On some big closed polygons

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On Some Big Closed Polygons

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Recently the following problem was proposed to me:

Show that there exists a polygon with 1990 equal angles, of which the edges have length 1^2, 2^2, ,..., 1990^2 (in some order).

This result is generalized in the following theorem (the special case is $N = 1990$, $k = 3$, $q_1 = 2$, $q_2 = 5$, $q_3 = 199$, $P(x) = (1+x)^2$).

Theorem. Let N and k be positive integers, and let

$$N = q_1 \cdots q_k,$$

where $q_1, \ldots, q_k$ are integers > 1, pairwise relatively prime. Furthermore, let P be a polynomial of degree $< k$, and let ζ be the primitive N-th root of unity

$$\zeta = e^{2\pi i/N}.$$

Then there exists a permutation π of the integers $0, \ldots, N-1$ such that

$$\sum_{n=0}^{N-1} P(\pi(n))\zeta^n = 0. \tag{1}$$

Proof. In order to avoid complicated notation, we restrict ourselves to the case $k = 3$; the general case is completely analogous.

Let the integer a run through the range $0, \ldots, q_1 - 1$, b through $0, \ldots, q_2 - 1$, c through $0, \ldots, q_3 - 1$. Then the combination

$$a \cdot \frac{N}{q_1} + b \cdot \frac{N}{q_2} + c \cdot \frac{N}{q_3} \tag{2}$$

runs through a complete system of residues mod N, and the combination

$$a q_2 q_3 + b q_3 + c \tag{3}$$

represents each one of the integers n with $0 \leq n < N$ exactly once.

We define the permutation π as follows. For given n with $0 \leq n < N$ we take a, b, c such that (2) is congruent to n mod N, and with these a, b, c the value of (3) is called $\pi(n)$.

We now evaluate the sum (1). In that sum we have to deal with

$$P(aq_2q_3 \; + \; bq_3 \; + \; c);$$

as a function of a, b, c this is a polynomial of degree < 3. We write it as a sum

$$\sum_{\lambda=0}^{2}\sum_{\mu=0}^{2}\sum_{\nu=0}^{2} C_{\lambda,\mu,\nu} a^\lambda b^\mu c^\nu.$$

The left hand side of (1) can now be expressed as follows:

$$(4) \qquad \sum_{n=0}^{N-1} P(\pi(n))\zeta^n \; = \; \sum_{\lambda=0}^{2}\sum_{\mu=0}^{2}\sum_{\nu=0}^{2} C_{\lambda,\mu,\nu} W(\lambda, \mu, \nu),$$

where

$$W(\lambda, \mu, \nu) \; = \; \sum_{a=0}^{q_1-1}\sum_{b=0}^{q_2-1}\sum_{c=0}^{q_3-1} a^\lambda b^\mu c^\nu \zeta^h,$$

where h stands for the value of (2). So

$$(5) \qquad W(\lambda, \mu, \nu) \; = \; \sum_{a=0}^{q_1-1} a^\lambda \zeta^{aN/q_1} \cdot \sum_{b=0}^{q_2-1} b^\mu \zeta^{bN/q_2} \cdot \sum_{c=0}^{q_3-1} c^\nu \zeta^{cN/q_3}.$$

Since the degree of P is less than 3, we have $\lambda+\mu+\nu < 3$, so at least one of λ, μ, ν is zero. The corresponding factor in the product (5) vanishes, and therefore $W(\lambda, \mu, \nu) = 0$. The theorem now follows from (4).