

## A new partial geometry with parameters $(s, t, \alpha) = (7, 8, 4)$

**Citation for published version (APA):**

Cohen, A. M. (1979). *A new partial geometry with parameters  $(s, t, \alpha) = (7, 8, 4)$* . Stichting Mathematisch Centrum.

**Document status and date:**

Published: 01/01/1979

**Document Version:**

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

**Please check the document version of this publication:**

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

**General rights**

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

[www.tue.nl/taverne](http://www.tue.nl/taverne)

**Take down policy**

If you believe that this document breaches copyright please contact us at:

[openaccess@tue.nl](mailto:openaccess@tue.nl)

providing details and we will investigate your claim.

**stichting  
mathematisch  
centrum**



---

AFDELING ZUIVERE WISKUNDE  
(DEPARTMENT OF PURE MATHEMATICS)

ZN 92/79 · NOVEMBER

A.M. COHEN

A NEW PARTIAL GEOMETRY WITH PARAMETERS  
 $(s, t, \alpha) = (7, 8, 4)$

---

**2e boerhaavestraat 49 amsterdam**

BIBLIOTHEEK MATHEMATISCH CENTRUM  
AMSTERDAM

CWI BIBLIOTHEEK



3 0054 00108 2347

A new partial geometry with parameters  $(s,t,\alpha) = (7,8,4)$

by

Arjeh M. Cohen

ABSTRACT

A partial geometry with parameters as given in the title is constructed by use of the 240 points closest to the origin in the lattice  $E_8$ .

KEYWORDS & PHRASES: *partial geometry, strongly regular graph, lattice  $E_8$ .*

## INTRODUCTION.

A *partial geometry*  $(V, L)$  with parameters  $(s, t, \alpha)$  is a finite nonempty set  $V$  of *points* together with a family  $L$  of subsets of  $V$  called *lines* such that

- (i) For any two points in  $V$  there is at most one line containing them both (If such a line exists, the two points are called *collinear*);
- (ii) Each line in  $L$  contains exactly  $s+1$  points ( $s \geq 1$ );
- (iii) Each point is contained in exactly  $t+1$  lines ( $t \geq 1$ );
- (iv) For any point  $x \in V$  and any line  $L \in L$  not containing  $x$  there are exactly  $\alpha$  points on  $L$  collinear with  $x$ .

A partial geometry  $(V, L)$  with parameters  $(s, t, \alpha)$  consists of  $v = (s+1)(st+\alpha)/\alpha$  points and  $b = (t+1)(st+\alpha)/\alpha$  lines. Furthermore, there are exactly  $k = s(t+1)$  points collinear with a given point and  $\mu = \alpha(t+1)$  points collinear with each of any two given mutually non-collinear ones.

From  $(V, L)$  a graph  $G = (V, E)$  can be constructed on the points of  $V$  such that two points are adjacent whenever they are collinear. Such a graph is strongly regular with parameters  $(v, k, \lambda, \mu)$ , where  $\lambda = t(\alpha-1)+s-1$  is the number of points collinear with each of any two given collinear ones. This amounts to saying that each point in the graph  $G$  has valency  $k$  and that any two connected (non-connected) points have  $\lambda$  ( $\mu$ ) common neighbors. More about partial geometries can be found in BOSE [3]. It will be clear that for any system  $(V, L)$  not necessarily satisfying all axioms (i), ..., (iv) a graph  $G = G(V, L)$  can be constructed in the way described above. This graph may very well be strongly regular while  $(V, L)$  is not a partial geometry. However, the following holds.

**LEMMA.** *If  $(V, L)$  is a pair consisting of a finite nonempty set  $V$  and a family  $L$  of subsets of  $V$  such that for given  $s, t \geq 1$ , the axioms (i), (ii), (iii) are satisfied and such that there is a natural number  $\alpha$  for which the graph  $G(V, L)$  is strongly regular with parameters  $(v, k, \lambda, \mu) = ((s+1)(st+\alpha)/\alpha, s(t+1), t(\alpha-1)+s-1, \mu)$ , then  $(V, L)$  is a partial geometry with parameters  $(s, t, \alpha)$ .*

PROOF. It suffices to check axiom (iv). Fix a line  $L \in \mathcal{L}$ . For  $x \in V$  outside  $L$  we denote by  $\alpha_x$  the number of points in  $L$  that are collinear with  $x$ . By counting arguments, we obtain

$$\sum_{x \notin L} \alpha_x = (s+1)ts \text{ and}$$

$$\sum_{x \notin L} \binom{\alpha_x}{2} = \binom{s+1}{2}(\lambda-s+1),$$

whence  $\sum_{x \in L} (\alpha_x - \alpha)^2 = 0$ . This implies that  $\alpha_x = \alpha$  for any  $x \notin L$ , so we are through.  $\square$

For a description of the selfdual unimodular lattice  $E_8$  of rank 8, the reader is referred to [2] or [4]. Consider the strongly regular graph  $G_0$  whose points are the 120 lines through the origin containing a nonzero vector of minimal distance to the origin in the lattice  $E_8$  (points being connected whenever they represent mutually orthogonal lines with respect to the bilinear form on  $E_8$ ). Mathon suggested that study of  $G_0$ , whose parameters are  $(v, k, \lambda, \mu) = (120, 63, 30, 36)$ , might lead to a new partial geometry with parameters  $(s, t, \alpha) = (7, 8, 4)$ . This note is concerned with the construction of such a partial geometry. The help of H.A. Wilbrink has been crucial for the outcome.

CONSTRUCTION.  $S_5$  ( $A_5$ ) denotes the symmetric (alternating) group on 5 letters. Moreover a formal element  $\tau$  outside  $A_5$  is chosen so as to obtain a copy  $\tau A_5 = \{\tau x \mid x \in A_5\}$  of  $A_5$ . Now put  $V = A_5 \cup \tau A_5$ . We shall use the permutation representations  $c$  of  $S_5$  on  $V$  by conjugation and  $r$  of  $A_5$  on  $V$  by right multiplication, both acting in such a way that  $\tau$  is fixed. Thus  $c(g)(\tau a) = \tau g a g^{-1}$  ( $a \in A_5, g \in S_5$ ) and  $r(h)(\tau a) = \tau a h$  ( $h, a \in A_5$ ). Conjugation of  $v \in V$  by  $g \in S_5$  will also be denoted by writing  $g$  in the exponent of  $v$ , i.e.  $v^g = c(g)v$ . Similarly for subsets  $X$  of  $V$ :

$$X^g = \{x^g \mid x \in X\}$$

We write down three lines of  $V$  explicitly:

$$\begin{aligned}
L_1 &= \{1, (15243), (13254), (12345), \tau(23)(15), \tau(34)(25), \tau(13)(45), \tau(124)\}, \\
L_5 &= \{1, (12)(34), (13)(24), (14)(23), \tau(142), \tau(243), \tau(134), \tau(123)\}, \\
L_6 &= \{1, (14)(25), (12)(45), (24)(15), \tau, \tau(14)(25), \tau(12)(45), \tau(24)(15)\}.
\end{aligned}$$

Finally, denoting by  $K$  the group generated by  $(124)$  and  $(14)(25)$  (isomorphic to  $A_4$ ), we can define the set  $L$  of all lines on  $V$ :

$$L = \{L_m^g \mid g \in K; h \in A_5; m = 1, 5, 6\}.$$

Clearly by construction  $c(K) \cdot r(A_5)$  is a group of automorphisms of  $(V, L)$  isomorphic to  $A_4 \times A_5$ . This is not all of  $\text{Aut}(V, L)$  as for instance

$$\pi \begin{cases} x & \mapsto \tau x \\ \tau x & \mapsto x \end{cases} \begin{matrix} (1245) \\ (1245) \end{matrix} \quad (x \in A_5)$$

defines an automorphism not contained in this subgroup. Let  $A$  be the group of automorphisms generated by  $c(K) \cdot r(A_5)$  and  $\pi$ .

THEOREM.  $(V, L)$  is a partial geometry with parameters  $(s, t, \alpha) = (7, 8, 4)$ . The corresponding graph is isomorphic to  $G_0$ .

PROOF OF THE THEOREM. First of all we shall establish a correspondence between the points closest to the origin in  $E_8$  and the points of  $V$ . In order to do so we present  $E_8$  in the following way. Take  $\tau = (1+\sqrt{5})/2$  and consider the skew field  $\mathbb{H}(\tau)$  of real quaternions with coefficients in  $\mathbb{Q}(\tau)$ . Choose the basis  $1, i, j, k$  such that  $i^2 = j^2 = k^2 = -1$  and  $ij = -ji = k$ . For any  $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\tau)$  ( $x_0, x_1, x_2, x_3 \in \mathbb{Q}(\tau)$ ), the conjugate  $\bar{x}$ , the norm  $N(x)$  and the real part  $\text{Re}(x)$  are defined by

$$x = x_0 - x_1 i - x_2 j - x_3 k,$$

$$N(x) = x\bar{x} \quad \text{and}$$

$$\text{Re}(x) = \frac{1}{2}(x + \bar{x}) = x_0 \quad \text{respectively.}$$

Thus any nonzero  $x \in \mathbb{H}(\tau)$  has inverse  $\bar{x} N(x)^{-1}$ . The subgroup of the multiplicative group of nonzero elements in  $\mathbb{H}(\tau)$  generated by  $i, j, \tau = \frac{1}{2}(-1 + (1-\tau)i - \tau j)$  will be denoted by  $Ic$ . It is isomorphic

to  $Sl_2(5)$  and of order 120. In fact there is an epimorphism  $Ic \rightarrow A_5$  determined by  $i \mapsto (12)(34)$ ,  $j \mapsto (13)(24)$ ,  $\zeta \mapsto (124)$ ,  $w \mapsto (235)$ . Thus each point in  $A_5$  can (and will) be identified with the two points in its inverse image under the epimorphism. In order to extend this identification to all of  $V$ , we just identify  $\tau x (x \in A_5)$  with  $\pm \tau x$ , a set of two elements in  $\tau Ic$ .

Next we will supply the subring  $\mathbb{Z}[Ic]$  of  $\mathbb{H}(\tau)$  generated by all elements of  $Ic$  with the structure of a  $\mathbb{Z}$ -lattice by defining a quadratic form  $q$  on  $\mathbb{Z}[Ic]$ . Write  $t(a+b\tau) = a$  for  $a, b \in \mathbb{Q}$ . The form  $q$  is then given by  $q(x) = 2(t \circ N(x))$  for  $x \in \mathbb{Z}[Ic]$ . The corresponding bilinear form is  $(x, y) = 2t(\text{Re } \bar{x}y)$  ( $x, y \in \mathbb{Z}[Ic]$ ).

Now  $(\mathbb{Z}[Ic], q)$  is an even unimodular 8-dimensional lattice, and therefore isomorphic to  $E_8$  (cf. [4], p.55). Moreover each element in  $V$  determines a unique line through the origin containing two nonzero points in  $E_8$  closest to the origin, and vice versa.

It is easily checked that if two points  $x, y \in V$  are collinear in  $(V, L)$ , they are perpendicular with respect to the bilinear form derived from  $q$ . Thus  $G(V, L)$  is a subgraph of  $G_0$  with the same number of points and the same number of edges and therefore coincides with  $G_0$ . As to the proof of the first statement of the theorem, clearly each line contains 8 points. There are exactly 9 lines containing the point 1, namely 4 in the  $A$ -orbit of  $L_1$ , 4 in the  $A$ -orbit of  $L_5$ , and  $L_6$ . They are denoted by  $L_1, L_2, L_3, L_4, L_5, L_7, L_8, L_9$  and  $L_6$  respectively and written out explicitly in table 1. As no point  $\neq 1$  occurs twice in this table, axioms (i), (iii) hold if one of the points concerned is 1. But the group  $A$  acts transitively of the 120 points of  $V$ , so the two axioms hold without restriction. Finally, as  $G(V, L)$  is strongly regular, axiom (iv) is a consequence of the Lemma.  $\square$

REMARKS (i) Let  $\Omega$  be the subset of  $A_5$  consisting of all elements in the  $A_5$ -conjugacy classes of  $(12345)$  and  $(12)(34)$ . The complete subgraph of  $G_0$  on the points of  $A_5$  is the Cayley graph  $\Gamma(A_5, \Omega)$  in the BIGGS' notation [1]. If  $\Omega_1$  is the union of the  $A_5$ -conjugacy classes  $(12354)$  and

table 1  
The lines in  $L$  containing 1

line	elements in the line							
$L_1$	1	(15243)	(13254)	(12345)	$\tau(23)(15)$	$\tau(34)(25)$	$\tau(13)(45)$	$\tau(124)$
$L_2$	1	(13425)	(12453)	(14352)	$\tau(12)(34)$	$\tau(24)(35)$	$\tau(13)(25)$	$\tau(145)$
$L_3$	1	(14523)	(15324)	(13542)	$\tau(13)(24)$	$\tau(14)(35)$	$\tau(23)(45)$	$\tau(152)$
$L_4$	1	(15432)	(12534)	(14235)	$\tau(14)(23)$	$\tau(34)(15)$	$\tau(12)(35)$	$\tau(254)$
$L_5$	1	(12)(34)	(13)(24)	(14)(23)	$\tau(142)$	$\tau(243)$	$\tau(134)$	$\tau(123)$
$L_6$	1	(14)(25)	(12)(45)	(24)(15)	$\tau.1$	$\tau(14)(25)$	$\tau(12)(45)$	$\tau(24)(15)$
$L_7$	1	(15)(23)	(12)(35)	(13)(25)	$\tau(132)$	$\tau(235)$	$\tau(125)$	$\tau(315)$
$L_8$	1	(23)(45)	(25)(34)	(24)(35)	$\tau(234)$	$\tau(354)$	$\tau(245)$	$\tau(253)$
$L_9$	1	(14)(35)	(15)(34)	(13)(45)	$\tau(143)$	$\tau(135)$	$\tau(154)$	$\tau(345)$

table 2  
The lines in  $L$  containing  $\tau$

line	elements in the line							
$L_{\tau 1}$	(142)	(13)(25)	(23)(45)	(15)(34)	$\tau$	$\tau(14325)$	$\tau(13452)$	$\tau(15423)$
$L_{\tau 2}$	(15)(23)	(12)(34)	(14)(35)	(245)	$\tau$	$\tau(14532)$	$\tau(15234)$	$\tau(12435)$
$L_{\tau 3}$	(34)(25)	(13)(24)	(12)(35)	(154)	$\tau$	$\tau(15342)$	$\tau(12543)$	$\tau(13524)$
$L_{\tau 4}$	(13)(45)	(14)(23)	(24)(35)	(125)	$\tau$	$\tau(13245)$	$\tau(14253)$	$\tau(12354)$
$L_{\tau 5}$	(124)	(234)	(143)	(132)	$\tau$	$\tau(12)(34)$	$\tau(13)(24)$	$\tau(14)(23)$
$L_{\tau 6}$	1	(14)(25)	(12)(45)	(15)(24)	$\tau$	$\tau(14)(25)$	$\tau(12)(45)$	$\tau(15)(24)$
$L_{\tau 7}$	(135)	(253)	(123)	(152)	$\tau$	$\tau(13)(25)$	$\tau(15)(23)$	$\tau(12)(35)$
$L_{\tau 8}$	(432)	(254)	(235)	(345)	$\tau$	$\tau(23)(45)$	$\tau(25)(34)$	$\tau(24)(35)$
$L_{\tau 9}$	(134)	(145)	(354)	(153)	$\tau$	$\tau(15)(34)$	$\tau(13)(25)$	$\tau(14)(35)$



(12) (34), then  $\Gamma(A_5, \Omega_1)$  is isomorphic to the complete subgraph of  $G(V, L)$  on the points of  $\tau A_5$ . Finally, for  $x, y \in A_5$  the points  $x, \tau y$  of  $V$  are joined in  $G(V, L)$  if and only if  $xy^{-1} \in \Omega_2$ , where  $\Omega_2$  is the union of the  $A_5$ -conjugacy classes of 1, (12) (34) and (123).

(ii) In view of (i) it will be clear that verification  $G_0$  is strongly regular and therefore the proof of the first statement of the theorem could be done without using quaternions or  $E_8$ .

(iii)  $A$  is a subgroup of  $\text{Aut}(V, L)$  of order  $2^5 \cdot 3^2 \cdot 5$ . On the other hand,  $\text{Aut}(V, L)$  is a subgroup of  $\text{Aut}(E_8)/\{\pm I\}$ , so the order of  $\text{Aut}(V, L)$  must divide  $2^{13} \cdot 3^5 \cdot 5^2 \cdot 7$ .

Note that  $\text{Aut}(V, L)$  cannot be all of  $\text{Aut}(G_0)$ , since the latter group acts transitively on the family of  $15 \times 135$  8-cliques in  $G_0$ , while no more than 135 8-cliques of  $G_0$  originate from lines in  $L$ .

(iv) Consideration of parameters might lead to the expectation that the choice of an appropriate sub-family  $L_0$  of lines in  $L$  provides a partial geometry on  $V$  with parameters  $(s, t, \alpha) = (7, 4, 2)$  or (even weaker) a strongly regular graph with parameters  $(v, k, \lambda, \mu) = (120, 35, 10, 10)$ . However no selection of  $A$ -orbits from  $L$  leads to such a family  $L_0$ .

#### REFERENCES

- [1] BIGGS, N., *Algebraic graph theory*, Cambridge University Press, Cambridge, 1974.
- [2] BOURBAKI, N., *Groupes et algèbres de Lie*, Chap IV, V, VI, Hermann, Paris, 1968.
- [3] BOSE, R.C., *Strongly regular graphs, partial geometries, and partially balanced designs*, Pac. J. Math. 13, (1963) 389-419.
- [4] SERRE, J.-P., *A course in Arithmetic*, Springer Verlag (Graduate Texts in Math No 7) 1973.