQ^m(a, e)$ *for* $m \geqslant 0$, $a \in A^m$, $e \in E^m$.

(b) *If the flow model is normalized and* (A4) *holds, then* $Q^m(a, e)$ *is stochastic for* $m \geqslant 0$, $a \in A^m$, $e \in E^m$.

Proof. Part (a) is immediate. Fix $m \geqslant 0$, $e \in E^m$, and $a \in A^m$. By normalization, $e\mathbf{1} = 1$, implying $Q^m(a, e)\mathbf{1} = \mathbf{1}$. By (A4), $[I - P^m(a)]\mathbf{1} \geqslant 0$, implying that $Q^m(a, e) \geqslant 0$. Part (b) follows.

Let MDP denote the Markovian decision process whose state space equals $\{1, ..., n\}$ such that for each period $m \geqslant 0$, the set of actions available in state i is $A^m_i \times E^m$, and the effect of selecting (a_i, e) in state i is an immediate cost $C^m_i(a_i, e)$ and a transition one period later to state j with probability $Q^m_{ij}(a_i, e)$, $1 \leqslant i, j \leqslant n$.

If for some discount factor $\beta > 0$ and each $m \geqslant 0$, $A^m = A$, $E^m = E$, $Q^m(a, e) \equiv Q(a, e)$, and $C^m(a, e) \equiv \beta^m C(a, e)$, then MDP is stationary. Evidently MDP is stationary if and only if the flow model is.

A *policy* for MDP is defined by a sequence, $(a^m, e^m(1), ..., e^m(n)) \in A^m \times E^m \times \cdots \times E^m$, $m \geqslant 0$, and the rule: When in state i during period m, select $(a^m_i, e^m(i))$. If $e^m(1) = \cdots = e^m(n) = e^m \in E^m$, $m \geqslant 0$, then the policy is *simple*. There is an obvious *correspondence* between simple policies and sequences, $(a^m, e^m) \in A^m \times E^m$, $m \geqslant 0$.

Recall that a policy for the flow model specifies a sequence of actions, $(a^m, e^m) \in A^m \times E^m$, $m \geqslant 0$, for each initial state $s \in \hat{S}$. (Feasibility is automatic by (A4).) It is natural to identify s with a probability distribution on the states of MDP and (a^m, e^m), $m \geqslant 0$, with the corresponding simple policy for MDP. Then the behavior of MDP under simple policies turns out to be the same as the behavior of the flow model under arbitrary policies. The next assumption ensures the existence of simple optimal policies for MDP.

(A5) There exist real $n \times 1$ vectors $v^m = (v_i^m)$, $0 \leqslant m \leqslant N-1$, such that an optimal policy for MDP results from selecting an $(a_i, e(i)) \in A_i^m \times E^m$ minimizing the expression, $C_i^m(a_i, e(i)) + \sum_{j=1}^n Q_{ij}^m(a_i, e(i))v_j^m$, when the state is i and the period is m, $1 \leqslant i \leqslant n$.

Note that (A5), which is the *only* restriction on the optimality criteria, is quite mild; e.g., it holds for (i) the total cost criterion when the horizon length N is finite, (ii) the discounted total cost criterion when MDP is stationary with $0 < \beta < 1$, and (iii) the averaging criterion when MDP is stationary with $\beta = 1$ and both A and E are finite (see Flynn [3]).

The next theorem summarizes the relationships between MDP and the flow model. Its proof follows that of Lemma 5.2.

THEOREM 5.1. *Suppose that the flow model is normalized and satisfies the invariant case and that* (A5) *holds.*

(a) *There exists a simple optimal policy for* MDP.

(b) *If* $(a^m, e^m) \in A^m \times E^m$, $m \geqslant 0$, *corresponds to a simple optimal policy for* MDP, *then the sequence of actions*, (a^m, e^m), $m \geqslant 0$, *is optimal for all initial states of the flow model.*

(c) *If for some* $s \in \hat{S}$, *the sequence of actions*, (a^m, e^m), $m \geqslant 0$, *is optimal for the flow model when the initial state is* s, *then the simple policy corresponding to* (a^m, e^m), $m \geqslant 0$, *is optimal for* MDP *when the initial probability distribution of the state is* s.

Under the hypotheses of Theorem 5.1, each simple optimal policy for MDP determines an optimal policy for the flow model under which the action depends only on the period $m \geqslant 0$ and not on the state (see Remark 3.3).

LEMMA 5.2. *Suppose that the flow model is normalized and satisfies the invariant case and that* (A5) *holds. Then there exist* $(a^m, e^m) \in A^m \times E^m$, $0 \leqslant\ \leqslant m \leqslant N-1$, *such that a simple optimal policy for* MDP *results from selecting* $(a_i^m, e^m) \in A_i^m \times E^m$ *in period* m *when the state is* i, $1 \leqslant i \leqslant n$.

Proof. Fix $0 \leqslant m \leqslant N-1$. Let v^m satisfy (A5) and let $e^m \in E^m$ minimize $e[c^m + v^m]$ among $e \in E^m$. Fix $1 \leqslant i \leqslant n$. Define $F(a_i, e) = C_i^m(a_i, e) + \sum_{j=1}^n Q_{ij}^m(a_i, e)v_j^m$, $a_i \in A_i^m$, and $e \in E^m$. Let $a_i^m \in A_i^m$ minimize $F(a_i, e^m)$ among $a_i \in A_i^m$. Since m and i are arbitrary, (A5) implies that the theorem is true if (a_i^m, e^m) minimizes $F(a_i, e)$ among $(a_i, e) \in A_i^m \times E^m$. Using the definitions of $C^m(a, e)$ and $Q^m(a, e)$, one can show that $F(a_i, e) = C_i^m(a_i) + \sum_{j=1}^n P_{ij}^m(a_i)v_j^m + [1 - \sum_{j=1}^n P_{ij}^m(a_i)]\, e[c^m + v^m]$, $a_i \in A^m$, and $e \in E^m$. But $\sum_{j=1}^n P_{ij}^m(a_i) \leqslant 1$, $a_i \in A_i^m$, by (A4) and normalization. It follows that for each $a_i \in A_i^m$, e^m minimizes $F(a_i, e)$ among $e \in E^m$. Hence (a_i^m, e^m) minimizes $F(a_i, e)$ among $(a_i, e) \in A_i^m \times E^m$.

Proof of Theorem 5.1. Part (a) is immediate from Lemma 5.2. Suppose that for MDP, the period is $m \geqslant 0$, the probability distribution of the state is $s \in \hat{S}$, and the decision rule is to select action (a_i, e) when the state is i, $1 \leqslant i \leqslant n$, for some $(a, e) \in A^m \times E^m$. Then the expected cost during period m is $sC^m(a, e)$, and the probability distribution of the state in period $(m+1)$ is $sQ^m(a, e)$, which by Lemma 5.1 equals C^m_{aes} and t^m_{aes}, respectively. These observations and (a) give us (b) and (c).

Remark 5.1. MDP is defined when the $Q^m(a, e)$ matrices are stochastic, which by Lemma 5.1(b) is true under (A4). Without (A4), it is possible for the $Q^m(a, e)$ matrices to be stochastic and for our equivalence to break down. As illustrated below, the solutions to MDP may correspond to action that are infeasible for the flow model.

COUNTEREXAMPLE 5.1. Let the flow model be stationary with $0 < \beta < 1$ and normalized. Let $n = 2$, $E = \{(1, 0)\}$, $A_1 = \{0, 1\}$, $A_2 = \{0\}$, $c = (0, 0)'$, $C(a) \equiv (-a_1, 0)'$, $P_{11}(a_1) \equiv a_1/2$, $P_{12}(a_1) \equiv a_1$, $P_{21}(a_2) \equiv P_{22}(a_2) \equiv 0$. One can compute $Q_{11}(a_1) \equiv 1 - a_1$, $Q_{12}(a_1) \equiv a_1$, $Q_{21}(a_2) \equiv 1$, $Q_{22}(a_2) \equiv 0$, and $C(a, e) \equiv (-a_1, 0)'$. Evidently an optimal policy for MDP must select $a_1 = 1$, $a_2 = 0$, and $e = (1, 0)$. The corresponding policy for the flow model always selects $a = (1, 0)$ and $e = (1, 0)$. But selecting $a = (1, 0)$ for the flow model is infeasible at any state $(s_1, s_2) \in \hat{S}$ with $s_1 > \frac{2}{3}$.

6. A Semi-Markovian Decision Process

Our interest shifts from normalized flow models that satisfy the invariant case to the infinite-horizon, stationary flow model of [4], where $|\sigma(\beta P(a))| < 1$ $(a \in A)$, the goal being to minimize the discounted total cost $(0 < \beta < 1)$ or the average cost $(\beta = 1)$. The following program plays a key role in the algorithms of [4, 5] (see Remark 6.1):

$$\begin{aligned} z^\beta &= \max z \\ \text{s.t. } v &\leqslant C(a) + \beta P(a)v - \beta P(a)uz, \qquad a \in A, \\ z &\leqslant e(1/\beta)c + ev, \qquad e \in E. \\ z &\in R^1 \quad \text{and} \quad v \in R^n. \end{aligned} \tag{6.1}$$

This section's main result is that z^β follows from the solution to an $(n+1)$-state semi-Markovian decision process, which helps explain why the algorithms of [4, 5] are so simple. (It does not help computation.)

By [4, Remark 6.5], $|\sigma(\beta P(a))| < 1$ $(a \in A)$, implies that

$$rz^\beta = \min\{F^\beta(a, e) : (a, e) \in A \times E\}, \tag{6.2}$$

where

$$F^{\beta}(a, e) = re[(1/\beta)c + (I - \beta P(a))^{-1}C(a)]/e(I - \beta P(a))^{-1}u. \quad (6.3)$$

Replacing c by $(1/\beta)c$ and $P(a)$ by $\beta P(a)$ in the definition of $F^1(a, e)$ yields $F^{\beta}(a, e)$. Thus one need only study (6.2) for the case where $\beta = 1$ and $P(a)$ is always transient. The properties of its soutions are then quite strong: rz^1 is a lower bound on the average cost, while

$$F^1(a^*, e^*) = \min\{F^1(a, e) : (a, e) \in A \times E\} \text{ and } (a^*, e^*) \in A \times E \quad (6.4)$$

imply that selecting (a^*, e^*) is average cost optimal at $s^* = re^*(I - P(a^*) + P(a^*)ue^*)^{-1}$—the unique state s^* with $t_{a^*e^*s^*} = s^*$—and that the optimal average cost at s^* is rz^* (see [4, Sect. 6]). The solutions to (6.4) correspond to the average cost optimal policies for the semi-Markovian decision process described below.

Assume that $P(a)$ is always transient and substochastic (see Remark 6.2) and that $\beta = 1$. Let SMDP denote the semi-Markovian decision process with state space $\{0, 1, ..., n\}$ and the following dynamics: At time 0 and then immediately after each transition, the current state i is observed. If $i = 0$, then an action $e \in E$ is selected, which results in a cost $(r/e\mathbf{1})ec$ and in a transition zero time units later to state j with probability $(1/e\mathbf{1})e_j$, $1 \leqslant j \leqslant n$. If $i \neq 0$, then an action $a_i \in A_i$ is selected, which results in a cost $rC_i(a_i)$ and in a transition u_i time units later to state 0 or state j with probabilities $[1 - P_{i1}(a_i) - \cdots - P_{in}(a_i)]$ and $P_{ij}(a_i)$, respectively, $1 \leqslant i, j \leqslant n$.

A *stationary policy* for SMDP is defined by an $(a, e) \in A \times E$ and the rule: Select e in state 0 and a_i in state i, $1 \leqslant i \leqslant n$. Since $P(a)$ is always transient, state 0 is recurrent under any stationary policy; moreover, under the one defined by (a, e), the expected total costs and expected time between visits to 0 equal $(r/e\mathbf{1})\, e[c + (I - P(a)]$ and $(1/e\mathbf{1})\, e(I - P(a))^{-1}u$, respectively. By arguments in Ross [9], (a^*, e^*) defines a stationary policy for MDP that minimizes the average cost if and only if it satisfies (6.4).

Remark 6.1. Flynn [5] does not require that $|\sigma(\beta P(a))| < 1$ $(a \in A)$.

Remark 6.2. There is no loss of generality in adding the requirement that $P(a)$ always be substochastic to the one that $P(a)$ always be transient. By [11, Sect. 2], the latter implies the existence of an $n \times n$ diagonal matrix B with positive diagonal elements with $B^{-1}P(a)B$ substochastic. Following Section 4, one can define an equivalent flow model with $P(a)$ always transient and substochastic.

Remark 6.3. The relationship between the flow model and SMDP described in this section is much weaker than the one in the invariant case between the flow model and MDP. One difficulty is that (a^*, e^*) of (6.4) may be infeasible at some states (see [4, Example 6.2]).

References

1. D. Bartholomew, "Stochastic Models for Social Processes," Wiley, New York, 1973.
2. E. Denardo and U. Rothblum, Affine structure and invariant policies for dynamic programs, *Math. Oper. Res.* **8** (1983), 342–366.
3. J. Flynn, On Optimality criteria for dynamic programs with long finite horizons, *J. Math. Anal. Appl.* **76** (1980), 202–208.
4. J. Flynn, A dynamic programming model of fractional flows with application to maintenance and replacement problems, *Math. Oper. Res.* **6** (1981), 405–419.
5. J. Flynn, An algorithm for a dynamic programming model of fractional flows, *Math. Oper. Res.*, in press.
6. R. Grinold and K. Marshall, "Manpower Planning Models," North-Holland, New York, 1977.
7. R. Grinold and R. Stanford, Optimal control of a graded manpower system, *Management Sci.* **8** (1974), 1201–1216.
8. S. Pliska, Optimization of multitype branching processes, *Management Sci.* **23** (1976), 117–124.
9. S. Ross, "Applied Probability Models with Optimization Applications," Holden–Day, San Francisco, 1970.
10. R. Stanford, Analytical solution of a dynamic transaction flow problem, *Math. Programming* **10** (1976), 214–229.
11. A. F. Veinott, Jr., Discrete dynamic programming with sensitive discount optimality criteria, *Ann. Math. Statist.* **40** (1969), 1635–1660.