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GEOMETRIC DETERMINATION OF COORDINATED CENTERS OF CURVATURE IN NETWORK MECHANISMS THROUGH LINKAGE REDUCTION

EVERT A. DIJKSMAN
Department of Mechanical Engineering, Eindhoven University of Technology, The Netherlands

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Abstract—Coordinated centers of curvature in a network mechanism may be found by way of *linkage reduction*. This has to be carried out twice, each time in a different way, namely, a first order reduction through *joint-joining* in order to determine the velocity poles,[1] and a second order reduction, that equally replaces binary bars, but this time preserves instantaneous motion up to the 2nd order.

For the reduced linkage, the problem of finding coordinated centers of curvature may be solved by successive application of Bobillier's Theorem in different four-bar loops.

In order to show applicability also for linkages not containing four-bars, the method will be demonstrated for an *eight*-bar linkage that contains only pentagonal loops. The method introduced is a purely geometric one and does not involve velocity or acceleration constructions. Notwithstanding that, the final result may be used also to determine accelerations in linkage mechanisms.

1. INTRODUCTION

Planar mechanisms, containing gear-wheels, chains, sliding- or rolling curves, may all be replaced, instantaneously, by network linkages. This may be done in such a way, that a second-order motion approximation is preserved, meaning that for the replaced mechanism as well as for the obtained linkage, three infinitesimally close positions of their corresponding links are identical. Such a replacement linkage mechanism may then be used to calculate velocities and accelerations or to determine curvatures of paths traced by points of the mechanism.

Linkages, however, may have a labyrinthine structure, for which it may be difficult to determine coordinated centers of curvature. It is the intention of this paper to solve this problem, even for linkages not containing four-bar loops.

The example to be used as demonstration, will be an 8-bar linkage as shown in Fig. 1(A). This network linkage resembles a one degree of freedom mechanism but which has no sub-chain with the mobility 1.

2. THE PRINCIPLE OF JOINT-JOINING THAT PRESERVES A 1st ORDER EQUIVALENCY OF MOTION[1]

Kinematic chains containing ternary links may be simplified by transforming such links into binary ones. This may be done by *joining* (or combining) two turning-*joints* of a ternary link in such a way that a double-joint is created. The location of the double-joint, however, is not arbitrary, but determined by the intersection (I) of the two binary links, turning about the joints we intend to combine. Clearly, the two binary links involved are then simultaneously replaced by other ones, in Fig. A to be recognized through the numbers that are provided with primes. The transformation, just explained, will be named a (first order) *joint-joining* operation. The operation preserves instantaneous (first order) motion between the links that appear in the original as well as in the reduced chain. The double-joint I, namely,† doesn't have any relative velocity against the ternary link and may therefore be regarded as a point rigidly attached to the ternary link now being transformed into a binary one.

Joint-joining operations applied on linkages turn, for instance, pentagonal loops into four-bar ones and, therefore, simplify labyrinthine chains in such a way that it becomes much easier to determine the velocity poles that exist in the chain. In a way, the joint-joining operation may be seen as an extension of the Aronhold–Kennedy Rule, thus far in use to determine the poles. Now, new intersection points I have been introduced that had no meaning before the transformation, but do obtain significance after the operation.

For most chains repeated intersection of Aronhold–Kennedy lines usually suffice to determine all the relative poles between the links that appear in the chain. Problems arise, however, if four-bar loops do not exist in the chain. Then, joint-joining operations become handy to find the poles through I-points. The Aronhold–Kennedy Rule repeatedly applied on the reduced linkage allows the designer to find the poles by way of subsequent intersection of lines not otherwise available.

3. JOINT-JOINING AND FIRST ORDER REDUCTION APPLIED TO THE 8-BAR[1]

In order to find the velocity poles of the linkage, we four times combine two joints of triangular links

†If we regard the newly created dyad-linkage as being adjoined to the original mechanism, it is easy to prove that its dyad-joint (I) has no velocity relative to the ternary link.

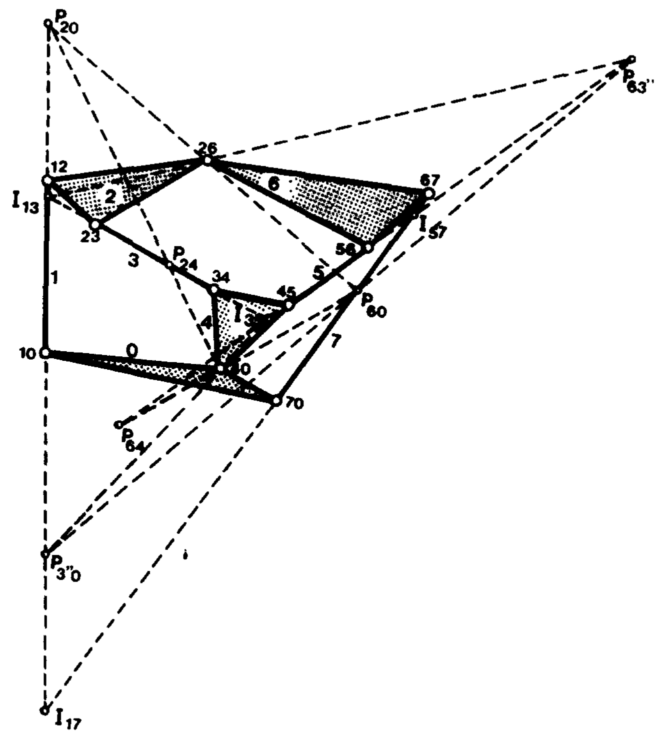

Fig. 1A. Initial 8-bar with determination of the poles $P_{60}P_{20}P_{24}$ and P_{64}.

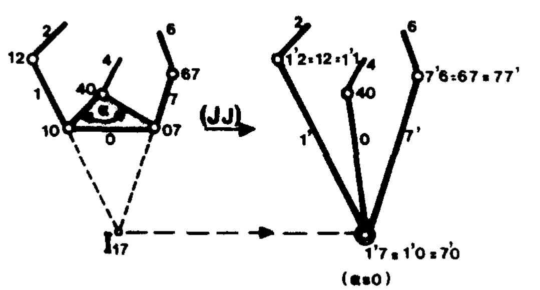

Fig. A. The 1st order *joint-joining* operation.

at the intersection points I of the binary links. Thus (see Fig. 1B)

$$I_{17} = (21 - 10) \times (67 - 70)$$
$$I_{35} = (65 - 54) \times (23 - 34)$$
$$I_{13} = (23 - 34) \times (21 - 10)$$
$$I_{57} = (65 - 54) \times (67 - 70).$$

This way, four times a joint-joining operation has been carried out, reducing the linkage into a basic four-bar with two dyads, each of them connecting opposite joints of the four-bar. The 8-bar, so reduced, still contains four binary links, having the same instantaneous motion of the former ternary links of the initial eight-bar. Successive application of

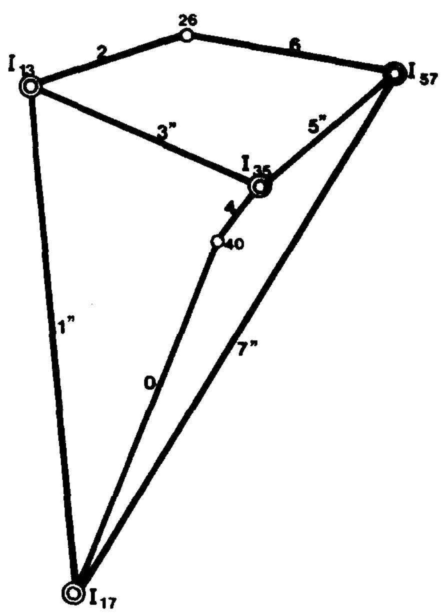

Fig. 1B. *Reduced* 8-*bar* with 1st order motion equivalency (after 4 joint-joinings of 1st order).

Aronhold–Kennedy's Theorem on the reduced linkage then gives all the locations of the poles that we are looking for. (See again Fig. 1A.)

For instance:

$$P_{63'} = (65 - 54) \times (62 - I_{13})$$
$$P_{3'0} = (21 - 10) \times (I_{35} - 40)$$
$$P_{60} = (67 - 70) \times (63'' - 3''0)$$
$$P_{64} = (60 - 04) \times (65 - 54)$$
$$P_{20} = (21 - 10) \times (26 - 60)$$
$$P_{24} = (23 - 34) \times (20 - 04), \text{etc.}$$

A more extensive explanation of the method of joint-joining may be read in the paper "Why joint-joining is applied on complex linkages"[1].

Clearly, the first-order reduction of the 8-bar through joint-joining has provided us with an easy, geometric method to determine all the relative poles of the 8-bar, we are investigating. It represents, in fact, the first step in understanding the motion of the linkage.

4. SECOND-ORDER REDUCTION OF THE 8-BAR THROUGH REPLACEMENT OF BINARY LINKS

For this kind of reduction, all the relative poles between the links must be known. The reduction is carried out in stages by successive relocation of the end joints of the binary links 1, 3, 5 and 7, in a special way. (See also the Figs. 2(A) 3 and 4.)

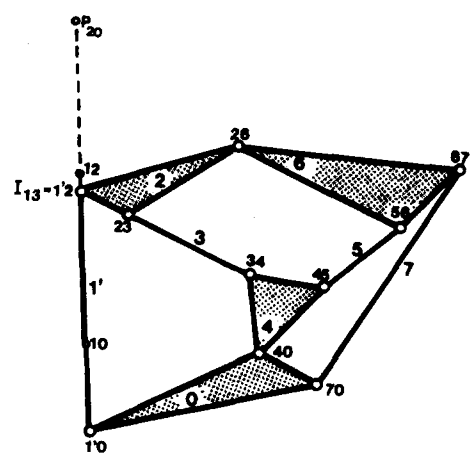

Fig. 2A. Substitution of bar 1 by bar 1′, preserving 2nd order motion between the remaining links.

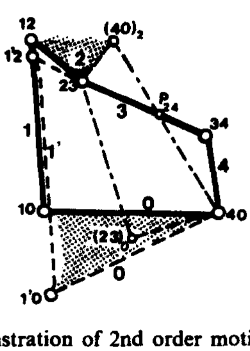

Fig. 2B. Demonstration of 2nd order motion equivalency.

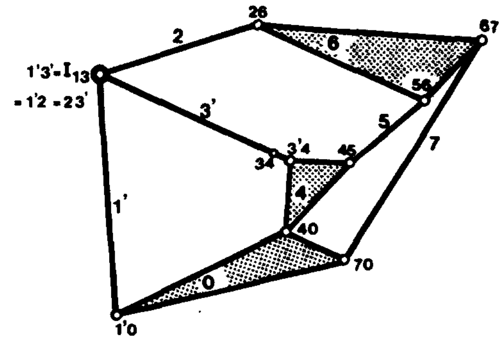

Fig. 3. 2nd order joint-joining applied on 8-bar linkage.

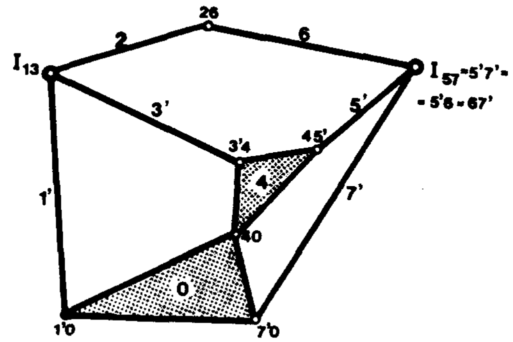

Fig. 4. 2nd order joint-joining twice applied. *Reduced* 8-*bar*, preserving 2nd order motion between even-numbered links. (Points 1′0, 3′4, 45′ and 7′0 to be obtained through calculation).

First, we let the substitute bars 1′, 3′, 5′ and 7′ run along the corresponding original bars. This alone will preserve the assumed configuration of the poles and thus maintain at least first-order equivalence of motion. A better, that is to say a 2nd-order, approximation of the motion may be obtained if, in addition, the replaced end joints of each bar are chosen interdependently. The way to do this will be by maintaining Euler–Savary's relationship between the joints. For instance, bar $1' \equiv 21' - 1'0 \equiv P_{21'}P_{1'0}$ may replace bar $1 \equiv 21 - 10 \equiv P_{21}P_{10}$ through two Euler–Savary relations, to wit

$$\frac{1}{P_{20}P_{21'}} - \frac{1}{P_{20}P_{1'0}} = \frac{1}{\delta_{20}\sin\theta_1} = \frac{1}{P_{20}P_{21}} - \frac{1}{P_{20}P_{10}}.$$

Thus, if

$$\frac{1}{P_{20}P_{21'}} - \frac{1}{P_{20}P_{1'0}} = \frac{1}{P_{20}P_{21}} - \frac{1}{P_{20}P_{10}},$$

Euler–Savary's relationship remains the same for bar 1 as well as for bar 1′. It means in fact, that by choosing the joint $P_{21'}$ on bar 1, the location of the other end-joint $P_{1'0}$ is completely determined by this relationship. Thus, unlike first order *joint-joining*, the 2nd joint (P_{10}) has to be relocated also.

4.1 *Proof of 2nd-order motion-equivalency*

If indeed relocation of joint 10 has been carried out as instructed, the instantaneous motion of link 2 with respect to 0 will remain the same up to the 2nd order. This means that the inflection circle of the relative motion of 2 with respect to 0 neither changes its size nor its position. Notwithstanding that we may still question the instantaneous motion between the remaining links. This may be clarified by comparing the relative motion between the links of the newly created pentagonal loop 0–1′–2–3–4 and between those of the initial one 0–1–2–3–4. (See Fig. 2(B.) Since the relative motion 2/0 has 2nd-order equivalency, joint 40 will have the same center of curvature embedded in link 2. Thus, for both pentagonals there exists a unique quadrilateral, involving the links 2,3,4 and the radius of curvature just found. From this, we derive a 2nd-order equivalency for the relative motion of 2/4.

Similarly, the choice between bar 1′ and bar 1 will have no influence on the location of the center of curvature $(23)_0$ of joint 23, which is embedded in link 0. From this, we recognize a unique quadrilateral involving the bars 0,4,3 and the radius of curvature, just obtained. Thus, a 2nd-order equivalency for the relative motion 3/0 appears.

Similar conclusions may be derived if we compare the pentagonal loops 0–1′–2–6–7 and 0–1–2–6–7. For the remaining relative motions, 2nd-order equivalency is then to be proved by considering the remaining loops.

Apart from all this, we are still free to choose one of the end joints of the new bar. As with the first-order joint-joining, we will use this freedom to take $P_{1'2}$ at I_{13}. Thus. $P_{1'2} = I_{13}$. Substitution in our last, so-called, "combined equation" then gives

$$\frac{1}{P_{20}P_{1'0}} = \frac{1}{P_{20}P_{10}} + \frac{1}{P_{20}I_{13}} - \frac{1}{P_{20}P_{12}} \qquad (1')$$

from which we may establish immediately the revised location of $P_{1'0}$. (See for its result Fig. 2A.)

We may now do the same for bar 3. That is to say, we replace bar 3 by 3′, using the known position of the pole P_{24}. We carry this out by again taking joint $23' \equiv P_{23'}$ at I_{13} and by calculating the new location of joint $3'4 \equiv P_{3'4}$ through the relation

$$\frac{1}{P_{24}P_{3'4}} = \frac{1}{P_{24}P_{34}} + \frac{1}{P_{24}I_{13}} - \frac{1}{P_{24}P_{23}}. \qquad (3')$$

As with the first order joint-joining method, joint $I_{13} = 1'2 = 23' = 3'1'$ now appears to be a double joint which is truly embedded in link 2. Hence, 1′3′ and 26 are to be interconnected. The connecting-line 1′3′–26 then represents link 2, which so becomes a binary bar having equivalent motion up to the 2nd-order as long as the joints 1′0 and 3′4 are taken at their appropriate and calculated positions. (See also Fig. 3.)

The replacement of the binary bars 1 and 3 by 1′ and 3′ respectively, will be called a "*2nd-order joint-joining*" method, provided of course the joints 1′0 and 3′4 are relocated according to the equations (1′) and (3′) from the above. If these joints had not been relocated, we would be using merely our 1st-order joint-joining method as was necessary to find the poles in the first place. Thus, if a 2nd-order joint-joining is applied, a triangular link turns into a binary one, simultaneously preserving 2nd-order motion-equivalency throughout the linkage mechanism.

A second and similar reduction is shown in Fig. 4. Here, the bars 5 and 7 are replaced by 5′ and 7′ respectively, but such that they have a common joint at I_{57}. Second-order motion-equivalency is hereby preserved by simultaneous relocation of the joints 5′4 and 7′0 through the formulae

$$\frac{1}{P_{65}P_{5'4}} = \frac{1}{P_{64}P_{54}} + \frac{1}{P_{64}I_{57}} - \frac{1}{P_{64}P_{65}} \qquad (5')$$

and

$$\frac{1}{P_{60}P_{7'0}} = \frac{1}{P_{60}P_{70}} + \frac{1}{P_{60}I_{57}} - \frac{1}{P_{60}P_{67}}. \qquad (7')$$

This *2nd-order joint-joining* joins the former joints 56 and 76 into a singular point I_{57}, that appears as a double joint of the reduced linkage. Also, the triangular link hereby turns into a binary bar, represented by the connected line 26–I_{57}. This bar maintains 2nd-order motion equivalency as long as 5′4 and 7′0 are relocated at their appropriate positions. Thus, Fig. 4 shows the result of 2nd-order joint-joining twice applied; the first time on the triangular link 2, and the second time on the triangular link 6. The two joint-joinings reduces the 8-bar into a reduced one, containing three independent loops, of which two are now four-bars.

The reduced linkage, in fact, consists of a Watt linkage and an adjoined dyad 2–6, which is I_{13}–26–I_{57} in this case.

Although deformed, the former ternary links, i.e. the even-numbered links 0,2,4 and 6, still appear in the reduced linkage. Since their instantaneous (relative) motion has been preserved to the 2nd order, the results of curvature problems that are solved for the reduced linkage, remain valid also for the 8-bar, we started from. This is true in so far as the curvature problems are restricted to the former ternary links. If, none the less, we want the curvatures of pathpoints attached to the former *binary* bars we simply insert the obtained curvature radii for the turning-joints of the ternary links into the original 8-bar. (See Section 5 for further details.)

Another, second possibility arises, if 2nd-order joint-joining is applied on the triangular links 4 and 0. Then, the links 4 and 0 turn into binary links instead of 2 and 6. In this case the joints $I_{35} = (23–34) \times (45–56)$ and $I_{17} = (07–76) \times (01–12)$ will become the double-joints of the mechanism.

The result is shown in Fig. 5. Here also, only the former ternary links still appear in the reduced

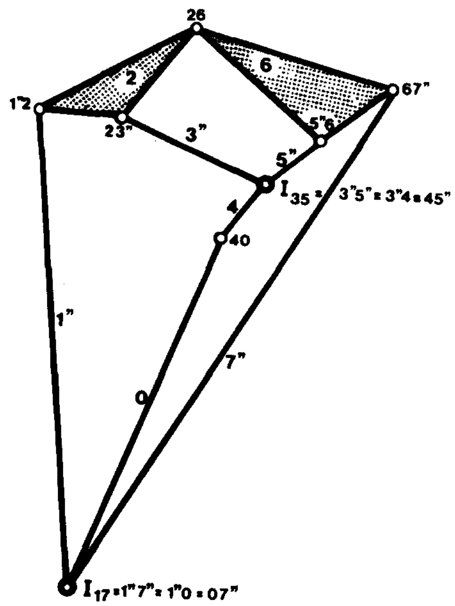

Fig. 5. Alternative way of 2nd order reduction.

linkage. Now, the dimensions of the linkage are to be derived from the equations

$$\frac{1}{P_{02}P_{1'2}} = \frac{1}{P_{02}P_{12}} + \frac{1}{P_{02}I_{17}} - \frac{1}{P_{02}P_{01}} \quad (1'')$$

$$\frac{1}{P_{24}P_{3'2}} = \frac{1}{P_{24}P_{32}} + \frac{1}{P_{24}I_{35}} - \frac{1}{P_{24}P_{34}} \quad (3'')$$

$$\frac{1}{P_{64}P_{5'6}} = \frac{1}{P_{64}P_{56}} + \frac{1}{P_{64}I_{35}} - \frac{1}{P_{64}P_{54}} \quad (5'')$$

$$\frac{1}{P_{60}P_{7'6}} = \frac{1}{P_{60}P_{76}} + \frac{1}{P_{60}I_{17}} - \frac{1}{P_{60}P_{70}}. \quad (7'')$$

5. DETERMINATION OF COORDINATED CENTERS OF CURVATURE IN LINKAGES[4]

The problem to find coordinated centers of curvature between the ternary links of our initial 8-bar is clearly simplified by the reduction carried out in the foregoing. It further appears that the remaining problem is to be solved by successive determination of coordinated centers of curvature in a series of four-bar loops. In order to demonstrate this, we will show how it is to be done for a four-bar, for a six-bar and, finally, for the reduced 8-bar, obtained in the last paragraph.

For a *four-bar*, we proceed as follows: Say, we need to find the center of curvature γ, coordinated to a coupler-point C attached to the coupler AB of a four-bar ($\alpha AB\beta$). To solve this, we use Bobillier's Theorem, reading: "*The bisector of the angle between two pole rays of the moving plane, coincides with the corresponding bisector of the angle between the pole tangent and the collineation axis*" (see Fig. 6).

The point γ, coordinated to C, may then be found if Bobillier's Theorem is applied twice. If we further

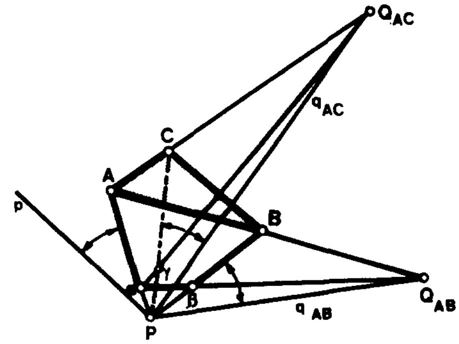

Fig. 6. Bobillier's Theorem twice applied.

use the same angle in the two applications of the theorem, a short cut is obtained, the instructions being:

(a) Intersect the two pole rays αA and βB at the pole P,

(b) Intersect the lines $\alpha\beta$ and AB at the collineation-point Q_{AB}

(c) Determine the line q_{AC} by making $\sphericalangle CPq_{AC} = \sphericalangle BPQ_{AB}$

(d) Interect AC and q_{AC} at Q_{AC}

(e) Intersect PC and αQ_{AC} at γ, the point we are looking for.

Thus, for a four-bar the problem is solved. Coordinated centers in a six-bar are found if we repeatedly solve the problem in subsequent four-bar loops. For Watt's six-bar, containing, for instance, the four-bar loops 0–1′–3′–4 and 0–4–5′–7′, we may carry this out as follows (see Fig. 7).

Suppose that we need to find the center of curvature (81′) of $I_{57} = 5'7'$ with respect to link 1′. In order to find this point, we first determine the center ($7^{\Delta}5'$) that is coordinated to the coupler point 1′0 attached to the coupler link 0 of the four-bar loop 0–4–5′–7′. For a three-point position analysis, it is allowed to connect the centers $7^{\Delta}5'$ and 1′0 by the bar 7^{Δ}.

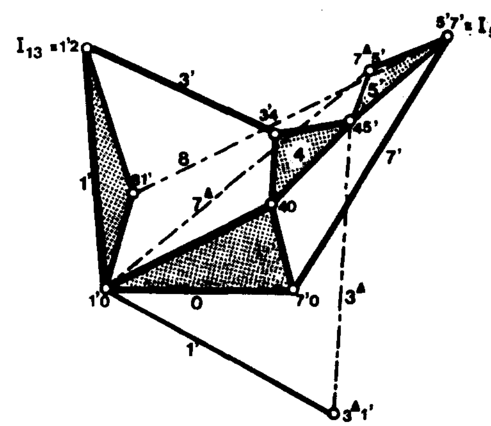

Fig. 7. Two, in series connected, 4-bar loops, replaced by a singular one with 2nd order motion equivalency. (for reasons of surveyability, the drawing shows only 1st order equivalency.)

Having done this, we further determine the center of curvature ($3^{\Delta}1'$) coordinated to the coupler point 45′ of the quadrilateral 0–1′–3′–4. This, introduces the bar 3^{Δ} connecting the points 45′ and $3^{\Delta}1'$. As a consequence, the quadrilateral $1'$–3^{Δ}–$5'$–7^{Δ} appears, from which we may extract the curvature center (81′) that is coordinated to the coupler-point I_{57} of link 5′. (See the result in Fig. 8.) Thus, the problem is solved for a series connection of two four-bars. What remains, is the problem of finding coordinated centers of curvature for the reduced 8-bar. This, we may now further solve as follows (see Fig. 9).

As a result of the foregoing, the introduced bar 8 did connect the joints 81′ and $I_{57} = 5'7'$. This then gives rise to the four-bar loop 6–2–1′–8 with the coupler-triangle (I_{13}–81′–1′0). Hence, we may extract the curvature center (96) that is coordinated to the coupler point 1′0. Clearly, the end-points of bar 9 represent a second pair of coordinated centers of curvature. The two pairs, represented by the end-points of the bars 7 (or 7′) and 9, form a four-bar loop 0–7–6–9 determining 2nd-order instantaneous motion of link 6 with respect to link 0.

Similarly, we may establish 2nd-order instantaneous four-bar motion of link 2 with respect to link 0. Also similarly, we may establish 2nd-order equivalent four-bars regarding the instantaneous motions of 6/4 and 2/4.

Thus, in principle, all coordinated centers of curvature between points of even-numbered (triangular) links of the 8-bar, we started from, are all established. However, the coordinated centers between points of the binary bars may still present a problem. For the motion of 5/1, for instance, we may proceed as follows:

From the foregoing, we may assume to have found the curvature center ($7^{\nabla}0$) that is coordinated to the coupler-point 56 of the four-bar 0–7–6–9 (see again Fig. 9). This, clearly, gives rise to the introduction of bar 7^{∇}. Moreover, since we may assume to have solved the problem for the relative motion of the triangular links, and for 4/2 in particular, we may similarly find the center 23^{∇} that is coordinated—through bar 3^{∇}—to joint 45 of the linkage. Hence, for 2nd-order equivalency of motion, we may replace the bars 3 and 7 by 3^{∇} and 7^{∇} respectively. This procedure reduces the linkage into one demonstrated in Fig. 10. The linkage then contains two double joints, respectively at 45 and at 56. Hence, the reduced linkage so accommodates two four-bar linkages, to wit □ 0–4–5–7^{∇} and □ 2–3^{∇}–5–6. The coupler-point 10 of the first four-bar has its coordinated center of curvature embedded in link 5. This then is the first pair of coordinated centers we are looking for. The second pair may be found if we observe joint 12 as the coupler-point of the second four-bar. Then, an additional coordinated center of curvature is found, also embedded in link 5. Thus, two pairs of coordinated centers of curvature are found, determining 2nd-order instantaneous motion of bar 5 with respect to bar 1, and conversely.

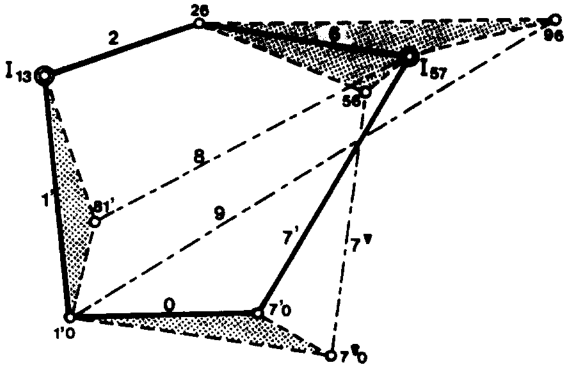

Fig. 9. Determination of 1′0–96 and of 56–$7^{\nabla}0$ thereafter.

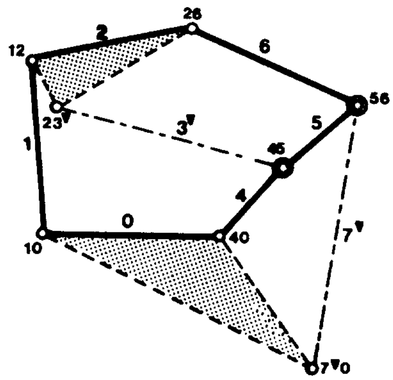

Fig. 10. 2nd-order equivalent motion of 1/5 determined.

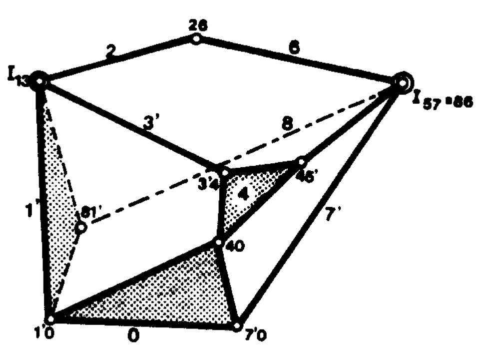

Fig. 8. Result obtained through reduction and successive application of Bobiller's theorem.

6. CONCLUSION

For the linkage network mechanism, demonstrated in Fig. 1, coordinated centers of curvature are all to be found through linkage reduction and successive creation of new four-bar loops replacing the motion instantaneously up to the 2nd-order. The example chosen guarantees that the method, presented, works also for other network linkages. Generally also, one first has to reduce the linkage, creating double joints by replacing binary links and then, in order to break down the remaining problem, one eventually applies solutions already found for the four-bar, six-bar, etc. The method presented will therefore be apt for a computer-program as it repeatedly uses the same sub-routines for the four-bar and, eventually, for the six-bar. What remains for the designer, would be to find the right sequence of four-bars to be used. We leave this to the designer, especially as this remaining problem appears to be an attractive one that belongs to the field of the technique of combinations and permutations.

REFERENCES

1. E. A. Dijksman, Why joint-joining is applied on complex linkages. *2nd IFToMM Int. Symp. on Linkages and Computer Aided Design Methods*, Vol. II, paper 17, pp. 185–212, SYROM '77, Bucuresti Romania.
2. E. A. Dijksman, *Motion Geometry of Mechanisms*. Cambridge University Press, Cambridge, London, New York, Melbourne (1976).
3. N. Rosenauer and A. H. Willis, *Kinematics of Mechanisms*. Dover, New York (1967).
4. K. Hain, Die Bahnkrümmungen der mehrgliedrigen Getriebe. *Industrie-Anzeiger* **93**(35), 797–801 (1971).
5. S. Molian, Mechanisms from square 1. *Machine Design and Control*, Part 5–An Introduction to Curvature Theory, pp. 12–15. Part 6–Further Notes on Curvature Theory, pp. 16–18 (1971).
6. E. A. Dijksman, *J. Mechanisms* **1**(1), 73–81 (1966).

Die geometrische Bestimmung der Bahnkrümmungen in labyrinthischen Gelenkgetrieben mittels Zusammenfügungen von jedesmahl zwei Einzelgelenke

von Evert A. Dijksman

Zusammenfassung Netwerkgetriebe zugehörende Krümmungsmittelpunkte möchten gefunden werden durch Getriebereduktion. Solche Reduktionen müssen aber zweimal durchgeführt werden und zwar auf verschiedener Weise; dass heisst, zuerst eine Reduktion von erster Ordnung mittels Zusammenfügungen von Drehgelenke zur Bestimmung der Geschwindigkeitspolen(1), und zweitens eine Reduktion zweiter Ordnung, die ebenfalls binären Gliedern ersetzt, aber so, dass jetzt auch die zweite Ordnung der Momentanbewegung unverändert bleibt.

Die Bestimmung der Krümmungsmittelpunkte für das reduzierte Gelenkgetriebe kann weiter gelöst werden durch aufeinanderfolgende Anwendungen des Bobillierschen Satzes in auf zu richten Gelenkvierecke. Die Anwendung für Gelenkgetriebe wird demonstriert an einem achtgliedrigem Gelenkgetriebe dass keine Gelenkvierecke sondern nur Gelenkfünfecke enthält. Das vorgeschlagene Verfahren ist rein geometrisch und arbeitet ohne Geschwindigkeits- und Beschleunigungskonstruktionen. Dennoch kann aber das erworbene Erfolg ohne weiteres verwendet werden für die Bestimmung von Beschleunigungen in Gelenkgetrieben.