

Upper bound for the length of the norm of an expression in lambda-typed lambda calculus

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1. For all notations used in this note we refer to [1]. Nevertheless it should be noted that for the material of the present note there is no substantial difference between the system $\Delta\Delta$ of [1] and the system Δ of [2].

2. The length of a lambda tree or of a subtree of a lambda tree is just the number of end-points. So in particular, expressed in terms of character strings,

$$\begin{aligned} \text{length}(x) &= 1, & \text{length}(\tau) &= 1. \\ \text{length}\langle U \rangle V &= \text{length}(U) + \text{length}(V), \\ \text{length}([x:U]V) &= \text{length}(U) + \text{length}(V). \end{aligned}$$

3. If V is a lambda tree, its norm (as defined in [1], sections 5.9 and 5.10) is denoted by $\text{norm}(V)$. The length of that norm, $\text{length}(\text{norm}(V))$, will be abbreviated to $\ln(V)$.

4. Theorem. If V is a lambda tree then

$$\ln(V) \leq \frac{\text{length}(V) - 1}{2}.$$

5. Sketch of a proof. We introduce the subdivided lambda trees (R,W,B) and (R,W,Y,B) as in [1], section 5.9. For these we have the norms $\text{norm}(R,W,B)$ and $\text{norm}(R,W,Y,B)$, which can be considered as the norms of the subtrees WB and WYB , taken with typing of free variables by means of the abstractors in R .

As above, we shall abbreviate $\text{length}(\text{norm}(R,W,B))$ to $\ln(R,W,B)$ and $\text{length}(\text{norm}(R,W,Y,B))$ to $\ln(R,W,Y,B)$.

We consider all the dummies x for which $(R, \xi, x) \in \text{Slam}3$. In other words, these x 's are the dummies attached to the abstractors in the main line of R . (We recall that R can be written as a sequence of applicator-abstractor pairs $\langle P \rangle [y:Q]$ and loose abstractors $[z:U]$). For each one of these we consider $\ln(R, \xi, x)$, and the largest one of these numbers will be denoted by $\text{mxln}(R)$. In case R is empty, we agree that $\text{mxln}(R) = 1$.

We are now in a position to announce

$$\ln(R,W,B) \leq f(R,B), \quad \ln(R,W,Y,B) \leq f(R,B), \quad (1)$$

where

$$f(R,B) = \frac{\text{length}(B) - 1}{2} \text{mxln}(R).$$

The inequalities (1) can be proved by recursion, if we just follow the list of clauses of [1], section 5.9. In the cases (i) and (vi) we have $\ln(R, \mathcal{E}, \tau) = 1$, $\ln(R, \mathcal{E}, \mathcal{E}, \tau) = 1$, and $\ln(R, \tau) = \text{mxln}(R) \geq 1$. We shall explain what has to be done in the other cases: as a typical case we take case (iv).

In case (iv) it is stated what the norm is of $(R, \mathcal{E}, [x:U]B)$, under the assumption that we know the norms of (R, \mathcal{E}, U) and $(R[x:U], \mathcal{E}, B)$. Accordingly we have to prove (1) for $(R, \mathcal{E}, [x:U]B)$, under the assumption that we know it for (R, \mathcal{E}, U) as well as for $(R[x:U], \mathcal{E}, B)$.

We shall not carry out all this here. The work is of a simple nature. During the course of this work, however, we need a few auxiliary results which have to be proved separately by recursion. As one of these we mention $\ln(R, W, B) \leq \ln(R, \mathcal{E}, B)$.

Once we have (1) for all cases, we have the theorem of section 4, just since the norm of the full lambda tree V was defined as $\text{norm}(\mathcal{E}, \mathcal{E}, V)$, and $\text{mxln}(\mathcal{E})$ was defined as 1.

6. The estimate in the theorem is best possible. The examples for which the upper bound are reached are given by

$$\begin{aligned} E_1 &= \quad , \\ E_2 &= [x_1:E_1]x_1, \\ E_3 &= [x_2:E_2]x_2, \\ E_4 &= [x_3:E_3]x_3, \\ &\dots\dots \end{aligned}$$

We have $\text{length}(E_n) = n$, $\ln(E_n) = 2^{n-1}$.

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