

# Saturation in $C[a,b]$ of a special sequence of linear positive operators

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# Saturation in $C[a,b]$ of a special sequence of linear positive operators

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## 1. Summary and introduction

In this note we investigate the saturation problem for a sequence of linear positive operators  $(L_n)_{n=1}^{\infty}$  defined in  $C[a,b]$ , which are related to distribution functions in the following way.

Let  $(Y_i(x))_{i=1}^{\infty}$  be a sequence of random variables depending on a parameter  $x \in [a,b]$ , mutually independent with a common distribution function  $F_{1,x}$  defined on  $\mathbb{R}$ , such that

$$\int_a^b dF_{1,x}(t) = 1, \quad \int_a^b t dF_{1,x}(t) = x.$$

By  $X_n(x)$  we denote the mean of  $Y_1(x), Y_2(x), \dots, Y_n(x)$ ,

$X_n(x) := \frac{1}{n}(Y_1(x) + \dots + Y_n(x))$ , and by  $F_{n,x}$  the distribution function of  $X_n(x)$ .

For expectations we use the notation  $E(X)$ , where  $X$  is a random variable.

Now the sequence  $(L_n)_{n=1}^{\infty}$  is defined as follows:

$$(1.1) \quad L_n(f;x) := E(f(X_n)) = \int_a^b f(\tau) dF_{n,x}(\tau),$$

where  $f \in C[a,b]$ .

A well-known example of such a sequence is the sequence of Bernstein operators defined in  $C[0,1]$ ,

$$B_n(f;x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

In this case we have

$$F_{n,x}(t) = \sum_{k \leq nt} \binom{n}{k} x^k (1-x)^{n-k}.$$

The saturation order of the Bernstein operators is given by the sequence  $\left(\frac{x(1-x)}{n}\right)_{n=1}^{\infty}$ , with trivial class the space of linear functions (cf. [2], p. 102), and saturation class the space of functions with a Lipschitz-continuous derivative. We shall prove that the sequence  $(L_n)$  has the same trivial class and saturation class as the Bernstein operators. The saturation

order is given by the sequence,

$$\left(\frac{\sigma^2(x)}{n}\right), \sigma^2(x) := E((X_n(x) - x)^2) .$$

## 2. Preliminary notes

We start with a definition of saturation of a sequence of operators  $(L_n)$  defined in  $C[a,b]$ .

Definition (2.1). A sequence of operators  $(L_n)$  defined on  $[a,b]$  is said to be saturated on  $[a,b]$ , if there exists a sequence of nonnegative functions  $(\varphi_n(x))$  on  $[a,b]$ , which tends to 0 uniformly on  $[a,b]$ , and a class  $T(L_n)$  of functions such that

$$(2.2) \quad f(x) - L_n(f;x) = o(\varphi_n(x)), \quad (n \rightarrow \infty) ,$$

uniformly on  $[a,b]$  if and only if  $f \in T(L_n)$ , and there exists a function  $f_0 \in C[a,b]$ ,  $f_0 \notin T(L_n)$  for which

$$(2.3) \quad f_0(x) - L_n(f_0;x) = O(\varphi_n(x)), \quad (n \rightarrow \infty) ,$$

uniformly on  $[a,b]$ . We let  $S(L_n)$  denote the set of functions for which (2.3) holds. The set  $S(L_n)$  is called the saturation class of  $(L_n)$  and the set  $T(L_n)$  is called the trivial class of  $(L_n)$ .

We remark that the definition given above is almost identical to the definition given in ([3], p. 123); we don't assume that  $L_n$  is an operator from  $C[a,b]$  into  $C[c,d]$  and that we only require that the functions  $\varphi_n(x)$  are nonnegative on  $[a,b]$  instead of positive on  $(a,b)$ .

Now we return to the special sequence  $(L_n)$  defined by (1.1). Since,

$$\int_a^b dF_{1,a}(t) = 1 \text{ and } \int_a^b t dF_{1,a}(t) = a ,$$

we have

$$0 \leq \int_a^b (t - a)^2 dF_{1,a}(t) \leq (b - a) \int_a^b (t - a) dF_{1,a}(t) = 0 ,$$

hence

$$(2.4) \quad \sigma^2(a) = 0 .$$

In a similar way we can prove

$$(2.5) \quad \sigma^2(b) = 0 .$$

From (2.4) and (2.5) it follows that

$$(2.6) \quad L_n(f;a) = f(a), L_n(f;b) = f(b), (n = 1, 2, \dots; f \in C[a, b]) ,$$

as illustrated in the proof of the following lemma.

Lemma 2.7. If  $x \in [a, b]$  is such that  $\sigma^2(x) = 0$ , then

$$L_n(f;x) = f(x), (n = 1, 2, \dots; f \in C[a, b]) .$$

Proof. Let  $\epsilon > 0$ . Because of the continuity of  $f$  at  $x$ , there exists a  $\delta > 0$  such that  $|f(x) - f(t)| < \epsilon$ , provided  $|x - t| < \delta$ . Therefore,

$$\begin{aligned} |L_n(f;x) - f(x)| &\leq \int_a^b |f(x) - f(t)| dF_{n,x}(t) = \\ &= \int_{|x-t| < \delta} |f(x) - f(t)| dF_{n,x}(t) + \int_{|x-t| \geq \delta} |f(x) - f(t)| dF_{n,x}(t) < \epsilon + \\ &+ 2M \int_{|x-t| \geq \delta} dF_{n,x}(t) \leq \epsilon + \frac{2M}{\delta^2} \int_a^b (x-t)^2 dF_{n,x}(t) = \epsilon , \end{aligned}$$

where  $M = \max\{|f(t)|, t \in [a, b]\}$ . □

Lemma 2.8. The function  $\sigma^2(x)$  is bounded on  $[a, b]$ .

Proof.

$$0 \leq \sigma^2(x) = \int_a^b (x-t)^2 dF_{1,x}(t) \leq (b-a)^2 . \quad \square$$

From lemma (2.8) it follows that

$$L_n(t^2;x) = E(X_n^2) = \frac{\sigma^2(x)}{n} + x^2$$

tend uniformly to  $x^2$  on  $[a, b]$  and since

$$L_n(1;x) = 1, L_n(t;x) = x \quad \text{for all } x \in [a, b],$$

we can apply Korovkin's theorem to the convergence of the sequence  $L_n(f;x)$  with the following result:

Theorem 2.9. Let  $f \in C[a,b]$ , then  $L_n(f;x) \rightarrow f(x)$ ,  $(n \rightarrow \infty)$  uniformly on  $[a,b]$ .

We will end this section with a qualitative result regarding the convergence of the sequence  $L_n(f;y)$  if the function  $f \in C[a,b]$  is twice continuously differentiable in some neighbourhood of a point  $y \in (a,b)$ .

Lemma 2.10. Let  $f \in C[a,b]$  have a continuous second derivative in a neighbourhood of some point  $y \in (a,b)$ . Then

$$L_n(f;y) = f(y) + \frac{f''(y)}{2n} \sigma^2(y) + o_y\left(\frac{\sigma^2(y)}{n}\right), \quad (n \rightarrow \infty).$$

Proof. First we compute  $L_n((t-y)^4; y)$ . From the first section of this note there follows that

$$L_n((t-y)^4; y) = E((X_n(y) - y)^4).$$

Setting  $\mu_4(y) = E((Y_1(y) - y)^4)$ , then a short calculation shows that

$$(2.11) \quad L_n((t-y)^4; y) = \frac{3(n-1)}{n^3} \sigma^4(y) + \frac{1}{n^3} \mu_4(y).$$

Hence,

$$(2.12) \quad \int_a^b (t-y)^4 dF_{n,y}(t) = o\left(\frac{1}{n^2}\right), \quad (n \rightarrow \infty).$$

If  $\sigma^2(y) = 0$ , then  $L_n(f;y) = f(y)$ , so in this case lemma 2.10 is trivial. We now assume that  $\sigma^2(y) \neq 0$ .

Let  $\epsilon > 0$ , then there exists a  $\delta > 0$  such that the function  $R(y,t)$ , defined by

$$f(t) = f(y) + f'(y)(t-y) + \frac{1}{2}f''(y)(t-y)^2 + R(y,t)(t-y)^2,$$

satisfies the inequality  $|R(y,t)| < \epsilon$ , provided  $|y-t| < \delta$ . Thus,

$$\begin{aligned} L_n(f;y) - f(y) &= \int_a^b (f(t) - f(y)) dF_{n,y}(t) = \\ &= \int_a^b f'(y)(t-y) dF_{n,y}(t) + \frac{1}{2} \int_a^b f''(y)(t-y)^2 dF_{n,y}(t) + \end{aligned}$$

$$+ \int_a^b R(y,t) (t-y)^2 dF_{n,y}(t) = \frac{f''(y)}{2n} \sigma^2(y) + \int_a^b R(y,t) (t-y)^2 dF_{n,y}(t).$$

The following estimation proofs the lemma.

$$\begin{aligned} \left| \int_a^b R(y,t) (t-y)^2 dF_{n,y}(t) \right| &\leq \int_{|t-y| < \delta} |R(y,t)| (t-y)^2 dF_{n,y}(t) + \\ &+ M \int_{|t-y| \geq \delta} |t-y|^2 dF_{n,y}(t) \leq \epsilon \frac{\sigma^2(y)}{n} + \frac{M}{\delta^2} \int_a^b (t-y)^4 dF_{n,y}(t) = \\ &= \epsilon \frac{\sigma^2(y)}{n} + O\left(\frac{1}{n^2 \delta^2}\right). \end{aligned}$$

Here  $M := \max\{|R(y,t)|, a \leq t \leq b\}$ . □

### 3. The saturation of the sequence $(L_n)$

We start with the definition of a special subset of  $C[a,b]$  denoted by  $Lip(1,M)$ , ( $M \geq 0$ ).

Definition 3.1.  $f \in Lip(1,M)$  if and only if

$$|f(x) - f(y)| \leq M|x - y| \quad \text{for all } x,y \in [a,b].$$

Now we state the main theorem of this note.

Theorem 3.2. The sequence of operators  $(L_n)$  defined by (1.1) are saturated with order  $\sigma^2(x)/n$  and trivial class  $(L_n)$  the set of linear functions on  $[a,b]$ .

If  $f \in C[a,b]$  then

$$|f(x) - L_n(f;x)| \leq \frac{M\sigma^2(x)}{2n},$$

if and only if  $f' \in Lip(1,M)$ .

Remark. In fact this theorem is more or less a direct consequence of theorem 5.4 in ([3], p. 136), we prefer to give the whole proof here.

In the proof of theorem 3.2 we need a characterization of those functions  $f \in C[a,b]$  for which  $f' \in Lip(1,M)$ .

Lemma 3.3. Let  $f \in C[a,b]$  then the following assertions are equivalent.

- i) The function  $f$  has a continuous derivative  $f'$  with  $f' \in \text{Lip}(1,M)$ .
- ii) For all  $x \in (a,b)$  and  $h > 0$  with  $x-h, x+h \in [a,b]$  the following inequality holds

$$\frac{1}{h^2} |f(x+h) - 2f(x) + f(x-h)| \leq M .$$

Proof. It is obvious that i) implies ii).

In order to prove i) from ii) we first show that in each subinterval  $(c,d) \subset [a,b]$  there exists a point where  $f$  is differentiable. Let  $\ell$  be the linear function such that  $\ell(c) = f(c)$  and  $\ell(d) = f(d)$ . For the function  $\varphi(x) := f(x) - \ell(x)$ , we have  $\varphi(c) = \varphi(d) = 0$ , and since  $\ell(x+h) - 2\ell(x) + \ell(x-h) = 0$  for all  $x$ , we have

$$(3.4) \quad \frac{1}{h^2} |\varphi(x+h) - 2\varphi(x) + \varphi(x-h)| \leq M .$$

The function  $\varphi$  attains an extreme value at an interior point  $\xi \in (c,d)$  and it follows from (3.4) that  $\varphi$  is differentiable in  $\xi$  with  $\varphi'(\xi) = 0$ . Hence,  $f$  is differentiable in  $\xi$ .

Let  $x, y$  be two arbitrary points in  $[a,b]$  with  $y = x + nh$  ( $n \in \mathbb{N}$ ). Then

$$(3.5) \quad \frac{1}{h}(f(x+h) - f(x)) = \frac{1}{h}(f(y) - f(y-h)) - \frac{1}{h} \sum_{k=2}^n (f(x+kh) + 2f(x+kh-h) + f(x+kh-2h)) .$$

It follows from ii) and (3.5) that

$$\frac{1}{h}(f(x+h) - f(x)) = \frac{1}{h}(f(y) - f(y-h)) + R(x,y,h) ,$$

where  $|R(x,y,h)| \leq M|x-y|$ , uniformly in  $h$ .

Let  $\epsilon > 0$  and let  $y \in [a,b]$  be such that  $|x-y| < \epsilon$  and  $f$  is differentiable at  $y$ . Then for all  $h_1, h_2 > 0$  sufficiently small we have

$$\left| \frac{f(x+h_1) - f(x)}{h_1} - \frac{f(x+h_2) - f(x)}{h_2} \right| < (M+1)\epsilon .$$

Hence,  $f$  has a right-hand derivative at  $x$ . Applying again ii) we conclude that  $f$  is differentiable at  $x$ . Moreover, according to (3.6),  $f' \in \text{Lip}(1,M)$ .  $\square$



Proof of theorem 3.2. If the function  $f \in C[a,b]$  has a continuous derivative  $f' \in \text{Lip}(1,M)$ , then

$$\begin{aligned} |f(x) - L_n(f;x)| &= \left| \int_a^b (f(x) - f(t)) dF_{n,x}(t) \right| = \\ &= \left| \int_a^b \left( \int_t^x f'(\tau) d\tau \right) dF_{n,x}(t) \right| = \left| \int_a^b \left( \int_t^x (f'(\tau) - f'(x)) d\tau \right) dF_{n,x}(t) \right| \leq \\ &\leq \frac{M}{2} \int_a^b (x-t)^2 dF_{n,x}(t) = \frac{M}{2n} \sigma^2(x) . \end{aligned}$$

Now let  $f \in C[a,b]$  be such that  $|L_n(f;x) - f(x)| \leq \frac{M}{2n} \sigma^2(x)$  and  $f' \notin \text{Lip}(1,M)$ . Then according to lemma 3.3 there exists a point  $x_0 \in (a,b)$  and a number  $h > 0$  such that  $|f(x_0-h) - 2f(x_0) + f(x_0+h)| > Mh^2$ .

We assume that

$$(3.7) \quad f(x_0-h) - 2f(x_0) + f(x_0+h) = M_1 h^2, \quad \text{where } M_1 < -M ,$$

otherwise we replace  $f$  by  $-f$ . The function  $\varphi(x) := f(x) - \ell(x)$ , where  $\ell$  is the linear function with  $\ell(x_0 \pm h) = f(x_0 \pm h)$ , satisfies the same relation (3.7), and in addition we have

$$(3.8) \quad |L_n(\varphi;x) - \varphi(x)| = |L_n(f;x) - f(x)| \leq \frac{M}{2n} \sigma^2(x), \quad x \in [a,b] .$$

Let  $\alpha$  be a positive number with  $M < \alpha < -M_1$  and let  $C$  be such that the quadratic function

$$(3.9) \quad Q(x) := \frac{-\alpha}{2}(x - x_0)^2 + C ,$$

satisfies the inequality

$$(3.10) \quad Q(x) > \varphi(x), \quad (x \in [x_0-h, x_0+h]) .$$

Now we have

$$\begin{aligned} Q(x_0 \pm h) - \varphi(x_0 \pm h) &= Q(x_0 \pm h) = -\frac{\alpha}{2} h^2 + C , \\ Q(x_0) - \varphi(x_0) &= \frac{M_1 h^2}{2} + C . \end{aligned}$$

So, the function  $v(x) := Q(x) - \varphi(x)$  on  $[x_0-h, x_0+h]$  attains its minimum value  $m$  at a point  $y \in (x_0-h, x_0+h)$ . The quadratic function  $Q^*$ , defined by

(3.11)  $Q^*(x) = Q(x) - m, \quad x \in [a, b]$  has the properties:

(3.12)  $Q^*(x) \geq \varphi(x), \quad x \in [x_0 - h, x_0 + h],$

$Q^*(y) = \varphi(y).$

Let

$a' = \min\{x: x \leq x_0 - h, Q^*(x) = \varphi(x)\}$

and

$b' = \min\{x: x \geq x_0 + h, Q^*(x) = \varphi(x)\},$

then  $a \leq a' < y < b' \leq b$  and  $Q^*(x) \geq \varphi(x)$  on  $[a', b']$ .

Let

$$h(x) := \begin{cases} 0, & x \in [a', b'] \\ \varphi(x) - Q^*(x), & x \notin [a', b'] \end{cases}.$$

Then  $\varphi(x) \leq Q^*(x) + h(x), \quad (x \in [a, b]).$

Hence

$$\begin{aligned} L_n(\varphi; y) - \varphi(y) &= L_n(\varphi; y) - Q^*(y) \leq L_n(Q^* + h; y) - Q^*(y) = \\ &= L_n(Q^*; y) - Q^*(y) + L_n(h; y) - h(y) = -\frac{\alpha}{2} \frac{\sigma^2(y)}{n} + o\left(\frac{\sigma^2(y)}{n}\right), \end{aligned}$$

according to lemma 2.10. This contradicts (3.8).

To prove that the set  $(L_n)$  consists of all linear function, we have only to remark that

$$|f(x) - L_n(f; x)| = o\left(\frac{\sigma^2(x)}{n}\right), \quad (n \rightarrow \infty), \text{ uniformly in } x,$$

implies  $f' \in \text{Lip}(1, \epsilon)$  for all  $\epsilon > 0$ . Then  $f'$  is constant and so  $f$  is linear.  $\square$

Remark. The parabola technique, applied in the proof of theorem (3.2) is introduced in [1] by B. Bajšanski and R. Bojanic.

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