On the zeros of a polynomial and of its derivative II

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1. In a previous paper [1]) (referred to as I), the following theorem was proved for some special classes of polynomials:

Theorem 1. *Let the polynomial $f(z)$ of degree $n > 1$, have the zeros $\xi_1, \ldots, \xi_n$, and let $\eta_1, \ldots, \eta_{n-1}$ be those of $f'(z)$. Then we have*

$$\frac{1}{n-1} \sum_{\nu=1}^{n-1} |Im\, \eta_\nu| \leq \frac{1}{n} \sum_{\nu=1}^{n} |Im\, \xi_\nu|, \quad \ldots \ldots \quad (1)$$

the sign of equality holding if and only if no two zeros of $f(z)$ are separated by the real axis.

Here we shall prove the theorem in the general case, namely for polynomials with arbitrary real or complex coefficients. In our proof we introduce an auxiliary function $f^*(z)$ obtained from $f(z)$ by replacing the zeros of $f(z)$ in the lower half-plane by their complex conjugates.

Theorem 1 can be generalized in several ways. In the first place we may ask for the class C of real continuous functions $\psi(z)$ of the complex variable z, such that

$$\frac{1}{n-1} \sum_{\nu=1}^{n-1} \psi(\eta_\nu) \leq \frac{1}{n} \sum_{\nu=1}^{n} \psi(\xi_\nu). \quad \ldots \ldots \quad (2)$$

holds for any polynomial $f(z)$. We have not been able to characterize this class C; it is, however, likely, that C consists of all convex functions $\psi(z)$ [2]). Anyhow, all functions of the class C are convex.

It is possible to derive from theorem 1, by superposition, a large subclass C* of functions $\psi(z)$ belonging to C. Important items are $\psi(z) = |z|^p$ and $\psi(z) = |Im\, z|^p$ $(p \geq 1)$. This will be shown in section **3.**

A second generalisation of theorem 1 is to rational functions with positive residues (section 4).

Other generalisations, concerning the zeros of "composition-polynomials", will be given in a next paper.

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[1]) N. G. DE BRUIJN, On the zeros of a polynomial and of its derivative, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 49, 1037—1044 (1946). In that paper, our theorem 1 was proved in the following two cases:

a) if all coefficients of $f(z)$ are real, and

b) if all zeros of $f(z)$ are purely imaginary.

[2]) $\psi(z)$ is called convex, if $\psi(\lambda_1 z_1 + \lambda_2 z_2) \leq \lambda_1 \psi(z_1) + \lambda_2 \psi(z_2)$ for all values of $z_1, z_2, \lambda_1, \lambda_2$, satisfying $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$.

2. Proof of theorem 1.

We remark, in the first place, that the theorem is trivial when all zeros of $f(z)$ lie in $Im\, z \geqslant 0$. For then, by the well-known Gausz–Lucas theorem, the same holds for the zeros of $f'(z)$, so that the imaginary parts of $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_{n-1}$ all have the same sign. The theorem then follows from the relation

$$\frac{1}{n-1} \sum_{\nu=1}^{n-1} \eta_\nu = \frac{1}{n} \sum_{\nu=1}^{n} \xi_\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This shows that in (1) the sign of equality holds in this case.

The general case is reduced to this one by means of the following

Lemma. *Let* $f(z) = a \prod_{\nu=1}^{k} (z - \xi_\nu) \prod_{\nu=k+1}^{n} (z - \xi_\nu)$ $(0 \leq k \leq n)$, *where* $Im\, \xi_\nu \geq 0$ $(\nu = 1, 2, \ldots, k)$, $Im\, \xi_\nu < 0$ $(\nu = k+1, \ldots, n)$.
Putting

$$f^*(z) = a \prod_{\nu=1}^{k} (z - \xi_\nu) \prod_{\nu=k+1}^{n} (z - \bar{\xi}_\nu), \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

we have

$$|f'(x)| \leqslant |f^{*\prime}(x)| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

for all real values of x.

There is equality for all real x *if and only if no two zeros of* $f(z)$ *are separated by the real axis.*

Proof. Writing

$$\sum_{\nu=1}^{k} \frac{1}{x - \xi_\nu} = P + Qi, \quad \sum_{\nu=k+1}^{n} \frac{1}{x - \xi_\nu} = R + Si \quad (x, P, Q, R, S \text{ real})$$

we have

$$\frac{f'(x)}{f(x)} = (P + R) + i(Q + S), \quad \frac{f^{*\prime}(x)}{f^*(x)} = (P + R) + i(Q - S).$$

Now it follows from

$$Im\, \xi_\nu \geqslant 0 \ (\nu = 1, \ldots, k), \quad Im\, \xi_\nu < 0 \quad (\nu = k+1, \ldots, n)$$

that $Q \geq 0$, $S \leq 0$. We obtain $|Q + S| \leq |Q - S|$, which gives

$$\left| \frac{f'(x)}{f(x)} \right| \leqslant \left| \frac{f^{*\prime}(x)}{f^*(x)} \right|.$$

For real x we have

$$|f(x)| = |f^*(x)|,$$

and hence

$$|f'(x)| \leqslant |f^{*\prime}(x)|.$$

There is equality (for all real x) only if either $Q = 0$ or $S = 0$, that is to say, if all zeros of $f(z)$ lie either in $Im\, z \leq 0$ or in $Im\, z \geq 0$.

With this lemma the proof of theorem 1 is quite simple. Since all zeros of $f^*(z)$ lie in $Im\, z \geq 0$, it follows from our remark above that

$$\frac{1}{n-1}\sum_{\nu=1}^{n-1} |Im\, \eta_\nu^*| = \frac{1}{n}\sum_{\nu=1}^{n} |Im\, \xi_\nu| \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where $\eta_1^*, \ldots, \eta_{n-1}^*$ denote the zeros of $f^{*\prime}(z)$. Further, by the lemma,

$$\int_{-A}^{A} \log |f'(x)|\, dx \leqslant \int_{-A}^{A} \log |f^{*\prime}(x)|\, dx \qquad (A > 0) \quad . \quad . \quad . \quad (7)$$

or

$$\sum_{\nu=1}^{n-1} \int_{-A}^{A} \log |x - \eta_\nu|\, dx \leqslant \sum_{\nu=1}^{n-1} \int_{-A}^{A} \log |x - \eta_\nu^*|\, dx \quad . \quad . \quad . \quad . \quad (8)$$

It is easily seen that

$$\int_{-A}^{A} \log |x - a|\, dx = 2\,(A \log A - A) + \pi\, |Im\, a| + O\left(\frac{1}{A}\right). \quad . \quad (9)$$

Substituting this into (8), and making $A \to \infty$ we find

$$\sum_{\nu=1}^{n-1} |Im\, \eta_\nu| \leqslant \sum_{\nu=1}^{n-1} |Im\, \eta_\nu^*| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Combining this inequality with (6) we obtain (1).

There is equality in (10) if and only if there is equality in (7), that is, if $|f'(x)| = |f^{*\prime}(x)|$ for all real values of x, and then, by the lemma, all zeros of $f(z)$ lie either in $Im\, z \geq 0$ or in $Im\, z \leq 0$.

Thus theorem 1 is completely proved.

3. Theorem 1 means, geometrically, that the zeros of $f'(z)$ lie, in the mean, closer to the real axis, than the zeros of $f(z)$. The same can be said about any line, that is to say, theorem 1 remains true, when we replace $|Im\, z|$ by $|Im(\alpha z + \beta)|$, α and β being complex numbers. (This is easily proved by applying theorem 1 to $f\left(\dfrac{z-\beta}{\alpha}\right)$). Hence the functions $|Im(\alpha z + \beta)|$ belong to the class C, defined in section **1.** Furthermore, it follows from (3) that the functions $Im\,(\alpha z + \beta)$ also belong to C. We can obtain new functions of C by superposition of these special ones. We thus obtain a sub-class C^* of C, which consists of all real continuous functions $\psi(z)$ of the complex variable z, which are sums of functions of the types $|Im(\alpha z + \beta)|$, $Im\,(\alpha z + \beta)$ with positive weights. For instance, C^* contains all convex functions of $Im\, z$. We have, namely

Theorem 2. *Let $\xi_1, \ldots, \xi_n$ be the zeros of $f(z)$, $\eta_1, \ldots, \eta_{n-1}$ those of*

$f'(z)$, and let $Im\,\xi_1 \leq Im\,\xi_2 \leq \ldots \leq Im\,\xi_n$. If $\psi(x)$ is a convex real function of x in the interval $Im\,\xi_1 \leq x \leq Im\,\xi_n$, and if

$$D(\psi, f) = \frac{1}{n}\sum_{\nu=1}^{n} \psi\,(Im\,\xi_\nu) - \frac{1}{n-1}\sum_{\nu=1}^{n-1} \psi\,(Im\,\eta_\nu)$$

then

$$D(\psi, f) \geqslant 0 \quad \ldots \ldots \ldots \quad (11)$$

$D(\psi, f) = 0$ holds only if *$\psi(x)$ is linear for $Im\,\xi_1 \leq x \leq Im\,\xi_n$* (which implies the case $Im\,\xi_1 = \ldots = Im\,\xi_n$).

Theorem 2 can be proved in the same way as theorem 7 in I. A special case is $\psi(x) = |x|^p$ $(p \geq 1)$, giving

Theorem 3. *With the notations of theorem 1, we have, if $p \geq 1$*

$$\frac{1}{n-1}\sum_{\nu=1}^{n-1} |Im\,\eta_\nu|^p \leqslant \frac{1}{n}\sum_{\nu=1}^{n} |Im\,\xi_\nu|^p \quad \ldots \ldots \quad (12)$$

There is equality in the following two cases only: a) if $p = 1$ and all zeros of $f(z)$ lie in the same half-plane $Im\,z \geq 0$ or $Im\,z \leq 0$, and b) if $p \geq 1$ and $Im\,\xi_1 = \ldots = Im\,\xi_n$.

Obviously this theorem remains true when $|Im(\alpha z + \beta)|$ is substituted for $|Im\,z|$. This remark is used for the proof of

Theorem 4. *With the assumptions of theorem 1, we have, if $p \geq 1$*

$$\frac{1}{n-1}\sum_{\nu=1}^{n-1} |\eta_\nu|^p \leqslant \frac{1}{n}\sum_{\nu=1}^{n} |\xi_\nu|^p \quad \ldots \ldots \quad (13)$$

There is equality in the following two cases only: a) if $p = 1$ and all zeros of $f(z)$ lie on the same half-line with endpoint 0, and b) if $\xi_1 = \ldots = \xi_n$.

Proof. The distance of the point z in the complex plane to the line through the point 0 making an angle φ with the positive real axis, is

$$|\cos\varphi \cdot Im\,z - \sin\varphi \cdot Re\,z|.$$

By theorem 3 we have

$$\frac{1}{n-1}\sum_{\nu=1}^{n-1} |\cos\varphi \cdot Im\,\eta_\nu - \sin\varphi \cdot Re\,\eta_\nu|^p \leqslant \frac{1}{n}\sum_{\nu=1}^{n} |\cos\varphi \cdot Im\,\xi_\nu - \sin\varphi \cdot Re\,\xi_\nu|^p.$$

By integrating this inequality we obtain

$$\frac{1}{n-1}\sum_{\nu=1}^{n-1}\int_0^{2\pi} |\cos\varphi \cdot Im\,\eta_\nu - \sin\varphi \cdot Re\,\eta_\nu|^p\,d\varphi \leqslant$$

$$\leqslant \frac{1}{n}\sum_{\nu=1}^{n}\int_0^{2\pi} |\cos\varphi \cdot Im\,\xi_\nu - \sin\varphi \cdot Re\,\xi_\nu|^p\,d\varphi$$

or

$$\frac{1}{n-1}\sum_{\nu=1}^{n-1} |\eta_\nu|^p \leqslant \frac{1}{n}\sum_{\nu=1}^{n} |\xi_\nu|^p.$$

The cases of equality are easily deduced from those of theorem 3.

A direct consequence of theorem 4 is

Theorem 5. *With the assumptions of theorem 1 and $p > 0$, we have*

$$\sum_{\nu=1}^{n-1} |\eta_\nu|^p \leqslant \frac{n-\frac{1}{k}}{n} \sum_{\nu=1}^{n} |\xi_\nu|^p, \text{ if } k \text{ is an integer} > \frac{1}{p} \quad . \quad . \quad (14)$$

This may be proved by application of theorem 4 to $f(z^k)$. (Cf. I, theorem 3).

It is probable, that, more generally, the inequality

$$\sum_{\nu=1}^{n-1} |\eta_\nu|^p \leqslant \frac{n-p}{n} \sum_{\nu=1}^{n} |\xi_\nu|^p \qquad (0 \leqslant p \leqslant 1) \quad . \quad . \quad . \quad (15)$$

holds. We have, however, not been able to prove this. Anyhow, if ϱ is a fixed number $(\varrho > p)$, an inequality

$$\sum_{\nu=1}^{n-1} |\eta_\nu|^p \leqslant \frac{n-\varrho}{n} \sum_{\nu=1}^{n} |\xi_\nu|^p \qquad (0 \leqslant p \leqslant 1) \quad . \quad . \quad . \quad . \quad (16)$$

cannot be true for arbitrary $f(z)$. This is easily seen by considering $f(z) = z^{n-1}(z-1)$ for large integers n.

4. Rational functions of the type

$$\varphi(z) = -az + b + \sum_{\nu=1}^{n} \frac{t_\nu}{z-a_\nu} \qquad (a \geqslant 0,\ t_\nu > 0,\ n \geqslant 0) \quad . \quad (17)$$

have properties analogous to that expressed in theorem 1 (Cf. I, theorem 2).

In the first place, we obtain

Theorem 5. *If $n > 1$,*

$$\varphi(z) = \sum_{\nu=1}^{n} \frac{t_\nu}{z-a_\nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and if $\beta_1, \ldots, \beta_{n-1}$ are the zeros of $\varphi(z)$, then we have [3])

$$\sum_{\nu=1}^{n-1} |Im\, \beta_\nu| \leqslant \sum_{\nu=1}^{n} |Im\, a_\nu| - \frac{\sum_{\nu=1}^{n} t_\nu |Im\, a_\nu|}{\sum_{\nu=1}^{n} t_\nu} \quad . \quad . \quad . \quad . \quad (19)$$

Proof. This may be proved by the same method as theorem 1, but it is also possible to deduce (19) directly from theorem 1. For, by applying (1)

[3]) The case $t_1 = \ldots = t_n = 1$ is embodied in theorem 1.

to the polynomial $\prod_{\nu=1}^{n} (z - a_\nu)^{k}$ (where the k_ν are natural numbers) we obtain

$$\sum_{\nu=1}^{n-1} |Im\, \beta_\nu| + \sum_{\nu=1}^{n} (k_\nu - 1)\, |Im\, a_\nu| \leqslant \frac{-1 + \sum_{\nu=1}^{n} k_\nu}{\sum_{\nu=1}^{n} k_\nu} \cdot \sum_{\nu=1}^{n} k_\nu\, |Im\, a_\nu|,$$

where $\beta_1, \ldots, \beta_{n-1}$ are the zeros of $\sum_{\nu=1}^{n} \frac{k_\nu}{x - a_\nu}$. Hence (19) follows for rational t_ν, and the general case follows by an argument of continuity.

Since (1) holds for all functions $\psi(z)$ of the class C, the same argument shows

Theorem 6. *Under the assumptions of theorem 5, we have*

$$\sum_{\nu=1}^{n-1} \psi(\beta_\nu) \leqslant \sum_{\nu=1}^{n} \psi(a_\nu) - \frac{\sum_{\nu=1}^{n} t_\nu\, \psi(a_\nu)}{\sum_{\nu=1}^{n} t_\nu}, \quad \ldots \quad (20)$$

for any function $\psi(z)$ of the class C.

To obtain a corresponding inequality for the function

$$\varphi(z) = b + \sum_{\nu=1}^{n} \frac{t_\nu}{z - a_\nu} \quad (b \neq 0,\ t_\nu > 0,\ n > 0) \quad \ldots \quad (21)$$

we apply theorem 6 to

$$\varphi_T(z) = \frac{T}{z + \frac{T}{b}} + \sum_{\nu=1}^{n} \frac{t_\nu}{z - a_\nu}.$$

From $\lim_{T \to +\infty} \varphi_T(z) = \varphi(z)$ it follows, that

$$\sum_{\nu=1}^{n} \psi(\beta_\nu) \leqslant \sum_{\nu=1}^{n} \psi(a_\nu) + \left(\sum_{\nu=1}^{n} t_\nu\right) \cdot \lim_{T \to +\infty} \frac{1}{T}\, \psi\left(-\frac{T}{b}\right)$$

($\beta_1, \ldots, \beta_n$ denoting the zeros of $\varphi(z)$) if $\psi(z)$ belongs to class C and the limit exists. This will occur, for example, if $\psi(z)$ is homogeneous (i.e. $\psi(\lambda z) = \lambda \psi(z)$ for all $\lambda \geqq 0$ and all complex z). We then have

$$\sum_{\nu=1}^{n} \psi(\beta_\nu) \leqslant \sum_{\nu=1}^{n} \psi(a_\nu) + \psi\left(-\frac{1}{b}\right) \cdot \left(\sum_{\nu=1}^{n} t_\nu\right) \quad \ldots \quad (22)$$

The most important applications are $\psi(z) = |Im\, z|$ and $\psi(z) = |z|$.

For the function

$$\varphi(z) = -az + b + \sum_{\nu=1}^{n} \frac{t_\nu}{z - a_\nu} \qquad (a > 0,\ t_\nu > 0,\ n \geqslant 0) \quad . \quad (23)$$

whose zeros be denoted by $\beta_1, \ldots, \beta_{n+1}$, we have $\varphi(z) = \lim_{T \to +\infty} \varphi_T(z)$, where

$$\varphi_T(z) = \frac{T^2}{z + pT - q} + \frac{T^2}{z - pT - q} + \sum_{\nu=1}^{n} \frac{t_\nu}{z - a_\nu} \qquad \left(p = \sqrt{\frac{2}{a}}, q = \frac{b}{a}\right).$$

Application of theorem 6 to $\varphi_T(z)$ yields

$$\sum_{\nu=1}^{n+1} \psi(\beta_\nu) \leqslant \sum_{\nu=1}^{n} \psi(a_\nu) + \lim_{T \to +\infty} \frac{T^2 + \sum_{\nu=1}^{n} t_\nu}{2T^2 + \sum_{\nu=1}^{n} t_\nu} \cdot \{\psi(pT + q) + \psi(-pT + q)\}.$$

This proves

Theorem 7. *If $\beta_1, \ldots, \beta_{n+1}$ are the zeros of* (23) $(a > 0, t_\nu > 0, n \geq 0)$, *and $\psi(z)$ is a convex function of $Im\, z$* (Cf. *Theorem* 2), *then we have*

$$\sum_{\nu=1}^{n+1} \psi(\beta_\nu) \leqslant \sum_{\nu=1}^{n} \psi(a_\nu) + \psi\left(\frac{b}{a}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

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