

A new partial geometry with parameters $(s, t, \alpha) = (7, 8, 4)$

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A NEW PARTIAL GEOMETRY WITH PARAMETERS
 $(s, t, \alpha) = (7, 8, 4)$

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A new partial geometry with parameters $(s,t,\alpha) = (7,8,4)$

by

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ABSTRACT

A partial geometry with parameters as given in the title is constructed by use of the 240 points closest to the origin in the lattice E_8 .

KEYWORDS & PHRASES: *partial geometry, strongly regular graph, lattice E_8 .*

INTRODUCTION.

A *partial geometry* (V, L) with parameters (s, t, α) is a finite nonempty set V of *points* together with a family L of subsets of V called *lines* such that

- (i) For any two points in V there is at most one line containing them both (If such a line exists, the two points are called *collinear*);
- (ii) Each line in L contains exactly $s+1$ points ($s \geq 1$);
- (iii) Each point is contained in exactly $t+1$ lines ($t \geq 1$);
- (iv) For any point $x \in V$ and any line $L \in L$ not containing x there are exactly α points on L collinear with x .

A partial geometry (V, L) with parameters (s, t, α) consists of $v = (s+1)(st+\alpha)/\alpha$ points and $b = (t+1)(st+\alpha)/\alpha$ lines. Furthermore, there are exactly $k = s(t+1)$ points collinear with a given point and $\mu = \alpha(t+1)$ points collinear with each of any two given mutually non-collinear ones.

From (V, L) a graph $G = (V, E)$ can be constructed on the points of V such that two points are adjacent whenever they are collinear. Such a graph is strongly regular with parameters (v, k, λ, μ) , where $\lambda = t(\alpha-1)+s-1$ is the number of points collinear with each of any two given collinear ones. This amounts to saying that each point in the graph G has valency k and that any two connected (non-connected) points have λ (μ) common neighbors. More about partial geometries can be found in BOSE [3]. It will be clear that for any system (V, L) not necessarily satisfying all axioms (i), ..., (iv) a graph $G = G(V, L)$ can be constructed in the way described above. This graph may very well be strongly regular while (V, L) is not a partial geometry. However, the following holds.

LEMMA. *If (V, L) is a pair consisting of a finite nonempty set V and a family L of subsets of V such that for given $s, t \geq 1$, the axioms (i), (ii), (iii) are satisfied and such that there is a natural number α for which the graph $G(V, L)$ is strongly regular with parameters $(v, k, \lambda, \mu) = ((s+1)(st+\alpha)/\alpha, s(t+1), t(\alpha-1)+s-1, \mu)$, then (V, L) is a partial geometry with parameters (s, t, α) .*

PROOF. It suffices to check axiom (iv). Fix a line $L \in \mathcal{L}$. For $x \in V$ outside L we denote by α_x the number of points in L that are collinear with x . By counting arguments, we obtain

$$\sum_{x \notin L} \alpha_x = (s+1)ts \text{ and}$$

$$\sum_{x \notin L} \binom{\alpha_x}{2} = \binom{s+1}{2}(\lambda-s+1),$$

whence $\sum_{x \in L} (\alpha_x - \alpha)^2 = 0$. This implies that $\alpha_x = \alpha$ for any $x \notin L$, so we are through. \square

For a description of the selfdual unimodular lattice E_8 of rank 8, the reader is referred to [2] or [4]. Consider the strongly regular graph G_0 whose points are the 120 lines through the origin containing a nonzero vector of minimal distance to the origin in the lattice E_8 (points being connected whenever they represent mutually orthogonal lines with respect to the bilinear form on E_8). Mathon suggested that study of G_0 , whose parameters are $(v, k, \lambda, \mu) = (120, 63, 30, 36)$, might lead to a new partial geometry with parameters $(s, t, \alpha) = (7, 8, 4)$. This note is concerned with the construction of such a partial geometry. The help of H.A. Wilbrink has been crucial for the outcome.

CONSTRUCTION. S_5 (A_5) denotes the symmetric (alternating) group on 5 letters. Moreover a formal element τ outside A_5 is chosen so as to obtain a copy $\tau A_5 = \{\tau x \mid x \in A_5\}$ of A_5 . Now put $V = A_5 \cup \tau A_5$. We shall use the permutation representations c of S_5 on V by conjugation and r of A_5 on V by right multiplication, both acting in such a way that τ is fixed. Thus $c(g)(\tau a) = \tau g a g^{-1}$ ($a \in A_5, g \in S_5$) and $r(h)(\tau a) = \tau a h$ ($h, a \in A_5$). Conjugation of $v \in V$ by $g \in S_5$ will also be denoted by writing g in the exponent of v , i.e. $v^g = c(g)v$. Similarly for subsets X of V :

$$X^g = \{x^g \mid x \in X\}$$

We write down three lines of V explicitly:

$$\begin{aligned}
L_1 &= \{1, (15243), (13254), (12345), \tau(23)(15), \tau(34)(25), \tau(13)(45), \tau(124)\}, \\
L_5 &= \{1, (12)(34), (13)(24), (14)(23), \tau(142), \tau(243), \tau(134), \tau(123)\}, \\
L_6 &= \{1, (14)(25), (12)(45), (24)(15), \tau, \tau(14)(25), \tau(12)(45), \tau(24)(15)\}.
\end{aligned}$$

Finally, denoting by K the group generated by (124) and $(14)(25)$ (isomorphic to A_4), we can define the set L of all lines on V :

$$L = \{L_m^g \mid g \in K; h \in A_5; m = 1, 5, 6\}.$$

Clearly by construction $c(K) \cdot r(A_5)$ is a group of automorphisms of (V, L) isomorphic to $A_4 \times A_5$. This is not all of $\text{Aut}(V, L)$ as for instance

$$\pi \begin{cases} x & \mapsto \tau x \\ \tau x & \mapsto x \end{cases} \begin{matrix} (1245) \\ (1245) \end{matrix} \quad (x \in A_5)$$

defines an automorphism not contained in this subgroup. Let A be the group of automorphisms generated by $c(K) \cdot r(A_5)$ and π .

THEOREM. (V, L) is a partial geometry with parameters $(s, t, \alpha) = (7, 8, 4)$. The corresponding graph is isomorphic to G_0 .

PROOF OF THE THEOREM. First of all we shall establish a correspondence between the points closest to the origin in E_8 and the points of V . In order to do so we present E_8 in the following way. Take $\tau = (1+\sqrt{5})/2$ and consider the skew field $\mathbb{H}(\tau)$ of real quaternions with coefficients in $\mathbb{Q}(\tau)$. Choose the basis $1, i, j, k$ such that $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$. For any $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\tau)$ ($x_0, x_1, x_2, x_3 \in \mathbb{Q}(\tau)$), the conjugate \bar{x} , the norm $N(x)$ and the real part $\text{Re}(x)$ are defined by

$$x = x_0 - x_1 i - x_2 j - x_3 k,$$

$$N(x) = x\bar{x} \quad \text{and}$$

$$\text{Re}(x) = \frac{1}{2}(x + \bar{x}) = x_0 \quad \text{respectively.}$$

Thus any nonzero $x \in \mathbb{H}(\tau)$ has inverse $\bar{x} N(x)^{-1}$. The subgroup of the multiplicative group of nonzero elements in $\mathbb{H}(\tau)$ generated by $i, j, \tau = \frac{1}{2}(-1 + (1-\tau)i - \tau j)$ will be denoted by Ic . It is isomorphic to $Sl_2(5)$ and of order 120. In fact there is an epimorphism $Ic \rightarrow A_5$ determined by $i \mapsto (12)(34), j \mapsto (13)(24), \zeta \mapsto (124), w \mapsto (235)$. Thus each point in A_5 can (and will) be identified with the two points in its inverse image under the epimorphism. In order to extend this identification to all of V , we just identify $\tau x (x \in A_5)$ with $\pm \tau x$, a set of two elements in τIc .

Next we will supply the subring $\mathbb{Z}[Ic]$ of $\mathbb{H}(\tau)$ generated by all elements of Ic with the structure of a \mathbb{Z} -lattice by defining a quadratic form q on $\mathbb{Z}[Ic]$. Write $t(a+b\tau) = a$ for $a, b \in \mathbb{Q}$. The form q is then given by $q(x) = 2(t \circ N(x))$ for $x \in \mathbb{Z}[Ic]$. The corresponding bilinear form is $(x, y) = 2t(\text{Re } \bar{xy})$ ($x, y \in \mathbb{Z}[Ic]$).

Now $(\mathbb{Z}[Ic], q)$ is an even unimodular 8-dimensional lattice, and therefore isomorphic to E_8 (cf. [4], p.55). Moreover each element in V determines a unique line through the origin containing two nonzero points in E_8 closest to the origin, and vice versa.

It is easily checked that if two points $x, y \in V$ are collinear in (V, L) , they are perpendicular with respect to the bilinear form derived from q . Thus $G(V, L)$ is a subgraph of G_0 with the same number of points and the same number of edges and therefore coincides with G_0 . As to the proof of the first statement of the theorem, clearly each line contains 8 points. There are exactly 9 lines containing the point 1, namely 4 in the A -orbit of L_1 , 4 in the A -orbit of L_5 , and L_6 . They are denoted by $L_1, L_2, L_3, L_4, L_5, L_7, L_8, L_9$ and L_6 respectively and written out explicitly in table 1. As no point $\neq 1$ occurs twice in this table, axioms (i), (iii) hold if one of the points concerned is 1. But the group A acts transitively on the 120 points of V , so the two axioms hold without restriction. Finally, as $G(V, L)$ is strongly regular, axiom (iv) is a consequence of the Lemma. \square

REMARKS (i) Let Ω be the subset of A_5 consisting of all elements in the A_5 -conjugacy classes of (12345) and $(12)(34)$. The complete subgraph of G_0 on the points of A_5 is the Cayley graph $\Gamma(A_5, \Omega)$ in the BIGGS' notation [1]. If Ω_1 is the union of the A_5 -conjugacy classes (12354) and

table 1
The lines in L containing 1

line	elements in the line							
L_1	1	(15243)	(13254)	(12345)	$\tau(23)(15)$	$\tau(34)(25)$	$\tau(13)(45)$	$\tau(124)$
L_2	1	(13425)	(12453)	(14352)	$\tau(12)(34)$	$\tau(24)(35)$	$\tau(13)(25)$	$\tau(145)$
L_3	1	(14523)	(15324)	(13542)	$\tau(13)(24)$	$\tau(14)(35)$	$\tau(23)(45)$	$\tau(152)$
L_4	1	(15432)	(12534)	(14235)	$\tau(14)(23)$	$\tau(34)(15)$	$\tau(12)(35)$	$\tau(254)$
L_5	1	(12)(34)	(13)(24)	(14)(23)	$\tau(142)$	$\tau(243)$	$\tau(134)$	$\tau(123)$
L_6	1	(14)(25)	(12)(45)	(24)(15)	$\tau.1$	$\tau(14)(25)$	$\tau(12)(45)$	$\tau(24)(15)$
L_7	1	(15)(23)	(12)(35)	(13)(25)	$\tau(132)$	$\tau(235)$	$\tau(125)$	$\tau(315)$
L_8	1	(23)(45)	(25)(34)	(24)(35)	$\tau(234)$	$\tau(354)$	$\tau(245)$	$\tau(253)$
L_9	1	(14)(35)	(15)(34)	(13)(45)	$\tau(143)$	$\tau(135)$	$\tau(154)$	$\tau(345)$

table 2
The lines in L containing τ

line	elements in the line							
$L_{\tau 1}$	(142)	(13)(25)	(23)(45)	(15)(34)	τ	$\tau(14325)$	$\tau(13452)$	$\tau(15423)$
$L_{\tau 2}$	(15)(23)	(12)(34)	(14)(35)	(245)	τ	$\tau(14532)$	$\tau(15234)$	$\tau(12435)$
$L_{\tau 3}$	(34)(25)	(13)(24)	(12)(35)	(154)	τ	$\tau(15342)$	$\tau(12543)$	$\tau(13524)$
$L_{\tau 4}$	(13)(45)	(14)(23)	(24)(35)	(125)	τ	$\tau(13245)$	$\tau(14253)$	$\tau(12354)$
$L_{\tau 5}$	(124)	(234)	(143)	(132)	τ	$\tau(12)(34)$	$\tau(13)(24)$	$\tau(14)(23)$
$L_{\tau 6}$	1	(14)(25)	(12)(45)	(15)(24)	τ	$\tau(14)(25)$	$\tau(12)(45)$	$\tau(15)(24)$
$L_{\tau 7}$	(135)	(253)	(123)	(152)	τ	$\tau(13)(25)$	$\tau(15)(23)$	$\tau(12)(35)$
$L_{\tau 8}$	(432)	(254)	(235)	(345)	τ	$\tau(23)(45)$	$\tau(25)(34)$	$\tau(24)(35)$
$L_{\tau 9}$	(134)	(145)	(354)	(153)	τ	$\tau(15)(34)$	$\tau(13)(25)$	$\tau(14)(35)$

(12) (34), then $\Gamma(A_5, \Omega_1)$ is isomorphic to the complete subgraph of $G(V, L)$ on the points of τA_5 . Finally, for $x, y \in A_5$ the points $x, \tau y$ of V are joined in $G(V, L)$ if and only if $xy^{-1} \in \Omega_2$, where Ω_2 is the union of the A_5 -conjugacy classes of 1, (12) (34) and (123).

(ii) In view of (i) it will be clear that verification G_0 is strongly regular and therefore the proof of the first statement of the theorem could be done without using quaternions or E_8 .

(iii) A is a subgroup of $\text{Aut}(V, L)$ of order $2^5 \cdot 3^2 \cdot 5$. On the other hand, $\text{Aut}(V, L)$ is a subgroup of $\text{Aut}(E_8)/\{\pm I\}$, so the order of $\text{Aut}(V, L)$ must divide $2^{13} \cdot 3^5 \cdot 5^2 \cdot 7$.

Note that $\text{Aut}(V, L)$ cannot be all of $\text{Aut}(G_0)$, since the latter group acts transitively on the family of 15×135 8-cliques in G_0 , while no more than 135 8-cliques of G_0 originate from lines in L .

(iv) Consideration of parameters might lead to the expectation that the choice of an appropriate sub-family L_0 of lines in L provides a partial geometry on V with parameters $(s, t, \alpha) = (7, 4, 2)$ or (even weaker) a strongly regular graph with parameters $(v, k, \lambda, \mu) = (120, 35, 10, 10)$. However no selection of A -orbits from L leads to such a family L_0 .

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