

Computation of flow in interrupted wall channels

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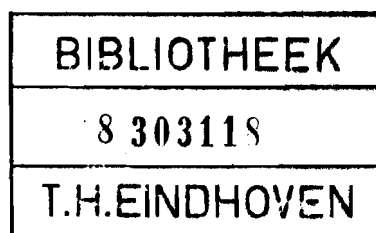
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Computing Centre Note 10

Computation of flow in
interrupted wall channels.

By: Lieuwe Sytse de Jong



1. Introduction.

Interrupted wall channels are employed in heat exchangers because of their high convective heat transfer coefficients. It is important to determine patterns of interruptions with high performance. The performance could be defined as the ratio of the heat exchanged and the material used. Numerous investigations have been done in relation to this subject. We mention especially the work of E.M. Sparrow and S.V. Patankar.

In this paper the emphasis is on the numerical computation of the air flow in such channels. We believe that we have found a rather simple numerical method that differs from the ones reported on in the literature. Starting point for the analysis are the boundary layer equations for fully developed, incompressible, laminar flow in a plane. For the patterns of interruptions investigated here, a transformation of the equations to dimensionless form is possible such that only a single dimensionless parameter combination remains. The equation of motion contains a pressure term. In literature this term is treated by numerical means, which may lead to an iterative scheme as in the Sparrow, Baliga, Patankar paper [4]. Here we eliminate the term by analytical means. The resultant numerical method is non-iterative. The flow velocity is computed by numerical integration along a finite number of streamlines.

For a particular pattern of interruptions an agreement of about 10 percent with actual measurements might be achieved using simple numerical methods such as Runge-Kutta second order.

2. Problem statement.

Although the coming analysis applies to many different configurations, it is convenient to restrict the attention to one particular pattern. We have chosen the one displayed in figure 1. The pattern is symmetric with respect to the lines $y = 0, \pm 3H, \pm 6H, \dots$ and periodic in x with period $3L$. Thus a zigzag overall pattern is obtained. We suppose that the pattern extends infinitely long in the z^+ and the z^- direction. Therefore we can regard the flow as a two-dimensional one.

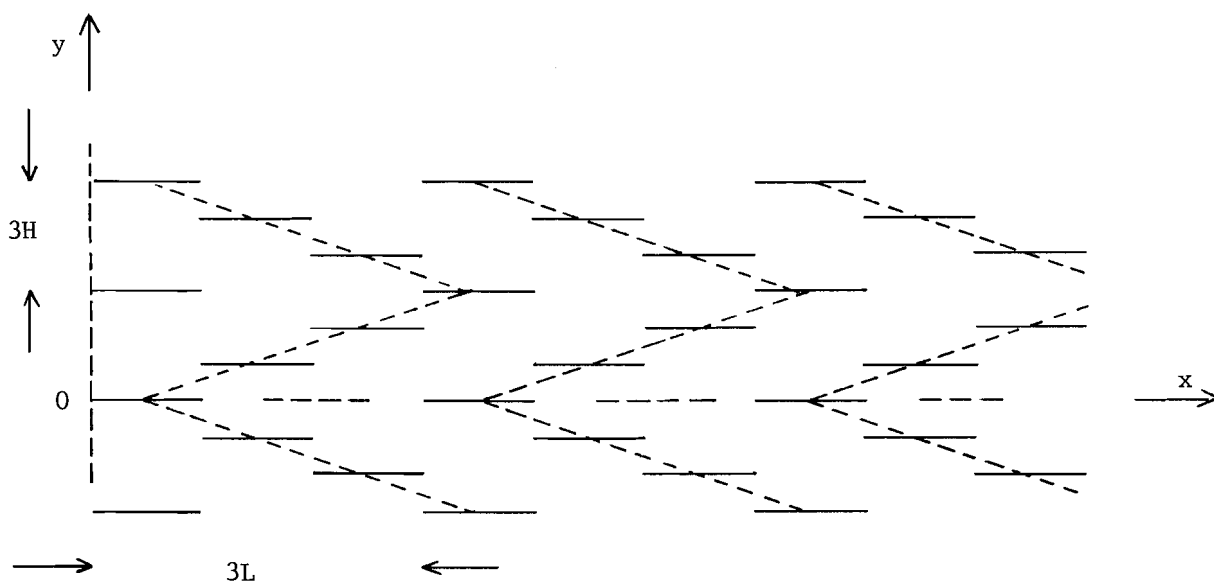


figure 1

Due to the symmetry and periodicity it is sufficient to consider the region $R = \{0 < x < 3L, 0 < y < 3H\}$. The channel walls contained in R shall be denoted as W_1, W_2, W_3 and W_4 . The region R minus the walls W_1, W_2, W_3 and W_4 is denoted as R' . According to Sparrow, Abdel-Nahed and Patankar [1], it is permitted to apply the boundary-layer equations to the flow in interrupted wall channels provided that the interrupted part in any wall is larger than the non-interrupted part.

The boundary-layer equations for stationary, laminar, incompressible flow are given by the continuity equation:

$$(2.1) \quad u_x + v_y = 0 \quad \text{for } (x,y) \in R',$$

and the equation of motion:

$$(2.2) \quad uu_x + vv_y = p_x/\rho + \nu u_{yy} \quad \text{for } (x,y) \in R'.$$

The velocity in the x -direction is denoted by u , the velocity in the y -direction is denoted by v , the pressure is denoted by p , whereas ρ and ν denote respectively the density and the kinematic viscosity of the flow. These equations are well-known in the literature, e.g. [3].

We assume the following initial condition at $x = 0$.

$$(2.3) \quad u = U, v = 0 \quad \text{for } x = 0 \text{ and } 0 < y < 3H.$$

The boundary conditions are

$$(2.4) \quad \begin{aligned} u_y = v = 0 & \quad \text{for } (L < x < 3L) \text{ and } (y = 0 \text{ or } y = 3H). \\ u = v = 0 & \quad \text{at the channel walls.} \end{aligned}$$

We shall describe a procedure for computing u and v with increasing x . The periodic flow as it develops in reality can be found by applying this procedure repeatedly with as new initial values for u and v the obtained end-values at $x = 3L$.

3. Transformation of the equations.

Applying the Von Mises transformation (a transformation to streamlines)

$$(3.1) \quad \xi := x/(3L); \quad \eta := \psi(x,y)/(3UH); \quad \psi_x = -v; \quad \psi_y = u$$

in combination with the scaling transformation

$$(3.2) \quad \tilde{u} := u/U; \quad \tilde{v} := vL/(UH); \quad \tilde{p} := p/(\rho U^2)$$

we obtain the equations in the following form

$$(3.3) \quad \begin{aligned} \tilde{u}_\xi - \tilde{v}\tilde{u}_\eta + \tilde{u}\tilde{v}_\eta &= 0 \quad (\text{equation of continuity}) \\ \tilde{u}_\xi &= -\tilde{p}_\xi/\tilde{u} + \gamma(\tilde{u}\tilde{u}_\eta)_\eta \quad (\text{equation of motion}) \end{aligned}$$

Here γ denotes the dimensionless parameter $U(3H^2/L)/\gamma$.

It is obvious that we have $0 < \xi < 1$ in R .

From the definition of η it follows that any line along which η is constant must be a streamline. It also follows that along streamlines η must be constant. Since the channel walls can be considered as streamlines, there exist constants η_i corresponding with the walls W_i ($i = 1, \dots, 4$).

Furthermore, $y = 0$ and $y = 3H$ must be streamlines due to the symmetry in the pattern. Thus, we conclude:

$$(3.4) \quad \begin{aligned} y = 0 & \text{ transforms into } \eta = \eta_1 \text{ for } 0 < \xi < 1 \\ y = H & \text{ transforms into } \eta = \eta_2 \text{ for } 1/3 < \xi < 2/3 \\ y = 2H & \text{ transforms into } \eta = \eta_3 \text{ for } 2/3 < \xi < 1 \\ y = 3H & \text{ transforms into } \eta = \eta_4 \text{ for } 0 < \xi < 1 \end{aligned}$$

From (3.4) we derive several relations for the η_1 . First, we have

$$(3.5) \quad \eta_4 - \eta_1 = \int_{\eta_1}^{\eta_4} d\eta = \int_0^{3H} \frac{u}{3UH} dy = \int_0^{3H} \frac{\tilde{u}}{3H} dy = 1 \quad \text{for } 0 < \xi < 1.$$

This relation tells that the flow-transport between $y = 0$ and $y = 3H$ is constant. Choosing $\eta_1 = 0$, we obtain $\eta_4 = 1$ and we conclude that $R = \{0 < \xi < 1, 0 < \eta < 1\}$.

Next we have

$$(3.6) \quad \eta_2 = \int_{\eta_1}^{\eta_2} d\eta = \int_0^H \frac{u}{3UH} dy = \int_0^H \frac{\tilde{u}}{3H} dy \quad \text{for } 1/3 < \xi < 2/3$$

and

$$(3.7) \quad \eta_3 = \int_{\eta_1}^{\eta_3} d\eta = \int_0^{2H} \frac{u}{3UH} dy = \int_0^{2H} \frac{\tilde{u}}{3H} dy \quad \text{for } 2/3 < \xi < 1.$$

In words, η_2 and η_3 are flow transports respectively between ($y = 0$ and W_2) and ($y = 0$ and W_3). So η_2 and η_3 can be computed simultaneously with \tilde{u} and \tilde{v} .

We can also derive an integral equation for \tilde{u} :

$$(3.8) \quad 1 = \int_0^{3H} \frac{dy}{3H} = \frac{1}{3H} \int_{\eta_1}^{\eta_4} \frac{\partial y}{\partial \eta} d\eta = \int_{\eta_1}^{\eta_4} \frac{d\eta}{\tilde{u}} \quad \text{for } 0 < \xi < 1/3$$

$$(3.9) \quad 1/3 = \int_0^H \frac{dy}{3H} = \int_{\eta_1}^{\eta_2} \frac{d\eta}{\tilde{u}} \quad \text{for } 1/3 < \xi < 2/3$$

$$(3.10) \quad 2/3 = \int_0^{2H} \frac{dy}{3H} = \int_{\eta_1}^{\eta_3} \frac{d\eta}{\tilde{u}} \quad \text{for } 2/3 < \xi < 1$$

We have not yet given the transformed initial and boundary conditions. These are

$$\begin{aligned}
 (3.11) \quad & \tilde{u} = 1, \tilde{v} = 0 && \text{for } \xi = 0, 0 < \eta < 1 \\
 & \tilde{u}\tilde{u}_\eta = 0, \tilde{v} = 0 && \text{for } 1/3 < \xi < 1, \eta = 0 \text{ or } \eta = 1 \\
 & \tilde{u} = \tilde{v} = 0 && \text{at the channel walls}
 \end{aligned}$$

4. Analysis of the problem.

In the sequel we deal with the equations (3.3) under the conditions (3.11) and use the relations (3.6) - (3.10). The tildes will be omitted. First we reconsider R . We divide R into subregions $R_{\lambda\mu}$ such that channel walls are encountered only at the boundaries of the subregions:

$$\begin{aligned}
 (4.1) \quad & R_{11} := \{0 < \xi < 1/3, 0 < \eta < 1\} \\
 & R_{21} := \{1/3 < \xi < 2/3, 0 < \eta < \eta_2\} \\
 & R_{22} := \{1/3 < \xi < 2/3, \eta_2 < \eta < 1\} \\
 & R_{31} := \{2/3 < \xi < 1, 0 < \eta < \eta_3\} \\
 & R_{32} := \{2/3 < \xi < 1, \eta_3 < \eta < 1\}
 \end{aligned}$$

In each region $R_{\lambda\mu}$ we shall formulate an initial boundary value problem. If $\lambda = 1$, the initial values in $R_{\lambda\mu}$ are given by (3.11); if $\lambda = 2$ or $\lambda = 3$, the initial values in $R_{\lambda\mu}$ are equal to the end-values in $R_{(\lambda-1), \mu}$.

At the η -boundaries of an $R_{\lambda\mu}$ it either holds that $u = v = 0$ if the η -boundary coincides with a channel wall or it holds that $u u_\eta = v = 0$ if the η -boundary coincides with a line of symmetry.

We assume that the initial/boundary value problem for $R_{\lambda\mu}$ has an unique solution. We also assume that any of the forthcoming expressions has sense. For instance, that occurring improper integrals are convergent.

The subregions $R_{\lambda\mu}$ shall be denoted as

$$R_{\lambda\mu} = \{\xi_\ell < \xi < \xi_b, \eta_\ell < \eta < \eta_b\}$$

$R'_{\lambda\mu}$ shall denote $R_{\lambda\mu}$ minus the channel walls at the boundaries of $R_{\lambda\mu}$.

Next, let us consider the equations (3.3).

Dividing the equation of continuity by u^2 , we obtain

$$(4.2) \quad u_{\xi} u^{-2} - v u_{\eta}^{-1} + v_{\eta} u^{-1} = 0 \quad \text{for } (\xi, \eta) \in R'_{\lambda\mu}.$$

After integration with respect to η , we have

$$(4.3) \quad v u^{-1} = - \int_{\eta_{\ell}}^{\eta} u_{\xi} u^{-2} d\eta + C(\xi) \quad \text{for } (\xi, \eta) \in R_{\lambda\mu}.$$

The term $C(\xi)$ is a constant of integration. Equation (4.3) determines the direction in the scaled (x-y) plane of a streamline passing through a point (ξ, η) . Keeping η constant and varying ξ , generates the differential equation for one particular streamline. Thus the streamline $\eta_s = \text{constant}$ (or $\theta = \theta(\xi, \eta_s)$) obeys the equation:

$$(4.4) \quad \frac{d\theta(\xi, \eta_s)}{d\xi} = - \int_{\eta_{\ell}}^{\eta_s} u_{\xi} u^{-2} d\eta + C(\xi) \quad \text{for } (\xi, \eta_s) \in R_{\lambda\mu}.$$

Taking $\eta_s = \eta_{\ell}$ and $\eta_s = \eta_b$ we find, respectively,

$$(4.5) \quad C(\xi) = 0$$

$$(4.6) \quad \int_{\eta_{\ell}}^{\eta_b} u_{\xi} u^{-2} d\eta = 0$$

$$\left. \begin{array}{l} (4.5) \\ (4.6) \end{array} \right\} \quad \text{for } (\xi_1 < \xi < \xi_b)$$

From this it follows that

$$(4.7) \quad v = - u \int_{\eta_{\ell}}^{\eta} u_{\xi} u^{-2} d\eta \quad \text{for } (\xi, \eta) \in R_{\lambda\mu}.$$

Integrating (4.4) and using that (4.5) and (4.6) hold for any $R_{\lambda\mu}$ we find for the streamlines the expression:

$$(4.8) \quad \theta(\xi, \eta_s) = \theta(0, \eta_s) + \int_0^{\eta_s} \frac{d\eta}{u} \quad \text{for } (\xi, \eta) \in R.$$

Thus, when u is known, v and θ follow from (4.7) and (4.8).

Let us now consider the equation of motion. Dividing it by u^2 gives

$$(4.9) \quad \frac{u_\xi}{u^2} = \frac{-p_\xi}{u^3} + \frac{\gamma(uu_\eta)_\eta}{u^2} \quad \text{for } (\xi, \eta) \in R'_{\lambda\mu}.$$

Integrating (4.9) with respect to η and using (4.6) results in the formula

$$(4.10) \quad \int_{\eta_\ell}^{\eta_b} \left(\frac{-p_\xi}{u^3} + \frac{\gamma(uu_\eta)_\eta}{u^2} \right) d\eta = 0 \quad \text{for } (\xi_\ell < \xi < \xi_b).$$

If u is known on an initial line $\xi = \xi_0$, then $(uu_\eta)_\eta$ is also known on that line. It is then possible to determine p_ξ and u_ξ at the line $\xi = \xi_0$ from (4.10) and (4.9). Consequently, the equation of motion can be reduced to a partial equation that only involves u and partial derivatives of u . This means that u can be solved. Before proceeding to the numerical details in section 5 we examine formula (4.10). Let us for the moment suppose that $\eta = \eta_\ell$ corresponds with a wall and that for some positive number k

$$u = O((\eta - \eta_\ell)^k) \quad \text{for } (\eta - \eta_\ell) \rightarrow 0.$$

Then the two terms in the integrand of (4.10) behave respectively as

$$O((\eta - \eta_\ell)^{-3k}) \quad \text{and} \quad O((\eta - \eta_\ell)^{-2}) \quad \text{for } (\eta - \eta_\ell) \rightarrow 0.$$

Since the last behaviour is not integrable, it must be annihilated by the first behaviour. This is possible only if $k = 2/3$. Hence, near a channel wall we have

$$(4.14) \quad u = O((\eta - \eta_\ell)^{2/3}) \quad \text{for } (\eta - \eta_\ell) \rightarrow 0.$$

One may show that this implies

$$(4.15) \quad u = O(d^2) \quad \text{for } d \rightarrow 0,$$

where d denotes the Euclidean distance to a channel wall.

In literature the pressure term in the equation of motion is not eliminated, but treated as an unknown function for which an equation must be present. In [4], the extra equation is our equation (3.8). Because that equation is non-linear, an iterative solution method is necessary in the numerical computations.

Our subsection 4 shows that an iterative solution is not necessary.

5. The numerical method.

We have the differential equation for u :

$$(5.1) \quad u_\xi = -p_\xi/u + \gamma(uu_\eta)_\eta \quad \text{with } (\xi, \eta) \in R'_{\lambda\mu}.$$

For the pressure gradient p_ξ we have the following equation

$$(5.2) \quad \int_{\eta_\ell}^{\eta_b} \frac{-p_\xi}{u^3} + \frac{\gamma(uu_\eta)_\eta}{u^2} d\eta = 0 \quad \text{where } \xi_\ell < \xi < \xi_b.$$

The initial value of u at $\xi = \xi_\ell$ is known. The following boundary conditions hold:

$$\begin{aligned} u &= 0 \quad \text{at a channel wall } (\eta = \eta_\ell \text{ or } \eta = \eta_b) \\ uu_\eta &= 0 \quad \text{at a symmetry line } (\eta = \eta_\ell \text{ or } \eta = \eta_b) \end{aligned}$$

The vertical velocity v satisfies the equation

$$(5.3) \quad v = (-u) * \int_{\eta_\ell}^{\eta} \frac{u_\xi}{u^2} d\eta = (-u) * \int_{\eta_\ell}^{\eta} \frac{-p_\xi}{u^3} + \frac{\gamma(uu_\eta)_\eta}{u^2} d\eta.$$

The streamlines follow from:

$$(5.4) \quad \theta(\xi, \eta) = \theta(0, \eta) + \int_{\eta}^{\eta} \frac{d\eta}{u}$$

If u is known on an initial line $\xi = \xi_0$, then also (uu_{η}) , p_{ξ} (via (5.2)), v (via (5.3)) and u_{ξ} (via (5.1)) are known on that same line. This suggests a numerical method in which u is found in the rectangle $R_{\lambda\mu}$ by marching in the positive ξ -direction. First we semidiscretize the equations with respect to η . We choose a division of $[\eta_a, \eta_b]$:

$$\eta_a = \eta_0 < \eta_1 < \eta_2 < \dots < \eta_M = \eta_b$$

and replace derivatives and integrals with respect to η by finite differences and finite sums, using only values at the lines $\eta = \eta_i$ ($i = 0, \dots, M$). Because $\eta = \text{constant}$ corresponds with a streamline, this means marching along streamlines. The velocities along a streamline $\eta = \eta_i$ will be indicated by subindexing: $u_i(\xi)$ denotes the value of $u(\xi, \eta_i)$. The derivatives with respect to η will be approximated by central differences. So $(uu_{\eta})_{\eta}$ is replaced as follows:

$$\begin{aligned} (uu_{\eta})_{\eta}(\xi, \eta_i) &\cong \frac{(uu_{\eta})(\xi, \eta_{i+\frac{1}{2}}) - (uu_{\eta})(\xi, \eta_{i-\frac{1}{2}})}{\eta_{i+\frac{1}{2}} - \eta_{i-\frac{1}{2}}} \\ &= \left(u_{i+\frac{1}{2}} * \frac{u_{i+1} - u_i}{\eta_{i+1} - \eta_i} - u_{i-\frac{1}{2}} * \frac{u_i - u_{i-1}}{\eta_i - \eta_{i-1}} \right) / (\eta_{i+\frac{1}{2}} - \eta_{i-\frac{1}{2}}) \end{aligned}$$

Replacing $u_{i+\frac{1}{2}}$ and $u_{i-\frac{1}{2}}$ by the mean of respectively u_{i+1} and u_i , u_i and u_{i-1} , we find the approximation:

$$(5.5) \quad (uu_{\eta})_{\eta}(\xi, \eta_i) \cong \left(\frac{u_{i+1}^2 - u_i^2}{\Delta\eta_i} - \frac{u_i^2 - u_{i-1}^2}{\Delta\eta_{i-1}} \right) / (\Delta\eta_i + \Delta\eta_{i-1})$$

for $1 < i < M - 1$.

Here $\Delta\eta_i$ is short for $(\eta_{i+1} - \eta_i)$. This formula cannot be used at the boundaries because u_{-1} and u_{M+1} are not defined. We do not need an approximation for $(uu_\eta)_\eta$ at a channel wall, but we do need an approximation at a symmetry line, because there u is not known on beforehand. For the moment let us suppose that $\eta = \eta_0$ is a line of symmetry. Defining u_{-1} by $u_{-1} := u_{-1}$ and $\eta_{-1} := \eta_0 - (\eta_1 - \eta_0)$ and applying (5.5) for $i = 0$ gives:

$$(5.6) \quad (uu_\eta)_\eta(\eta = \eta_0) \cong \frac{u_1^2 - u_0^2}{(\Delta\eta_0)^2}$$

By a similar procedure it is possible to find an approximation for $(uu_\eta)_\eta$ if $\eta = \eta_M$ is a line of symmetry.

It can be shown that the discretization error in (5.5) and (5.6) is of the order of $(\Delta\eta_i - \Delta\eta_{i-1})$. If the lines $\eta = \eta_i$ are equidistant, then the error is even of the order of $(\Delta\eta)^2$.

The integrals with respect to η are replaced by finite sums. Our

approximation to an integral $\int_{\eta_1}^{\eta_i} S d_\eta$ becomes formally

$$(5.7) \quad \sum_{i=1}^{M-1} \Delta\eta_i * W_i$$

Each term in this sum corresponds with a subinterval $[\eta_i, \eta_{i+1}]$.

For W_i we take $\frac{1}{2}(S_i + S_{i+1})$, if there are no singularities in the subinterval. If there is a singularity (say in η_i), then we take for W_i : $3S_{i+1}$. Doing so, we account for the $\eta^{-2/3}$ behaviour of the integrands near a singularity. It can be shown that the discretization error is of the order of $(\Delta\eta)^2$.

Effectuation of formulas (5.5) through (5.7) yields a system of ordinary differential equations for $u_i(\xi)$. This system is solved by standard numerical integration methods.

Before proceeding to the results, let us mention a last problem.

In section 4 we suggested that the initial values of u in a rectangle $R_{\lambda\mu}$ can be obtained from the end-values of u in the previous rectangle. Inspecting (5.1) we see that the initial value of u may not be equal to zero, because in the right-hand side division by u occurs.

Indeed, the initial value of u is zero on streamlines rising just after a channel wall. We remedied for this by taking a non-zero initial value for u . We let the non-zero value correspond with the values of u at neighbouring streamlines (e.g. by interpolating those values).

6. Results.

The numerical results were produced by applying a Runge-Kutta integration method to the system of ordinary differential equations resulting from the previous section. The order of the method was 2. The ξ -step was fixed and so chosen that the beginning and the end of each wall section coincided with an integer number of ξ -steps. From experiments with various ξ -steps it followed that a ξ -step of 0.0416 gave results accurate within two figures (that is accurate with respect of the system of ordinary differential equations).

The measured values were obtained by John Baken of the University of Technology of Eindhoven. He measured velocities in a model using a Laser-Doppler method.

The following pictures show the profile of the velocity at $\xi = 1/12, 3/12, 5/12, 7/12, 9/12$ and $11/12$ respectively.

The dotted profiles were obtained from the measurements.

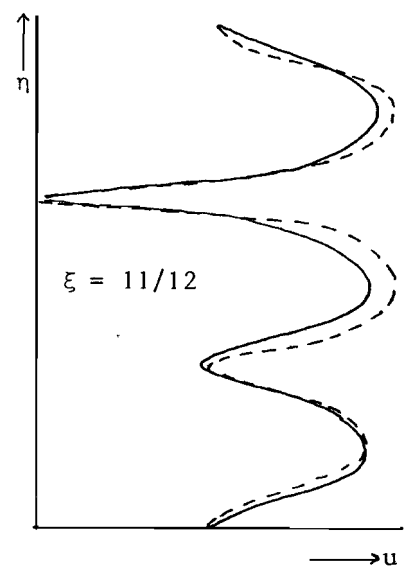
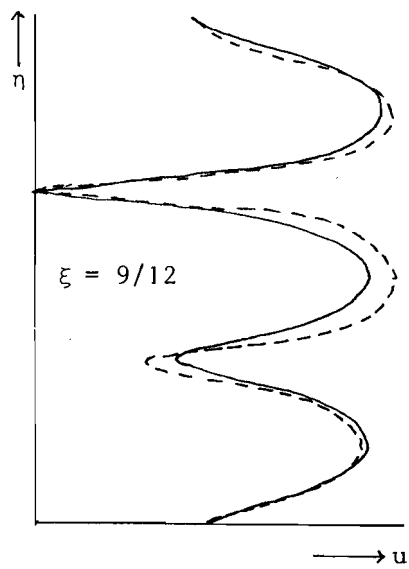
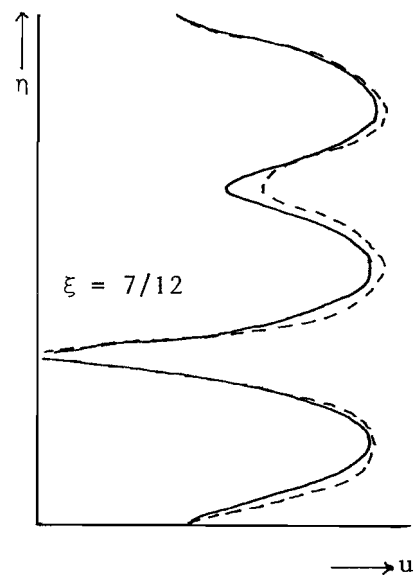
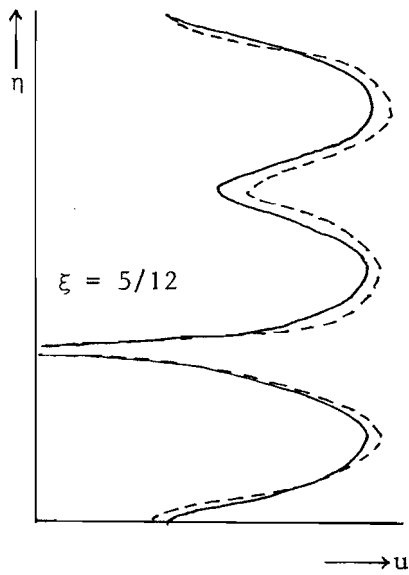
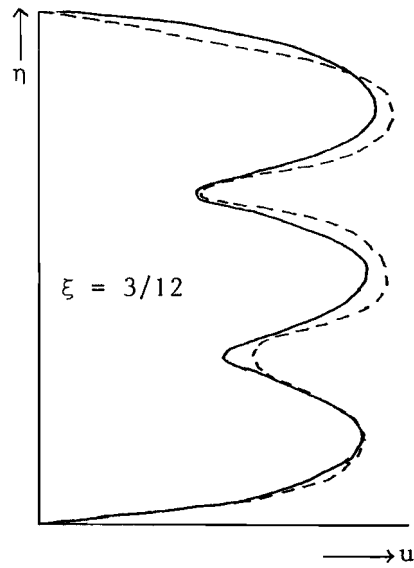
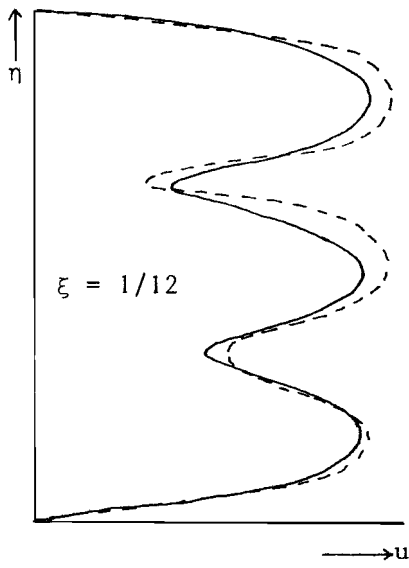


figure 2

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