On the set of divisors of a number

ON THE SET OF DIVISORS OF A NUMBER

BY

N. G. DE BRUIJN,

(Delft)

C$^{\text{A}}$. VAN EBBENHORST TENGBERGEN

(Den Haag)

and D. KRUYSWIJK

(Amsterdam)

Theorem 1 below was stated as a conjecture and proposed as a prize problem of the Wiskundig Genootschap for 1949. Correct solutions were given by the second and the third author. The proof presented here depends on theorem 2, which is of some interest in itself. The theorems are purely combinatorial, but they are stated in an arithmetical form for the sake of a simple terminology.

If m is a natural number, and if $m = p_1^{\lambda_1} \dots p_\nu^{\lambda_\nu}$ is its canonical factorisation, then the sum $\lambda_1 + \dots + \lambda_\nu$ will be called the *degree* of m.

Consider the set U of all divisors of a fixed number m whose degree equals $[\frac{1}{2}n]$, where n is the degree of m. It is clear that no element of this set divides any other element of the set. Theorem 1 expresses that the set just considered is a maximal set with this property.

Theorem 1. *Let m be a natural number, and let n be its degree. Let $d_1, \dots, d_l$ be a set of divisors of m with the property that no d_i divides any d_j $(i \neq j)$. Then we have $l \leqslant L$, where L is the number of divisors of m whose degree equals $[\frac{1}{2}n]$.*

The special case of squarefree m, that is, m is the product of n different primes, is a theorem of E. Sperner [1]).

[1]) Math. Zeitschrift **27**, 544—548 (1928).

A sequence $d_1, \ldots., d_h$ of divisors of m will be called a *symmetrical chain* whenever it joins the following properties:

a. The degree of d_1 equals the one of m/d_h.

b. For $1 \leqslant i < h$ the quotient d_{i+1}/d_i is a prime [2]).

Theorem 2. *The set of divisors of m can be completely divided into a number of disjoint symmetrical chains.*

Proof. We proceed by induction with respect to the number $\nu(m)$ of different primes dividing m. For $\nu(m) = 0$ the theorem is trivial. Now assume that the theorem has been proved for the number m; we proceed to prove it for the number $m_1 = mp^\lambda$ where p does not divide m.

Let $d_1, \ldots., d_h$ be a symmetrical chain of divisors of m. Then consider the divisors $d_i p^\alpha$ $(i = 1, \ldots, h;\ \alpha = 0, \ldots., \lambda)$ of the number m_1. Our induction will be accomplished by showing that this set can be sub-divided into symmetrical chains of divisors of m_1. This is exhibited by the following scheme:

$$\begin{array}{c|c|ccc}
d_1 & d_2 & \ldots\ldots & & d_h \\
d_1 p & d_2 p & \ldots\ldots & & d_h p \\
\vdots & \vdots & & & \vdots \\
\vdots & \vdots & & & \vdots \\
\cline{3-5}
d_1 p^{\lambda-1} & d_2 p^{\lambda-1} & \ldots\ldots & & d_h p^{\lambda-1} \\
\cline{2-5}
d_1 p^\lambda & d_2 p^\lambda & \ldots\ldots & & d_h p^\lambda
\end{array}$$

It is easily seen that the sequences $\{d_1, d_1 p, \ldots., d_1 p^\lambda, d_2 p^\lambda, \ldots., d_h p^\lambda\}$, $\{d_2, d_2 p, \ldots., d_2 p^{\lambda-1}, d_3 p^{\lambda-1}, \ldots., d_h p^{\lambda-1}\}$, etc. are symmetrical chains.

Proof of theorem 1. Let the set of divisors of m be divided into symmetrical chains, and let U be the set of the L divisors of m of degree $[\frac{1}{2}n]$. Any element of U is contained in just one chain, and from the definition of the chain it follows that each chain contains just one element of U. Consequently the number of chains is L. Finally, if the set $d_1, \ldots., d_h$ has the property mentioned in theorem 1, it is

[2]) If $h = 1$, the latter condition vanishes. Hence, if n is even, any divisor of degree $\frac{1}{2}n$ is a symmetrical chain of one element.

clear that each chain contains at most one of these d's. Hence $l \leqslant L$.

The following theorem is a simple application of theorem 2. A set S of divisors of m will be called *convex* whenever

$$d_1 \in S,\ d_2 \in S,\ d_1 \mid d_3 \mid d_2 \text{ imply } d_3 \in S.$$

Theorem 3. Let ν be the number of different primes dividing m, and let S be a convex set of divisors of m. Then we have, if μ indicates the Möbius function:

$$\left| \sum_{d \in S} \mu(d) \right| \leqslant \binom{\nu}{[\frac{1}{2}\nu]}.$$

Proof. Since $\mu(d) = 0$ whenever d is not squarefree, we may restrict ourselves to the case that m is squarefree. Thus ν is the degree of m, and clearly $L = \binom{\nu}{[\frac{1}{2}\nu]}$ is the number of elements in the set U. Hence it equals the number of chains into which the set of divisors of m can be divided. Now write $S = S_1 + \dots + S_L$, where S_i is a subset of the i-th chain. S being convex, we infer that each S_i is either empty or it consists of a number of consecutive elements of the chain. If d runs through the elements of a chain, then $\mu(d)$ assumes the values $+1$ and -1 alternately. Hence $\Sigma\, \mu(d) = 0$ or 1 or -1 whenever d runs through any set S_i. Finally

$$\left| \sum_{S} \mu(d) \right| \leqslant \sum_{i=1}^{L} \left| \sum_{S_i} \mu(d) \right| \leqslant L.$$

The special case where S consists of all divisors of m belonging to some interval $a \leqslant d \leqslant b$, was established by P. ERDÖS [3]. His argument can also be used in order to prove the general case. To this end one has to add the following remark: if S is a convex subset of the set U of all divisors of m, then we can find two convex sets V_1 and V_2 such that $1 \in V_1$, $m \in V_2$, $V_1 \cap V_2 = S$, $V_1 \cup V_2 = U$.

(Ingekomen 6.10.50).

[3] Math. Student **17**, 32—33 (1950).