Invariance of the analyticity domains of self-adjoint operators subjected to perturbations

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INVARIANCE OF THE ANALYTICITY DOMAINS OF SELF-ADJOINT OPERATORS SUBJECTED TO PERTURBATIONS

by

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Abstract

The analyticity domain $C^{\omega}(A)$ of a self-adjoint operator A in a Hilbert space X is dense. If A is positive, $C^{\omega}(A)$ is equal to the test space $S_{X,A}$ of De Graaf's distribution theory. Here the question is investigated for which Hilbert spaces Y and self-adjoint, positive operators B in Y.

$$S_{X,A} \subset S_{Y,B} \text{ or } S_{Y,B} \subset S_{X,A} :$$

Especially the case that B is derived from A by a perturbation will have our attention.

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An element f in a Hilbert space X is called an analytic vector for an operator R in X if there are constants a,b > 0 with

$$\| R^n f \| \leq ab^n \, n!$$

The linear subspace of all analytic vectors of R is called the analyticity domain of R and is denoted by $C^\omega(R)$. Nelson [1] observed that a symmetric operator P in X has a dense analyticity domain if and only if P is essentially self-adjoint.

An operator T in X is said to analytically dominate R, if $C^\omega(T) \subset C^\omega(R)$. If T has a dense analyticity domain and if R is symmetric, analytical dominance of T implies that R is essentially self-adjoint. Faris in [2], ch.VI, gives conditions on the commutator of T and R which lead to analytical dominance.

In the sequel A will denote a positive, self-adjoint operator in X. Nelson [1] showed that a vector $f \in$ X is in $C^\omega(A)$ if and only if there are $t > 0$ and $w \in$ X with $f = e^{-tA}w$. Instead of the notation $C^\omega(A)$ we shall employ the terminology of De Graaf [3] and denote the analyticity domain of A by $S_{X,A}$. In [3], De Graaf takes $S_{X,A}$ as the test function space in his theory of generalized functions. It would be nice if we could classify the spaces $S_{X,A}$. We remark that each Hilbert space X and each positive self-adjoint operator A in X generates a test function space $S_{X,A}$.
To this end, in this paper the problem will be investigated whether there exist conditions on Hilbert spaces Y and positive self-adjoint

operators B such that

$$S_{X,A} \subset S_{Y,B} \quad \text{or} \quad S_{Y,B} \subset S_{X,A}.$$

This investigation yields results with respect to the so-called classification problem, which can be translated in analytic vector terminology, of course.

As an immediate consequence of the theory in [3] we derive

(1) <u>Theorem</u>

Let G be a positive self-adjoint operator in X which generates a group $t \to e^{tG}$, $t \in \mathbb{R}$, of continuous linear mappings on $S_{X,A}$. Then

$$S_{X,A} \subset S_{X,G} .$$

<u>Proof</u>

Let $\alpha,\beta > 0$. Then following [3], $e^{\alpha G}e^{-\beta A}$ is a bounded operator on X. Let $f \in S_{X,A}$. Then there exists $t > 0$ such that $e^{tA}f \in X$. So we have for each $\alpha > 0$

$$f = e^{-\alpha G} \, (e^{\alpha G}e^{-tA})e^{tA}f$$

and thus $f \in S_{X,G}$. □

(2) <u>Theorem</u>

Let G be an operator in X which generates a holomorphic group $z \mapsto e^{zG}$, $z \in \mathbb{C}$, of continuous linear mappings on $S_{X,A}$. Then

$$S_{X,A} \subset C^{\omega}(G)$$

or, equivalently, A analytically dominates G.

Proof

For every $\alpha > 0$, the operators $e^{zG}e^{-\alpha A}$, $z \in \mathbb{C}$, are bounded, and the operator valued function

$$z \mapsto e^{zG}e^{-\alpha A}$$

is holomorphic. So there exists $M > 0$ such that

$$\| e^{zG}e^{-\alpha A} \| \le M \ , \quad |z| = 1 \ ,$$

uniformly. Further

$$G^n e^{-\alpha A} = \frac{n!}{2\pi i} \int_{|z|=1} \frac{e^{zG}e^{-\alpha A}}{z^{n+1}} dz \ ,$$

hence $\| G^n e^{-\alpha A} \| \le \frac{M}{2\pi} n!$

Let $f \in S_{X,A}$. Then there is $\alpha > 0$ such that $e^{\alpha A}f \in X$ and

$$\| G^n f \| = \| G^n e^{-\alpha A} e^{\alpha A} f \| \le \frac{M}{2\pi} \| e^{\alpha A} f \| \, n! \ .$$

Therefore $f \in C^{\omega}(G)$. □

We note that in [4] there are given conditions on operators G in X in order that G generates a holomorphic group of continuous linear mappings on $S_{X,A}$.

We now start with the main part of this paper and discuss the problem in case B is obtained from A by a perturbation. The results are contained in two lemmas.

(3) Lemma

Let P be an operator in X with $D(P) \supset S_{X,A}$ which satisfies the following conditions:

(i) There exists a Hilbert space Y such that $S_{X,A} \subset Y$ and $A + P$ is a positive, essentially self-adjoint operator in Y.

(ii) There exist $d > 0$ and $k > 0$ such that for all $\alpha > 0$

$$\| e^{\alpha A} PA^{-1} e^{-\alpha A} \| \le d\, e^{k\alpha} .$$

Then $S_{X,A} \subset S_{Y,A+P}$.

Proof

Let $t > 0$, and let $0 < \tau < t$ be fixed. We want to estimate the operator norm of $e^{\tau A}(A+P)^n e^{-tA}$. Therefore we factor as follows

$$e^{\tau A}(A+P)^n\, e^{-tA} =$$

$$\prod_{j=0}^{n-1} \exp\left[\left(\frac{n-j}{n}\tau + \frac{j}{n}t\right)A\right] (I + PA^{-1}) \exp\left[-\left(\frac{n-j}{n}\tau + \frac{j}{n}t\right) A\right] \circ$$

$$\circ\, A \exp\left[-\frac{1}{n}(t-\tau)A\right] .$$

This factorization yields the following estimate

$$\| e^{\tau A}(A+P)^n\, e^{-tA} \| \le$$

$$\le \| A e^{-\frac{1}{n}(t-\tau)A} \|^n \prod_{j=0}^{n-1} \| e^{(\frac{n-j}{n}\tau + \frac{j}{n}t)A} (I + PA^{-1})\, e^{-(\frac{n-j}{n}\tau + \frac{j}{n}t)A} \|$$

$$\le n!\, (t-\tau)^{-n} \prod_{j=0}^{n-1} \left(1 + d \exp\left(k\tau\, \frac{n-j}{n} + \frac{ktj}{n}\right)\right)$$

$$\leq \; n!\,(t-\tau)^{-n}\,(1+d)^n \exp(\tfrac{1}{2}nk(t+\tau)) \; .$$

Now it is obvious that for all σ, $0 < \sigma < (1+d)^{-1}(t-\tau)\exp(-kt)$,

$$\| e^{\tau A}\, e^{\sigma(A+P)}\, e^{-tA} \| < \infty$$

where we take

$$e^{\sigma(A+P)} = \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} (A+P)^n \; .$$

We remark that the operator norms are taken with respect to $\mathcal{B}(X)$, the algebra of bounded operators on X.

Let $f \in \mathcal{S}_{X,A}$. Then there exist $t > 0$ and τ, $0 < \tau < t$, such that $e^{tA} f \in X$. Now f can be written as

$$f = e^{-\sigma(A+P)}\, e^{-\tau A}\, (e^{\tau A}\, e^{\sigma(A+P)}\, e^{-tA})\, e^{tA}\, f.$$

Since $e^{-\tau A}(e^{\tau A}\, e^{\sigma(A+P)}\, e^{-tA})\, e^{tA}\, f \in \mathcal{S}_{X,A} \subset Y$, the result $f \in \mathcal{S}_{Y,A+P}$ is obtained, completing the proof. □

A corollary of Lemma (3) contains conditions on P such that A analytically dominates the perturbed operator $A + P$.

(4) Corollary

Let P be an operator in X with $\mathcal{D}(P) \supset \mathcal{S}_{X,A}$ for which there are constants $d > 0$ and $k > 0$ such that

(ii) $$\| e^{\alpha A}\, P A^{-1}\, e^{-\alpha A} \| \leq d\, e^{k\alpha} \; , \qquad \alpha > 0 \; .$$

Then $S_{X,A} \subset C^{\omega}(A + P)$ $(\subset X)$.

Proof

As in the proof of Lemma (3) the following estimate can be derived

$$\| (A + P)^n e^{-tA} \| \leq n!\, t^{-1} (1 + d)^n \exp(\tfrac{1}{2} nkt) \;, \quad t > 0.$$

Hence for $f \in S_{X,A}$,

$$\| (A + P)^n f \| \leq \| (A + P)^n e^{-tA} \| \, \| e^{tA} f \|$$

$$\leq n!\, a^n \| e^{tA} f \|$$

where $a = (1 + d)\, t^{-1} \exp(\tfrac{1}{2} kt)$, and t is chosen sufficiently small. □

Remark: In Lemma (3) and Corollary (4) we could as well demand

(ii') $\quad \| e^{\alpha A} A^{-1} P e^{-\alpha A} \| \leq d\, e^{k\alpha} \quad , \alpha > 0$.

As a preparation to the following lemma the function space $\hat{B}_+(\mathbb{R})$ is introduced

$$\psi \in \hat{B}_+(\mathbb{R}) \;:\Leftrightarrow\; \text{(i)}\; \exists_{\varepsilon>0} \forall_{x\in\mathbb{R}} : \psi(x) > 1+\varepsilon \;,$$

$$\text{(ii)}\; \forall_{t>0} \sup_{x\geq 0} (\psi(x)\, e^{-tx}) < \infty \;.$$

For $\psi \in \hat{B}_+(\mathbb{R})$, the operator $\log \psi(A)$ is self-adjoint, and it has a bounded everywhere defined inverse. Here we want to estimate the norm,

$$\| [\log\psi(A)]^n e^{-tA} \|$$

for all $n \in \mathbb{N}$ and all $t > 0$. To this end, let $t > 0$, $n \in \mathbb{N}$ and take τ, $0 < \tau < t$. Then there exists $M_\tau > 1$ such that

$$\psi(x)\ e^{-\tau x} < M_\tau\ , \qquad x \geq 0\ .$$

Hence

$$(\log\psi(x))^n \leq (\log M_\tau + x\tau)^n\ .$$

From the latter inequality we derive

$$\sup_{x\geq 0} [(\log\psi(x))^n\ e^{-tx}] \leq \sup_{x\geq 0} [(\log M_\tau + x\tau)^n\ e^{-tx}]$$

$$\leq n!\ (\tfrac{\tau}{t})^n\ (M_\tau)^{t/\tau}\ .$$

So the following estimate holds true

(5) $$\| (\log\psi(A))^n\ e^{-tA} \| \leq n!\ (\tfrac{\tau}{t})^n\ M_\tau^{t/\tau}$$

where $0 < \tau < t$ and $M_\tau = \sup_{x\geq 0} (\psi(x)\ e^{-\tau x})$.

(6) <u>Lemma</u>

Let P be a well-defined operator in X with $D(P) \supset S_{X,A}$ which satisfies

(i) There exists a Hilbert space Y with $S_{X,A} \subset Y$ such, that $A + P$ is a positive, essentially self-adjoint operator in Y, and $S_{Y,A+P} \subset X$.

(ii) There exists $\psi \in \hat{B}_+(\mathbb{R})$, $d > 0$ and $k > 0$ such, that

$$\| e^{\alpha A}(\log\psi(A))^{-1} P e^{-\alpha A} \| \le d\, e^{k|\alpha|}$$

for all $\alpha \in \mathbb{R}$. (Here we take the operator norm with respect to X.)

Then $S_{Y,A+P} \subset S_{X,A}$.

Proof

With the aid of Duhamel's iteration principle the operator $e^{-t(A+P)}$, $t > 0$, can be expressed by a series

$$e^{-t(A+P)} = \sum_{n=0}^{\infty} P_n(t)$$

where

$$P_0(t) = e^{-tA},$$

$$P_n(t) = (-1)^n \int_0^t \cdots \int_0^{t_{n-1}} e^{-tA}(e^{t_1 A} P e^{-t_1 A}) \cdots (e^{t_n A} P e^{-t_n A})\, dt_1 \ldots dt_n$$

$n = 1, 2, \ldots$

We shall prove that for every $t > 0$ the series

$$\sum_{n=0}^{\infty} \| e^{\frac{1}{2}tA} P_n(t) \|$$

converges. To this end we factor the integrand as follows

$$e^{-tA}(e^{t_1 A} P e^{-t_1 A}) \ldots (e^{t_n A} P e^{-t_n A}) =$$

$$= e^{-\frac{t}{2}A} \prod_{j=1}^{n} \left(e^{-\frac{t}{2n}A} \log\psi(A))(e^{(t_j - \frac{n-j}{2n}t)A} (\log\psi(A))^{-1} P e^{-(t_j - \frac{n-j}{2n}t)A}\right).$$

Straightforward computation yields

$$\| e^{-\frac{t}{2}A} (e^{t_1 A} P e^{-t_1 A}) \ldots (e^{t_n A} P e^{-t_n A}) \| \le$$

$$\le \| \log(\psi(A)) e^{-\frac{t}{2}A} \|^n \prod_{j=1}^{n} \| e^{(t_j - \frac{n-j}{2n} t)A} (\log\psi(A))^{-1} P e^{-(t_j - \frac{n-j}{2n} t)A} \| .$$

Since (by 5) for all τ, $0 < \tau < t$,

$$\| (\log\psi(A)) e^{-\frac{t}{2n}A} \|^n = \| \log\psi(A))^n \; e^{-\frac{t}{2}A} \| \le n! \; (\tfrac{\tau}{t})^n (M_{\tau/2})^{t/\tau}$$

and since

$$\prod_{j=0}^{n} \| e^{(t_j - \frac{n-j}{2n} t)A} (\log\psi(A))^{-1} P e^{-(t_j - \frac{n-j}{n} t)A} \| \le$$

$$\le \prod_{j=0}^{n} d \; e^{k|t_j - \frac{n-j}{n} t|}$$

$$\le d^n \; e^{2nkt} \quad ,$$

we derive

$$\| e^{\frac{1}{2}t A} P_n(t) \| \le C_{t,\tau} (\tfrac{\tau}{t})^n \; n! \; e^{2nkt} \left(\int_0^t \ldots \int_0^{t_{n-1}} dt_1 \ldots dt_n \right)$$

$$\le C_{t,\tau} \; \tau^n \; d^n \; e^{2nkt} \quad .$$

If we take $0 < \tau < d^{-1} \; e^{-2kt}$, then the series satisfy

$$\sum_{n=0}^{\infty} \| e^{\frac{1}{2}t A} P_n(t) \| \le C_{t,\tau} \sum_{n=0}^{\infty} (\tau \; d \, e^{2kt})^n < \infty;$$

and hence

$$\| e^{\frac{1}{2}tA} \; e^{-t(A+P)} \| < \infty \; .$$

Now let $f \in Y$, and let $t > 0$. Then for $0 < \tau < t$

$$e^{-t(A+P)} f = e^{-\frac{1}{2}\tau A} \; (e^{\frac{1}{2}\tau A} \; e^{-\tau(A+P)}) e^{-(t-\tau)(A+P)} f$$

and

$$e^{-(t-\tau)(A+P)} f \in X \; ,$$

by assumption. This yields the result

$$e^{-t(A+P)} f \in S_{X,A} \; .$$

□

A combination of Lemma (3) and Lemma (6) yields the following theorem.

(7) Theorem

Let P be a well-defined operator in X, $D(P) \supset S_{X,A}$, which satisfies the following conditions

(i) There exists a Hilbert space Y such that $S_{X,A} \subset Y$, $A + P$ is a positive, essentially self-adjoint operator in Y and $S_{Y,A+P} \subset X$.

(ii) There exists $\psi \in \hat{B}_+(\mathbb{R})$, $d > 0$ and $k > 0$ such that

$$\| e^{\rho A} (\log \psi(A))^{-1} \, P \, e^{-\rho A} \| \leq d \; e^{k|\rho|}$$

for all $\rho \in \mathbb{R}$.

Then $\quad S_{X,A} = S_{Y,A+P}$.

Proof. Combine Lemma (3) and Lemma (6) . □

An application of the theory of this paper can be found in [5] where it is proved that

$$S_{X_\alpha, A_\alpha} = S_{X_\beta, A_\beta}$$

for all $\alpha,\beta > -1$. Here we take

$$X_\gamma = L_2(0,\infty), \; x^{2\gamma+1}\, dx\,)$$

$$A_\gamma = -\frac{d^2}{dx^2} + x^2 - (2\gamma + 1)\,\frac{1}{x}\,\frac{1}{dx}\ ,$$

where $\gamma > -1$.

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