

## Isosceles point sets in $\mathbb{R}^d$

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Isosceles point sets in  $R^d$

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# ISOSCELES POINT SETS IN $R^d$

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## Abstract.

Let  $X$  be a set of points in  $R^d$  such that the triangle determined by an arbitrary triple from  $X$  is isosceles, then  $\text{card}(X) \leq \frac{1}{2}(d+1)(d+2)$ .

## Introduction.

An isosceles set  $X$  in  $R^d$  is a set of points such that any triple among them determines an isosceles triangle. Isosceles sets were introduced by Kelly, the problem goes back to Erdős [1].

Throughout the article  $X$  will denote an isosceles set in  $R^d$ ,

$X = \{x_1, x_2, \dots, x_v\}$  and we assume

$$\text{aff}(X) = \left\{ \sum_{i=1}^v a_i x_i \mid \sum a_i = 1 \right\} = R^d.$$

For any subset  $X_1 \subset X$ ,  $\dim(X_1)$  denotes the dimension of  $\text{aff}(X_1)$ .

By  $A(X)$  we mean the set of distances between points of  $X$ .

For  $a \in A(X)$  we denote by  $X_a$  the graph with point set  $X$  and edges the pairs of points at distance  $a$ .  $X$  is called decomposable if it is possible to partition  $X$  in sets  $X_1$  and  $X_2$  with  $|X_2| > 1$  such that a point of  $X_1$  has the same distance to all points of  $X_2$ . (This distance may be different for several points of  $X_1$  though.)

Finally if  $\text{card}(A(X)) = 2$ ;  $X$  is called a two-distance set.

## The structure of isosceles sets.

Lemma 1. If  $X$  is decomposable, and  $(X_1, X_2)$  is a decomposition for  $X$ , then  $\dim(X_1) + \dim(X_2) \leq \dim(X)$ .

Proof. Let  $P$  be the orthogonal projection on  $\text{Aff}(X_2)$ . Then for any  $x_1 \in X_1$ ,  $Px_1$  is the center of a sphere in  $\text{Aff}(X_2)$  containing  $X_2$ . Since  $X_2$  spans  $\text{Aff}(X_2)$   $P$  maps  $X_1$  onto a single point. Therefore the flats  $\text{Aff}(X_1)$  and  $\text{Aff}(X_2)$  are orthogonal and the result follows.

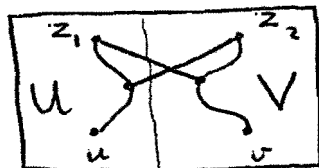
Lemma 2. If  $X$  is indecomposable then it is a two-distance set.

The proof is split into three parts, first we examine the case that there is some distance  $a$  for which  $X_a$  is disconnected. Then we look at the case where there is some  $a$  for which  $X_a$  has diameter larger than two. And finally we consider the case that  $X_a$  has diameter two for each  $a \in A(X)$ .

Case 1. Suppose there is an  $a \in A(X)$  such that  $X_a$  is disconnected, then  $X$  is decomposable, for let  $X_2$  be a component of  $X_a$  having more than 1 point. From the isosceles property it now follows that any point not in  $X_2$  has the same distance to all points in  $X_2$ .

Case 2. Now suppose  $X_a$  is connected for all  $a \in A(X)$  and let  $b$  be a distance such that there are two points,  $u$  and  $v$  at distance 3 in  $X_b$ . Let  $a$  be the euclidean distance between  $u$  and  $v$ . We claim that  $X$  is a two-distance set. Let  $U$  be the set of points in  $X$  that are closer to  $u$  than to  $v$  in the graph  $X_b$  and let  $V = X \setminus U$ . For any  $z$  in  $U$  there is a  $(u, z)$  path entirely in  $U$  so by the isosceles property  $v$  and  $z$  have distance  $a$ . Similarly  $u$  has (Euclidean) distance  $a$  to any point in  $V$ . Now take  $z_1 \in U$  and  $z_2 \in V$  and let  $P_1$  be a shortest  $(z_1, u)$  path and  $P_2$  a shortest  $(z_2, v)$  path. If  $z_1$  is adjacent to  $z_2$  in  $X_b$  they have distance  $b$  which is okay. If  $z_1$  is not adjacent to any point on  $P_2$  then they have distance  $a$  by the isosceles property, similarly if  $z_2$  is not adjacent to any point of  $P_1$ . Now if both points do have a neighbour on the other path it is clear from the picture that the following holds :

$$d_b(v, z_1) \leq d_b(v, z_2) \leq d_b(u, z_2) \leq d_b(u, z_1),$$



contradiction. Now for any further distance  $c$  the graph cannot be connected since  $U$  and  $V$  are only joined by distances  $a$  and  $b$ . Therefore  $X$  is a two-distance set.

Case 3. Finally we suppose that  $X_a$  is connected for every distance  $a$  and has diameter 2. Now suppose there are three distances, call them  $a, b$  and  $c$ . We will construct an infinite subset of  $X$  thus obtaining a contradiction.

Let  $z$  be an arbitrary point in  $X$  and  $a_1$  a point at distance  $a$  from  $z$ . In  $X$  there is a point  $b_1$  having distance  $b$  to both  $z$  and  $a_1$  for the diameter of  $X_b$  is 2. Similarly we can find a point  $c_1$  having distance  $c$  to both  $z$  and  $b_1$ . Since  $c_1 a_1$  is part of the triangle  $c_1 a_1 b_1$ ,  $c_1 a_1$  is either  $c$  or  $b$ , but since it is also a side of the triangle  $c_1 a_1 z$  it is either  $a$  or  $c$ , and therefore it has to be  $c$ . Now let  $a_2$  be a point at distance  $a$  from both  $c_1$  and  $z$  and define  $b_2, c_2, a_3, \dots$  in the way indicated above, we will show that at each step at the construction of the infinite set the last constructed point has the same distance to all previous constructed points. Suppose the last point we added was  $a_k$ , we assume that our induction assumption holds for all points preceding  $a_k$ , i.e. if  $d_j$  is a point of the sequence, where  $d = a, b$  or  $c$  and  $j < k$  then  $d_j$  has distance  $d$  to all points preceding  $d_j$ . By definition  $a_k$  has distance  $a$  to  $z$  and  $c_{k-1}$ . Comparing the triangles  $z a_k b_j$  and  $c_{k-1} a_k b_j$  we see that  $a_k b_j$  is  $a$ . Similarly, comparing the triangles  $z a_k c_j$  and  $b_{j+1} a_k c_j$  (where  $j+1 < k$ ) we conclude that  $a_k c_j$  is  $a$ . Finally the triangles  $b_{k-1} a_k a_j$  and  $c_{k-1} a_k a_j$  force  $a_k a_j$  to be  $a$ . Since every point has a different distance to it's immediate successor and it's predecessors all points we obtain in this way are new, therefore we constructed an infinite subset of  $X$ , a contradiction. Therefore  $X$  is two-distance set.

Theorem. Let  $X$  be an isosceles set in  $R^d$ , then

$$\text{card}(X) \leq \frac{1}{2}(d+1)(d+2),$$

equality implies that  $X$  is a two-distance set, or a spherical two-distance set together with the center.

Proof. The proof is by induction. If  $d=1$  then 3 is the maximal cardinality and  $X$  is a spherical set together with its center. For  $d=2$  Kelly [1] proved that the maximum is 6 realized only by the centered regular pentagon.

Now let  $d > 2$ . If  $X$  is a two-distance set then we have the required inequality (see [3]). Now suppose  $X$  is decomposable,  $(X_1, X_2)$  being a decomposition.

Case 1.  $\dim X_1 \neq 0$ . Since  $|X_2| > 1$  we have  $0 < \dim X_1 < d$ . Let  $d_1 = \dim X_1$ , then by induction we have:

$$|X| \leq \frac{1}{2}(d_1+1)(d_1+2) + \frac{1}{2}(d-d_1+1)(d-d_1+2) < \frac{1}{2}(d+1)(d+2).$$

Case 2.  $\dim X_1 = 0$ . In this case  $X_1$  is a single point and therefore  $X_2$  is spherical. If  $X_2$  is not a two-distance set it is decomposable say  $X_2 = (X_2', X_2'')$ . But now  $(X_1 \cup X_2', X_2'')$  is a decomposition of  $X$  as in Case 1. This finishes the proof.

#### Final Remarks.

Cases 2 and 3 in the proof of Lemma 2. can be considered as the proof of the following pure graph-theoretic theorem:

Let  $K_n$  (the complete graph on  $n$  vertices) be edge-colored with  $k$  colors, such that (i) every triangle has at most two colors, and (ii) for each color, the induced graph on that color is connected. Then  $k=2$ .

I wonder what the implications and possible generalizations of a theorem like this are in graph-theory.

#### References.

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