

Some linear and some quadratic recursion formulas. I

Citation for published version (APA):

Bruijn, de, N. G., & Erdős, P. (1951). Some linear and some quadratic recursion formulas. I. *Proceedings of the Section of Sciences of the Koninklijke Nederlandse Akademie van Wetenschappen. Series A, mathematical sciences*, 54(5), 374-382.

Document status and date:

Published: 01/01/1951

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy

If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

SOME LINEAR AND SOME QUADRATIC RECURSION FORMULAS.

I

BY

N. G. DE BRUIJN AND P. ERDÖS

(Communicated by Prof. H. D. KLOOSTERMAN at the meeting of November 24, 1951)

§ 1. *Introduction*

We shall mainly deal with linear recursion formulas of the type

$$(1.1) \quad f(1) = 1; \quad f(n) = \sum_{k=1}^{n-1} c_k f(n-k) \quad (n = 2, 3, \dots),$$

and with quadratic formulas of the type

$$(1.2) \quad f(1) = 1; \quad f(n) = \sum_{k=1}^{n-1} d_k f(k)f(n-k) \quad (n = 2, 3, \dots).$$

We assume that $c_k > 0$, $d_k > 0$ ($k = 1, 2, \dots$). In a previous paper [1] we discussed (1.1) under the condition $\sum_1^\infty c_k = 1$, and further special assumptions. Presently we deal with it more generally. We shall show that $\lim \{f(n)\}^{-1/n}$ always exists, and we shall give several sufficient conditions for the existence of $\lim f(n)/f(n+1)$. Some of the results can be applied to (1.2) (see § 6), and some of the methods can be extended to recurrence relations with coefficients c depending on n also (see § 3 and § 7).

In [1] as well as in the earlier paper of ERDÖS, FELLER and POLLARD [3], referred to below, the condition on the c_k was $c_k \geq 0$ ($k = 1, 2, \dots$), whereas the g.c.d. of the k 's with $c_k = 0$ was assumed to be 1. For convenience we assume $c_k > 0$ throughout. Consequently we have, both for (1.1) and for (1.2), $f(n) > 0$ ($n = 1, 2, \dots$).

Dealing with the linear relation (1.1) we put formally

$$(1.3) \quad C(x) = \sum_1^\infty c_n x^n, \quad F(x) = \sum_1^\infty f(n) x^n,$$

and we have formally

$$(1.4) \quad F(x) = x + C(x) F(x).$$

Furthermore, if ϱ is a positive number, and if we put

$$(1.5) \quad f(n) = \varrho^{-n+1} g(n),$$

then we have

$$(1.6) \quad g(n) = \sum_{k=1}^{n-1} b_k g(n-k), \quad g(1) = 1,$$

where $b_k = c_k \varrho^k$. Formula (1.6) is again of the type (1.1), and $b_k > 0$ for all k .

§ 2. *Linear recursions, different cases*

We discern amongst 5 different cases with respect to the behaviour of the series $C(x)$ (see (1.3)). Let R be the radius of convergence ($0 \leq R \leq \infty$) and let γ be the l.u.b. of the numbers a with $C(a) \leq 1$.

Case 1. $\gamma = R = 0$.

Case 2. $0 < \gamma < R \leq \infty$, $C(\gamma) = 1$.

Case 3. $0 < \gamma = R < \infty$, $C(\gamma) = 1$, $0 < C'(\gamma) < \infty$.

Case 4. $0 < \gamma = R < \infty$, $C(\gamma) = 1$, $C'(\gamma) = \infty$.

Case 5. $0 < \gamma = R < \infty$, $0 < C(\gamma) < 1$.

Since the coefficients c_k are positive it is easily seen that all possibilities are listed here.

§ 5 will be specially devoted to case 1; nevertheless case 1 is not excluded in §§ 2, 3, 4 unless explicitly stated.

In all cases we can show (§ 3)

$$(2.1) \quad (f(n))^{-\frac{1}{n}} \rightarrow \gamma,$$

In case 1 we infer that also $F(x)$ has 0 as its radius of convergence. In the other cases we can transform by (1.5), taking $\varrho = \gamma$. Apart from case 5, this leads to (1.6) with $\sum b_k = 1$. Therefore we can apply the results of ERDÖS, FELLER and POLLARD [3], and we obtain

$$(2.2) \quad \lim_{n \rightarrow \infty} f(n) \gamma^n \begin{cases} = \{C'(\gamma)\}^{-1} & \text{in cases 2 and 3,} \\ = 0 & \text{in case 4.} \end{cases}$$

If the limit is $= 0$, we have not yet an asymptotic formula for $f(n)$, and such a formula seems to be hard to obtain without introducing very special assumptions (see [1]).

In case 5 we have, just as in case 4, $f(n)\gamma^n \rightarrow 0$. For, it follows from (1.4) that

$$(2.3) \quad \sum_1^\infty f(n) \gamma^n = \gamma / (1 - C(\gamma));$$

hence the series on the left is divergent in cases 2, 3, 4 but convergent in case 5.

In case 2 it can be shown that for some $C > 0$ and some $\delta > \gamma$ we have

$$(2.4) \quad f(n) = C \gamma^{-n} + O(\delta^{-n}).$$

For, the coefficients of $C(x)$ being positive, we have $C(x) \neq 1$ ($|x| \leq \gamma$, $x \neq \gamma$) and $C'(\gamma) \neq 0$. Now (1.4) shows that $F(x)$ is regular in $|x| \leq \gamma$ apart from a simple pole at $x = \gamma$. This proves (2.4).

Apart from case 1 we have $\gamma > 0$, $C(\gamma) \leq 1$ and so, by induction

$$(2.5) \quad f(n) \leq \gamma^{1-n} \quad (n = 1, 2, 3, \dots).$$

In all cases we put

$$\liminf_{n \rightarrow \infty} \frac{f(n)}{f(n+1)} = \alpha \quad , \quad \limsup_{n \rightarrow \infty} \frac{f(n)}{f(n+1)} = \beta \quad ,$$

and we have

$$(2.6) \quad 0 \leq \alpha \leq \gamma \leq \beta \leq c_1^{-1} < \infty .$$

For, (2.1) shows that $\alpha \leq \gamma \leq \beta$, and $\beta \leq c_1^{-1}$ follows from the inequality $f(n+1) \geq c_1 f(n)$, which immediately follows from (1.1).

§ 3. *Linear recursion; existence of $\lim \{f(n)\}^{-1/n}$*

We shall show (theorem 2) that $\{f(n)\}^{-1/n}$ tends to a finite limit in all cases. Denoting the limit by L , it is easily proved afterwards that $L = \gamma$.

The existence of the limit will be shown for a slightly more general recursion formula.

Theorem 1. Let $0 < c_{k,k+1} \leq c_{k,k+2} \leq c_{k,k+3} \leq \dots$ ($k = 1, 2, 3, \dots$).

$$(3.1) \quad f(1) = 1 \quad , \quad f(n) = \sum_{k=1}^{n-1} c_{k,n} f(n-k) \quad (n = 2, 3, \dots).$$

Then we have

$$(3.2) \quad f(n+k-1) \geq f(n) f(k) \quad (k, n = 1, 2, 3, \dots).$$

Proof. We apply induction with respect to n . If $n = 1$, (3.2) is trivial. Now assume that (3.2) holds for $n = 1, \dots, N$. Then we have

$$\begin{aligned} f(N+k) &= \sum_{l=1}^{N+k-1} c_{l,N+k} f(N+k-l) \geq \\ &\geq \sum_1^N c_{l,N+k} f(N+k-l) \geq \sum_1^N c_{l,N+1} f(N+k-l) \geq \\ &\geq \sum_1^N c_{l,N+1} f(N+1-l) f(k) = f(N+1) f(k), \end{aligned}$$

and the induction is complete.

Theorem 2. Under the assumptions of theorem 1 we have, putting

$$\inf \{f(n+1)\}^{-1/n} = L \quad (0 \leq L < \infty),$$

that

$$\lim_{n \rightarrow \infty} \{f(n+1)\}^{-1/n} = L.$$

Proof. Clearly we have $f(n) > 0$ ($n = 1, 2, \dots$). Putting

$$g(n) = -\log f(n+1),$$

we infer from (3.2) that $g(n)$ is sub-additive:

$$g(n+k) \leq g(n) + g(k) \quad (n, k = 0, 1, 2, \dots).$$

It follows that

$$-\infty \leq \inf \frac{g(n)}{n} = \lim_{n \rightarrow \infty} \frac{g(n)}{n} < \infty .$$

(See [4], vol. 1, p. 17 and 171. An extension of this theorem will be given in § 7).

We next show for the equation (1. 1) that $L = \gamma$. We have $f(n) \geq c_{n-1}$ for all $n > 1$; therefore the radius of convergence of $F(x)$ is $\leq R$, and so $L \leq R$. In case 1 this means $L = 0 = \gamma$.

In case 2 we have $L = \gamma$ by (2. 4).

In the remaining cases we have $R = \gamma$, and so $L \leq \gamma$. On the other hand (2. 5) gives $L \geq \gamma$.

§ 4. Linear recursion; existence of $\lim f(n)/f(n+1)$

If $\lim f(n)/f(n+1)$ exists, it equals γ (see (2. 6)). In the cases 2 and 3 the limit exists (by (2. 2)). In the other cases $f(n)/f(n+1)$ can be oscillating, and we can even have (with the notations of (2. 6)) $\beta > \alpha = 0$.

In cases 4 and 5 we construct an example as follows. Let σ be a number, $0 < \sigma \leq 1$; and let $p_1 + p_2 + \dots$ be a series of positive terms whose sum is $\frac{1}{2}\sigma$. We shall construct a series $c_1 + c_2 + \dots$ with $c_k \geq p_k$, whose sum is σ , and such that $c_n/f(n)$ is not bounded.

Let $\varepsilon_1, \varepsilon_2, \dots$ be a sequence with $\varepsilon_k > 0$, $\varepsilon_k \rightarrow 0$. Take $c_k = p_k$ for $k = 1, 2, \dots, K_1 - 1$, where K_1 is the first k with $f(k) < \frac{1}{4}\varepsilon_1\sigma$. The existence of this k follows from the inequality

$$(4. 1) \quad f(1) + \dots + f(m) < \left\{1 - \sum_1^{m-1} c_k\right\}^{-1},$$

which is obtained by addition of the formulas (1. 1) with $n = 1, 2, \dots, m$, respectively.

Now take $c_k = \frac{1}{4}\sigma + p_k$ if $k = K_1$, which does not alter the values of $f(1), \dots, f(K_1)$. If $k = K_1 + 1, \dots, K_2 - 1$ we take $c_k = p_k$ again, where K_2 is the first $k > K_1$ with $f(k) < \frac{1}{8}\varepsilon_2\sigma$. For $k = K_2$ take $c_k = \frac{1}{8}\sigma + p_k$ etc. If $k = K_1, K_2, \dots$ we have $c_k/f(k) > \varepsilon_1^{-1}, \varepsilon_2^{-1}, \dots$, respectively. As $f(k+1) > c_k$ for all k , we also find that $f(k+1)/f(k)$ is not bounded. Therefore $\alpha = 0$. On the other hand we have $\beta > 0$ by (2. 6), since γ is positive. It can be shown that $\gamma = 1$, $O(\gamma) = \sigma$.

A sufficient condition for α to be positive is that $\sum c_k/f(k) < \infty$. For, writing down (1. 1) with $n = N + 1$ and $n = N$, respectively, we infer

$$\frac{f(N+1)}{f(N)} \leq \max_{1 \leq k < N} \frac{f(k+1)}{f(k)} + \frac{c_N}{f(N)},$$

whence $f(n+1) = O\{f(n)\}$.

In case 1 the series $\sum c_k/f(k)$ does not converge since it would lead to $\alpha > 0$. In cases 2 and 3 the series always converges (see (2. 2)). In case 4 the condition may be useful, and we can show that it implies $\alpha = \beta$ (theorem 11). In case 5 however the condition never applies:

Theorem 3. In case 5 we have $\sum c_k/f(k) = \infty$.

Proof. We have $\sum_1^\infty c_k \gamma^k < 1$. Assume $\sum c_k/f(k) < \infty$.

Put $1 - \sum_1^\infty c_k \gamma^k = 2\varepsilon$. Choose l such that $2\gamma \sum_{i+1}^\infty c_k/f(k) < \varepsilon$, and $\delta > 0$ such that $e^{\delta l} \sum_1^l c_k \gamma^k < 1 - \varepsilon$, $e^\delta < 2$. Then we can show by induction

$$(4.2) \quad f(k) \leq 2e^{-\delta k} \gamma^{1-k}.$$

If $k = 1$, (4.2) is trivial. Next assume (4.2) to be true for $k = 1, \dots, n-1$. Then by (1.1)

$$f(n) \leq \sum_1^s c_k f(n-k) + \sum_{s+1}^{n-1} \frac{c_k}{f(k)} f(k) f(n-k),$$

where $s = \min(n-1, l)$, and the second sum is empty if $n-1 \leq l$. It follows that

$$\begin{aligned} f(n) &\leq \sum_1^s c_k e^{\delta k} \gamma^k \cdot 2e^{-\delta n} \gamma^{1-n} + 4 \sum_{s+1}^{n-1} \frac{c_k}{f(k)} e^{-\delta n} \gamma^{2-n} \leq \\ &\leq 2e^{-\delta n} \gamma^{1-n} \{e^{\delta l} \sum_1^l c_k \gamma^k + 2\gamma \sum_{i+1}^\infty c_k/f(k)\} < 2e^{-\delta n} \gamma^{1-n}. \end{aligned}$$

This proves (4.2). However, (4.2) contradicts (2.1). Therefore our assumption $\sum c_k/f(k) < \infty$ is false.

We next discuss the condition $c_k = o\{f(k)\}$. We do not know whether this guarantees the existence of $\lim f(n)/f(n+1)$. On the other hand it is a necessary condition in cases 2, 3 and 4 (theorem 4), but it is not necessary in case 5.

In case 5 we can give an example where

$$(4.3) \quad \frac{f(n+1)}{f(n)} \rightarrow 1, \quad \frac{c_{n+1}}{c_n} \rightarrow 1, \quad \frac{c_n}{f(n)} \rightarrow \frac{1}{4}.$$

In order to construct this example, require (1.1) and $c_n = \frac{1}{4}f(n)$ for all n . Then we have $F(x) - x = \frac{1}{4}F^2(x)$, and so

$$F(x) = 2\{1 - (1-x)^{\frac{1}{2}}\}, \quad f(n) = \frac{4^{-n}}{2n-1} \frac{(2n)!}{n! n!}.$$

We are in case 5 indeed, for the radius of convergence of $C(x) = \frac{1}{4}F(x)$ equals 1, and

$$\sum_1^\infty c_k = M \cdot F(1) = \frac{1}{2}.$$

Theorem 4. If, in case 2, 3 or 4, $\lim f(n)/f(n+1)$ exists ¹⁾, then we have $c_n = o\{f(n)\}$.

Proof. If the limit exists, we know that it equals γ . And, if $n > k+1$, we have

$$(4.4) \quad f(n+1) \geq c_1 f(n) + \dots + c_{k+1} f(n-k) + c_n.$$

Dividing by $f(n)$ and making $n \rightarrow \infty$, we infer

$$\begin{aligned} \gamma^{-1} &\geq c_1 + c_2 \gamma + \dots + c_k \gamma^{k-1} + \limsup c_n/f(n), \\ \limsup c_n/f(n) &\leq \gamma^{-1} \{1 - c_1 \gamma - c_2 \gamma^2 - \dots - c_k \gamma^k\}. \end{aligned}$$

This holds for every k . Since $\sum c_k \gamma^k = 1$ we infer $c_n = o\{f(n)\}$.

¹⁾ In case 2 or 3 the limit exists automatically.

Theorem 5. If, in case 2, 3, or 4, $\lim c_{n+1}/c_n$ exists, then we have $c_n = o\{f(n)\}$.

Proof. The limit of c_{n+1}/c_n equals γ^{-1} , of course. If $n > k$, we have

$$f(n) \geq c_n f(1) + c_{n-1} f(2) + \dots + c_{n-k} f(k).$$

Dividing by c_n and making $n \rightarrow \infty$, we infer

$$\liminf f(n)/c_n \geq f(1) + f(2)\gamma + \dots + f(k)\gamma^{k-1}.$$

The theorem follows from the fact that $\sum f(k)\gamma^{k-1} = \infty$ (see (2.3)).

The following simple theorem applies to the cases 2, 3, 4, 5 (in case 1 the condition is never satisfied).

Theorem 6. If, for some fixed k , we have $c_n = O(c_{n-1} + c_{n-2} + \dots + c_{n-k})$, then $f(n+1) = O\{f(n)\}$, that is $\alpha > 0$.

Proof. For $n > k$ we have

$$\frac{c_{n-k} f(k+1) + \dots + c_n f(1)}{c_{n-k} f(k) + \dots + c_{n-1} f(1)} \leq \max_{1 \leq j \leq k} \frac{f(j+1)}{f(j)} + \frac{C(c_{n-k} + \dots + c_{n-1})}{c_{n-k} f(k) + \dots + c_{n-1} f(1)} < B,$$

B not depending on n . Furthermore, if $n > k$,

$$\begin{aligned} f(n+1) &= \sum_1^n c_j f(n+1-j) \leq \\ &\leq \sum_1^{n-k-1} c_j f(n-j) \cdot \max_{1 \leq l < n} \frac{f(l+1)}{f(l)} + B \sum_{n-k}^{n-1} c_j f(n-j) \leq \\ &\leq f(n) \max \left\{ B, \max_{1 \leq l < n} \frac{f(l+1)}{f(l)} \right\}. \end{aligned}$$

It follows by induction that $f(n+1) \leq Bf(n)$ for all n .

We shall give a necessary and sufficient condition for the existence of $\lim f(n)/f(n+1)$ in the cases 2, 3, 4, 5. That is, we assume

$$(4.5) \quad \gamma > 0, \sum_1^\infty c_k \gamma^k \leq 1; \quad 1 < \sum_1^\infty c_k x^k \leq \infty \text{ if } x > \gamma.$$

Put, if $1 \leq k < n$,

$$(4.6) \quad \left\{ \begin{aligned} &\frac{\gamma \{c_k f(n-k+1) + \dots + c_n f(1)\} - \{c_k f(n-k) + \dots + c_{n-1} f(1)\}}{f(n)} = S_{n,k}; \\ &\limsup_{n \rightarrow \infty} |S_{n,k}| = \varphi(k) \leq \infty. \end{aligned} \right.$$

Theorem 7. In the cases 2, 3, 4, 5 a necessary and sufficient condition for the existence of $\lim f(n)/f(n+1)$ is that $\varphi(k) \rightarrow 0$ when $k \rightarrow \infty$.

Proof. We have, if $1 \leq k < n$,

$$(4.7) \quad \gamma f(n+1) - f(n) = \gamma \sum_1^{k-1} c_j f(n+1-j) - \sum_1^{k-1} c_j f(n-j) + f(n) S_{n,k}.$$

If $f(n)/f(n+1) \rightarrow \gamma$, it easily follows by making $n \rightarrow \infty$ that $\varphi(k) = 0$ for all k .

We next show that $\varphi(k) \rightarrow 0$ is also sufficient. We have (see (2.6))

$0 \leq a \leq \beta < \infty$. First we prove that $a > 0$. We have $f(l+1) \geq c_l f(l)$ for all l . Hence, dividing (4.7) by $f(n)$ we obtain

$$\gamma \frac{f(n+1)}{f(n)} \leq 1 + \sum_1^{k-1} c_j c_1^{1-j} + |S_{n,k}|.$$

Choose k such that $\varphi(k) < \infty$, and make $n \rightarrow \infty$. It follows that $f(n+1) = O(f(n))$, that is $a > 0$.

Let $\{n_i\}$ be a sequence for which

$$(4.8) \quad f(n_i)/f(n_i+1) \rightarrow a \quad (i \rightarrow \infty).$$

Then we have, for any fixed $l \geq 0$, also

$$(4.9) \quad f(n_i-l)/f(n_i+1-l) \rightarrow a \quad (i \rightarrow \infty).$$

The same holds if a is replaced by β both times. We only prove it for the lower limit; the other case can be proved analogously.

Assume (4.9) false for some $l > 0$. Then there is a subsequence $\{m_i\}$ and a number δ ($\delta > a$) such that

$$f(m_i-l) > \delta f(m_i+1-l) \quad (i = 1, 2, \dots).$$

Further, if $\varepsilon > 0$ and $i > i_0(\varepsilon, k)$ then we have

$$f(m_i-j) > (a-\varepsilon) f(m_i+1-j) \quad (1 \leq j < k)$$

It follows, if $k > l$, $i > i_0(\varepsilon, k)$, that

$$\begin{aligned} \sum_{j=1}^{k-1} c_j \{ \gamma f(m_i+1-j) - f(m_i-j) \} &< \\ < \sum_{j=1}^{k-1} c_j (\gamma - a + \varepsilon) f(m_i+1-j) - c_l (\delta - a) f(m_i+1-l) < \\ < (\gamma - a + \varepsilon) f(m_i+1) - c_l (\delta - a) f(m_i+1-l), \end{aligned}$$

and so, by (4.7),

$$(a-\varepsilon) f(m_i+1) + c_l (\delta - a) f(m_i+1-l) \leq f(m_i) \{ |S_{m_i,k}| + 1 \}.$$

If $i \rightarrow \infty$, we have $f(m_i)/f(m_i+1) \rightarrow a$, $\liminf f(m_i+1-l)/f(m_i+1) \geq a^l$. Therefore

$$a - \varepsilon + c_l (\delta - a) a^l \leq a + a\varphi(k),$$

which holds whenever $k > l$, $\varepsilon > 0$. Making $k \rightarrow \infty$, $\varepsilon \rightarrow 0$ we obtain $\delta = a$, and a contradiction has been found. This proves (4.9).

We can now show that $a = \gamma$. Assume $a < \gamma$, and let the sequence $\{n_i\}$ satisfy (4.8). Now write down (4.7) with $n = n_i$, divide by $f(n_i+1)$ and make $i \rightarrow \infty$ (k is fixed). We obtain

$$|\gamma - a - \sum_1^{k-1} c_j (\gamma a^j - a^{j+1})| \leq a\varphi(k),$$

which leads to

$$|1 - \sum_1^{k-1} c_j a^j| \leq \frac{a\varphi(k)}{\gamma - a}.$$

Making $k \rightarrow \infty$ we infer $C(a) = 1$, which is impossible since $a < \gamma$.

In the same way the assumption $\beta > \gamma$ leads to $C(\beta) = 1$. Thus the proof of theorem 7 is completed.

For some applications we can better deal with $T_{n,k}$, where, if $n > k \geq 1$,

$$(4.10) \quad T_{n,k} = S_{n,k} - \gamma \frac{c_k f(n-k+1)}{f(n)} = \frac{1}{f(n)} \sum_{j=k}^{n-1} f(n-j) \{\gamma c_{j+1} - c_j\},$$

and put $\limsup_{n \rightarrow \infty} |T_{n,k}| = \psi(k) \leq \infty$.

Theorem 8. In the cases 2, 3, 4, 5 a necessary and sufficient condition for the existence of $\lim f(n)/f(n+1)$ is that $\psi(k) \rightarrow 0$ as $k \rightarrow \infty$.

Proof. In the first place, if $f(n)/f(n+1) \rightarrow \gamma$ is given, then we deduce

$$\lim_{n \rightarrow \infty} |T_{n,k} - S_{n,k}| = c_k \gamma^k,$$

and $c_k \gamma^k \rightarrow 0$ since $\sum c_k \gamma^k$ converges. Hence $\psi(k) \rightarrow 0$.

Next assume $\psi(k) \rightarrow 0$. As in the beginning of the proof of theorem 7 we deduce $f(n+1) < Cf(n)$ for some C and all n . Therefore we have, if $n > 2K$

$$\min_{K \leq k \leq 2K} \frac{\gamma c_k f(n-k+1)}{f(n)} \leq \frac{\gamma}{K f(n)} \sum_{K}^{2K} c_k f(n-k+1) \leq \frac{\gamma C}{K},$$

and hence

$$(4.11) \quad \lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} \min_{K \leq k \leq 2K} |S_{n,k}| = 0.$$

It is easily seen that with this condition, instead of $\varphi(k) \rightarrow 0$, we are also able to give the remaining part of the proof of theorem 7.

Theorem 9. In all cases the condition $c_n/c_{n+1} \rightarrow \gamma$ implies

$$f(n)/f(n+1) \rightarrow \gamma.$$

Proof. We exclude case 1 here; the proof for case 1 will be given in § 5.

If $\varepsilon > 0$, then for $j > A(\varepsilon)$ we have

$$|\gamma c_{j+1} - c_j| < \varepsilon c_j.$$

Hence, for $k > A(\varepsilon)$, $n > k$, we have by (4.10),

$$f(n) |T_{n,k}| < \sum_k^{n-1} \varepsilon c_j f(n-j) < \varepsilon f(n).$$

Therefore $\psi(k) \rightarrow 0$ as $k \rightarrow \infty$, and theorem 8 can be applied.

Theorem 10. In the cases 2, 3, 4, 5, the condition

$$\sum_2^{\infty} \frac{|\gamma c_n - c_{n-1}|}{f(n)} < \infty$$

implies $f(n)/f(n+1) \rightarrow \gamma$.

Proof. By (4.10) and by theorem 1 we have, if $n > k > 1$,

$$f(n) |T_{n,k}| < \sum_k^{n-1} \frac{f(n)}{f(j+1)} |\gamma c_{j+1} - c_j| < f(n) \sum_{k-1}^{\infty} \frac{|\gamma c_j - c_{j-1}|}{f(j)}.$$

Consequently $\psi(k) \rightarrow 0$ as $k \rightarrow \infty$, and theorem 8 can be applied.

Theorem 11. If $\Sigma c_n/f(n) < \infty$, then $f(n)/f(n+1) \rightarrow \gamma$.

Proof. As was remarked before, the convergence of the series implies $f(n+1) = O\{f(n)\}$, and it excludes case 1. Thus we may apply theorem 10, since

$$\sum_2^{\infty} \frac{c_{n-1}}{f(n)} = \sum_1^{\infty} \frac{c_n}{f(n+1)} < \sum_1^{\infty} \frac{c_n}{c_1 f(n)} < \infty.$$

Possibly the condition

$$(4.15) \quad \sum_1^{\infty} \left| \frac{c_{n+1}}{f(n+1)} - \frac{c_n}{f(n)} \right| < \infty$$

is also sufficient for $f(n)/f(n+1) \rightarrow \gamma$, but we could not decide this.

A sufficient condition which applies to all cases, is

Theorem 12. If $c_{n+1} c_{n-1} \geq c_n^2$ ($n > 1$), then $f(n)/f(n+1) \rightarrow \gamma$.

Proof. It was proved in [1] that $c_{n+1} c_{n-1} \geq c_n^2$ ($n > 1$) implies $f(n+1) \cdot f(n-1) \geq f^2(n)$ ($n > 1$). (The proof did not depend on the assumption $\Sigma c_k = 1$ which was made throughout that paper). Consequently $f(n)/f(n+1)$ is non-increasing, and its limit exists.

REFERENCES

1. N. G. DE BRUIJN and P. ERDŐS. On a recursion formula and on some Tauberian theorems. Submitted to J. Research Nat. Bur. Standards.
3. P. ERDŐS, W. FELLER and H. POLLARD. A property of power series with positive coefficients. Bull. Am. Math. Soc. 55, 201-204 (1949).
4. G. PÓLYA and G. SZEGŐ. Aufgaben und Lehrsätze aus der Analysis, vol. 1, (New York, 1945).

(To be continued)