Saturation in C[a,b] of a special sequence of linear positive operators

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1. Summary and introduction

In this note we investigate the saturation problem for a sequence of linear positive operators $(L_n)_{n=1}^{\infty}$ defined in C[a,b], which are related to distribution functions in the following way.
Let $(Y_i(x))_{i=1}^{\infty}$ be a sequence of random variables depending on a parameter $x \in [a,b]$, mutually independent with a common distribution function $F_{1,x}$ defined on $\mathbb{R}$, such that

$$\int_a^b dF_{1,x}(t) = 1, \quad \int_a^b t\,dF_{1,x}(t) = x .$$

By $X_n(x)$ we denote the mean of $Y_1(x), Y_2(x), \ldots, Y_n(x)$,
$X_n(x) := \frac{1}{n}(Y_1(x) + \ldots + Y_n(x))$, and by $F_{n,x}$ the distribution function of $X_n(x)$.
For expectations we use the notation $E(X)$, where X is a random variable.
Now the sequence $(L_n)_{n=1}^{\infty}$ is defined as follows:

$$(1.1) \qquad L_n(f;x) := E(f(X_n)) = \int_a^b f(\tau)\,dF_{n,x}(\tau) ,$$

where $f \in C[a,b]$.
A well-known example of such a sequence is the sequence of Bernstein operators defined in C[0,1],

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} .$$

In this case we have

$$F_{n,x}(t) = \sum_{k \le nt} \binom{n}{k} x^k (1-x)^{n-k} .$$

The saturation order of the Bernstein operators is given by the sequence $\left(\frac{x(1-x)}{n}\right)_{n=1}^{\infty}$, with trivial class the space of linear functions (cf. [2], p. 102), and saturation class the space of functions with a Lipschitz-continuous derivative. We shall prove that the sequence (L_n) has the same trivial class and saturation class as the Bernstein operators. The saturation order is given by the sequence,

$$\left(\frac{\sigma^2(x)}{n}\right),\ \sigma^2(x) := E((X_n(x) - x)^2) .$$

2. Preliminary notes

We start with a definition of saturation of a sequence of operators (L_n) defined in $C[a,b]$.

Definition (2.1). A sequence of operators (L_n) defined on $[a,b]$ is said to be saturated on $[a,b]$, if there exists a sequence of nonnegative functions $(\varphi_n(x))$ on $[a,b]$, which tends to 0 uniformly on $[a,b]$, and a class $T(L_n)$ of functions such that

$$(2.2)\qquad f(x) - L_n(f;x) = o(\varphi_n(x)),\quad (n \to \infty) ,$$

uniformly on $[a,b]$ if and only if $f \in T(L_n)$, and there exists a function $f_0 \in C[a,b]$, $f_0 \notin T(L_n)$ for which

$$(2.3)\qquad f_0(x) - L_n(f_0;x) = O(\varphi_n(x)),\quad (n \to \infty) ,$$

uniformly on $[a,b]$. We let $S(L_n)$ denote the set of functions for which (2.3) holds. The set $S(L_n)$ is called the saturation class of (L_n) and the set $T(L_n)$ is called the trivial class of (L_n).

We remark that the definition given above is almost identical to the definition given in ([3], p. 123); we don't assume that L_n is an operator from $C[a,b]$ into $C[c,d]$ and that we only require that the functions $\varphi_n(x)$ are nonnegative on $[a,b]$ instead of positive on (a,b).
Now we return to the special sequence (L_n) defined by (1.1). Since,

$$\int_a^b dF_{1,a}(t) = 1 \text{ and } \int_a^b t\,dF_{1,a}(t) = a ,$$

we have

$$0 \le \int_a^b (t-a)^2 dF_{1,a}(t) \le (b-a)\int_a^b (t-a)\,dF_{1,a}(t) = 0 ,$$

hence

$$(2.4)\qquad \sigma^2(a) = 0 .$$

In a similar way we can prove

(2.5) $\sigma^2(b) = 0$.

From (2.4) and (2.5) it follows that

(2.6) $L_n(f;a) = f(a),\ L_n(f;b) = f(b),\ (n = 1,2,\ldots;\ f \in C[a,b])$,

as illustrated in the proof of the following lemma.

Lemma 2.7. If $x \in [a,b]$ is such that $\sigma^2(x) = 0$, then

$$L_n(f;x) = f(x),\ (n = 1,2,\ldots;\ f \in C[a,b]) .$$

Proof. Let $\varepsilon > 0$. Because of the continuity of f at x, there exists a $\delta > 0$ such that $|f(x)-f(t)| < \varepsilon$, provided $|x - t| < \delta$. Therefore,

$$|L_n(f;x) - f(x)| \le \int_a^b |f(x) - f(t)|dF_{n,x}(t) =$$

$$= \int_{|x-t|<\delta} |f(x) - f(t)|dF_{n,x}(t) + \int_{|x-t|\ge\delta} |f(x) - f(t)|dF_{n,x}(t) < \varepsilon +$$

$$+ 2M \int_{|x-t|\ge\delta} dF_{n,x}(t) \le \varepsilon + \frac{2M}{\delta^2}\int_a^b (x-t)^2 dF_{n,x}(t) = \varepsilon ,$$

where $M = \max\{|f(t)|,\ t \in [a,b]\}$. □

Lemma 2.8. The function $\sigma^2(x)$ is bounded on [a,b].

Proof.

$$0 \le \sigma^2(x) = \int_a^b (x - t)^2 dF_{1,x}(t) \le (b - a)^2 .$$ □

From lemma (2.8) it follows that

$$L_n(t^2;x) = E(X_n^2) = \frac{\sigma^2(x)}{n} + x^2$$

tend uniformly to x^2 on [a,b] and since

$$L_n(1;x) = 1,\ L_n(t;x) = x \quad \text{for all } x \in [a,b],$$

we can apply Korovkin's theorem to the convergence of the sequence $L_n(f;x)$ with the following result:

Theorem 2.9. Let $f \in C[a,b]$, then $L_n(f;x) \to f(x)$, $(n \to \infty)$ uniformly on $[a,b]$.

We will end this section with a qualitative result regarding the convergence of the sequence $L_n(f;y)$ if the function $f \in C[a,b]$ is twice continuously differentiable in some neighbourhood of a point $y \in (a,b)$.

Lemma 2.10. Let $f \in C[a,b]$ have a continuous second derivative in a neighbourhood of some point $y \in (a,b)$. Then

$$L_n(f;y) = f(y) + \frac{f''(y)}{2n}\sigma^2(y) + o_y\left(\frac{\sigma^2(y)}{n}\right), \quad (n \to \infty).$$

Proof. First we compute $L_n((t-y)^4;y)$. From the first section of this note there follows that

$$L_n((t-y)^4;y) = E((X_n(y) - y)^4) .$$

Setting $\mu_4(y) = E((Y_1(y) - y)^4)$, then a short calculation shows that

$$(2.11) \qquad L_n((t-y)^4;y) = \frac{3(n-1)}{n^3}\sigma^4(y) + \frac{1}{n^3}\mu_4(y) .$$

Hence,

$$(2.12) \qquad \int_a^b (t-y)^4 dF_{n,y}(t) = O\left(\frac{1}{n^2}\right), \quad (n \to \infty) .$$

If $\sigma^2(y) = 0$, then $L_n(f;y) = f(y)$, so in this case lemma 2.10 is trivial. We now assume that $\sigma^2(y) \neq 0$.

Let $\varepsilon > 0$, then there exists a $\delta > 0$ such that the function $R(y,t)$, defined by

$$f(t) = f(y) + f'(y)(t-y) + \tfrac{1}{2}f''(y)(t-y)^2 + R(y,t)(t-y)^2 ,$$

satisfies the inequality $|R(y,t)| < \varepsilon$, provided $|y - t| < \delta$. Thus,

$$L_n(f;y) - f(y) = \int_a^b (f(t) - f(y))dF_{n,y}(t) =$$

$$= \int_a^b f'(y)(t-y)dF_{n,y})t) + \tfrac{1}{2}\int_a^b f''(y)(t-y)^2 dF_{n,y}(t) +$$

$$+ \int_a^b R(y,t)(t-y)^2 dF_{n,y}(t) = \frac{f''(y)}{2n}\sigma^2(y) + \int_a^b R(y,t)(t-y)^2 dF_{n,y}(t).$$

The following estimation proofs the lemma.

$$\left| \int_a^b R(y,t)(t-y)^2 dF_{n,y}(t) \right| \le \int_{|t-y|<\delta} |R(y,t)|(t-y)^2 dF_{n,y}(t) +$$

$$+ M \int_{|t-y|\ge\delta} |t-y|^2 dF_{n,y}(t) \le \varepsilon \frac{\sigma^2(y)}{n} + \frac{M}{\delta^2} \int_a^b (t-y)^4 dF_{n,y}(t) =$$

$$= \varepsilon \frac{\sigma^2(y)}{n} + O\left(\frac{1}{n^2\delta^2}\right) .$$

Here $M := \max\{|R(y,t)|,\ a \le t \le b\}$. □

3. The saturation of the sequence (L_n)

We start with the definition of a special subset of $C[a,b]$ denoted by $Lip(1,M)$, $(M \ge 0)$.

Definition 3.1. $f \in Lip(1,M)$ if and only if

$$|f(x) - f(y)| \le M|x-y| \quad \text{for all } x,y \in [a,b] .$$

Now we state the main theorem of this note.

Theorem 3.2. The sequence of operators (L_n) defined by (1.1) are saturated with order $\sigma^2(x)/n$ and trivial class (L_n) the set of linear functions on $[a,b]$.

If $f \in C[a,b]$ then

$$|f(x) - L_n(f;x)| \le \frac{M\sigma^2(x)}{2n} ,$$

if and only if $f' \in Lip(1,M)$.

Remark. In fact this theorem is more or less a direct consequence of theorem 5.4 in ([3], p. 136), we prefer to give the whole proof here.

In the proof of theorem 3.2 we need a characterization of those functions $f \in C[a,b]$ for which $f' \in Lip(1,M)$.

Lemma 3.3. Let $f \in C[a,b]$ then the following assertions are equivalent.

i) The function f has a continuous derivative f' with $f' \in \mathrm{Lip}(1,M)$.

ii) For all $x \in (a,b)$ and $h > 0$ with $x-h, x+h \in [a,b]$ the following inequality holds

$$\frac{1}{h^2}|f(x+h) - 2f(x) + f(x-h)| \le M .$$

Proof. It is obvious that i) implies ii).
In order to prove i) from ii) we first show that in each subinterval $(c,d) \subset [a,b]$ there exists a point where f is differentiable. Let ℓ be the linear function such that $\ell(c) = f(c)$ and $\ell(d) = f(d)$. For the function $\varphi(x) := f(x) - \ell(x)$, we have $\varphi(c) = \varphi(d) = 0$, and since $\ell(x+h) - 2\ell(x) + \ell(x-h) = 0$ for all x, we have

$$\frac{1}{h^2}|\varphi(x+h) - 2\varphi(x) + \varphi(x-h)| \le M . \tag{3.4}$$

The function φ attains an extreme value at an interior point $\xi \in (c,d)$ and it follows from (3.4) that φ is differentiable in ξ with $\varphi'(\xi) = 0$. Hence, f is differentiable in ξ.
Let x,y be two arbitrary points in [a,b] with $y = x + nh$ $(n \in \mathbb{N})$. Then

$$\frac{1}{h}(f(x+h) - f(x)) = \frac{1}{h}(f(y) - f(y-h)) - \frac{1}{h}\sum_{k=2}^{n} (f(x+kh) + \\ - 2f(x+kh-h) + f(x+kh-2h)) . \tag{3.5}$$

It follows from ii) and (3.5) that

$$\frac{1}{h}(f(x+h) - f(x)) = \frac{1}{h}(f(y) - f(y-h)) + R(x,y,h) ,$$

where $|R(x,y,h)| \le M|x-y|$, uniformly in h.
Let $\varepsilon > 0$ and let $y \in [a,b]$ be such that $|x - y| < \varepsilon$ and f is differentiable at y. Then for all $h_1, h_2 > 0$ sufficiently small we have

$$\left|\frac{f(x+h_1) - f(x)}{h_1} - \frac{f(x+h_2) - f(x)}{h_2}\right| < (M+1)\varepsilon .$$

Hence, f has a right-hand derivative at x. Applying again ii) we conclude that f is differentiable at x. Moreover, according to (3.6), $f' \in \mathrm{Lip}(1,M)$.□

Proof of theorem 3.2. If the function $f \in C[a,b]$ has a continuous derivative $f' \in \mathrm{Lip}(1,M)$, then

$$|f(x) - L_n(f;x)| = \left| \int_a^b (f(x) - f(t))\, dF_{n,x}(t) \right| =$$

$$= \left| \int_a^b \left(\int_t^x f'(\tau)\, d\tau \right) dF_{n,x}(t) \right| = \left| \int_a^b \left(\int_t^x (f'(\tau) - f'(x))\, d\tau \right) dF_{n,x}(t) \right| \le$$

$$\le \frac{M}{2} \int_a^b (x-t)^2 F_{n,x}(t) = \frac{M}{2n} \sigma^2(x) .$$

Now let $f \in C[a,b]$ be such that $|L_n(f;x) - f(x)| \le \frac{M}{2n} \sigma^2(x)$ and $f' \notin \mathrm{Lip}(1,M)$. Then according to lemma 3.3 there exists a point $x_0 \in (a,b)$ and a number $h > 0$ such that $|f(x_0 - h) - 2f(x_0) + f(x_0 + h)| > Mh^2$.
We assume that

$$(3.7) \qquad f(x_0 - h) - 2f(x_0) + f(x_0 + h) = M_1 h^2, \quad \text{where } M_1 < -M ,$$

otherwise we replace f by $-f$. The function $\varphi(x) := f(x) - \ell(x)$, where ℓ is the linear function with $\ell(x_0 \pm h) = f(x_0 \pm h)$, satisfies the same relation (3.7), and in addition we have

$$(3.8) \qquad |L_n(\varphi;x) - \varphi(x)| = |L_n(f;x) - f(x)| \le \frac{M}{2n} \sigma^2(x), \qquad x \in [a,b] .$$

Let α be a positive number with $M < \alpha < -M_1$ and let C be such that the quadratic function

$$(3.9) \qquad Q(x) := \frac{-\alpha}{2}(x - x_0)^2 + C ,$$

satisfies the inequality

$$(3.10) \qquad Q(x) > \varphi(x), \quad (x \in [x_0 - h, x_0 + h]) .$$

Now we have

$$Q(x_0 \pm h) - \varphi(x_0 \pm h) = Q(x_0 \pm h) = -\frac{\alpha}{2} h^2 + C ,$$

$$Q(x_0) - \varphi(x_0) = \frac{M_1 h^2}{2} + C .$$

So, the function $v(x) := Q(x) - \varphi(x)$ on $[x_0 - h, x_0 + h]$ attains its minimum value m at a point $y \in (x_0 - h, x_0 + h)$. The quadratic function Q^*, defined by

(3.11) $Q^*(x) = Q(x) - m, \quad x \in [a,b]$ has the properties:

(3.12) $Q^*(x) \geq \varphi(x), \quad x \in [x_0-h, x_0+h]$,

$Q^*(y) = \varphi(y)$.

Let

$$a' = \min\{x: x \leq x_0 - h,\ Q^*(x) = \varphi(x)\}$$

and

$$b' = \min\{x: x \geq x_0 + h,\ Q^*(x) = \varphi(x)\} ,$$

then $a \leq a' < y < b' \leq b$ and $Q^*(x) \geq \varphi(x)$ on $[a',b']$.

Let

$$h(x) := \begin{cases} 0, & x \in [a',b'] \\ \varphi(x) - Q^*(x), & x \notin [a',b'] \end{cases} .$$

Then $\varphi(x) \leq Q^*(x) + h(x)$, $(x \in [a,b])$.

Hence

$$L_n(\varphi;y) - \varphi(y) = L_n(\varphi;y) - Q^*(y) \leq L_n(Q^* + h;y) - Q^*(y) =$$

$$= L_n(Q^*,y) - Q^*(y) + L_n(h;y) - h(y) = -\frac{\alpha}{2}\frac{\sigma^2(y)}{n} + o_y\left(\frac{\sigma^2(y)}{n}\right) ,$$

according to lemma 2.10. This contradicts (3.8).

To prove that the set (L_n) consists of all linear function, we have only to remark that

$$|f(x) - L_n(f;x)| = o\left(\frac{\sigma^2(x)}{n}\right), \quad (n \to \infty), \text{ uniformly in } x ,$$

implies $f' \in Lip(1,\varepsilon)$ for all $\varepsilon > 0$. Then f' is constant and so f is linear.□

Remark. The parabola technique, applied in the proof of theorem (3.2) is introduced in [1] by B. Bajšanski and R. Bojanic.

References

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