

# Invariance of the analyticity domains of self-adjoint operators subjected to perturbations

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INVARIANCE OF THE ANALYTICITY DOMAINS OF SELF-ADJOINT  
OPERATORS SUBJECTED TO PERTURBATIONS

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*divide et impera*

Abstract

The analyticity domain  $C^\omega(A)$  of a self-adjoint operator  $A$  in a Hilbert space  $X$  is dense. If  $A$  is positive,  $C^\omega(A)$  is equal to the test space  $S_{X,A}$  of De Graaf's distribution theory. Here the question is investigated for which Hilbert spaces  $Y$  and self-adjoint, positive operators  $B$  in  $Y$ .

$$S_{X,A} \subset S_{Y,B} \quad \text{or} \quad S_{Y,B} \subset S_{X,A} :$$

Especially the case that  $B$  is derived from  $A$  by a perturbation will have our attention.

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An element  $f$  in a Hilbert space  $X$  is called an analytic vector for an operator  $R$  in  $X$  if there are constants  $a, b > 0$  with

$$\|R^n f\| \leq ab^n n!$$

The linear subspace of all analytic vectors of  $R$  is called the analyticity domain of  $R$  and is denoted by  $C^\omega(R)$ . Nelson [1] observed that a symmetric operator  $P$  in  $X$  has a dense analyticity domain if and only if  $P$  is essentially self-adjoint.

An operator  $T$  in  $X$  is said to analytically dominate  $R$ , if  $C^\omega(T) \subset C^\omega(R)$ . If  $T$  has a dense analyticity domain and if  $R$  is symmetric, analytical dominance of  $T$  implies that  $R$  is essentially self-adjoint. Faris in [2], ch. VI, gives conditions on the commutator of  $T$  and  $R$  which lead to analytical dominance.

In the sequel  $A$  will denote a positive, self-adjoint operator in  $X$ . Nelson [1] showed that a vector  $f \in X$  is in  $C^\omega(A)$  if and only if there are  $t > 0$  and  $w \in X$  with  $f = e^{-tA} w$ . Instead of the notation  $C^\omega(A)$  we shall employ the terminology of De Graaf [3] and denote the analyticity domain of  $A$  by  $S_{X,A}$ . In [3], De Graaf takes  $S_{X,A}$  as the test function space in his theory of generalized functions. It would be nice if we could classify the spaces  $S_{X,A}$ . We remark that each Hilbert space  $X$  and each positive self-adjoint operator  $A$  in  $X$  generates a test function space  $S_{X,A}$ .

To this end, in this paper the problem will be investigated whether there exist conditions on Hilbert spaces  $Y$  and positive self-adjoint

operators  $B$  such that

$$S_{X,A} \subset S_{Y,B} \quad \text{or} \quad S_{Y,B} \subset S_{X,A}.$$

This investigation yields results with respect to the so-called classification problem, which can be translated in analytic vector terminology, of course.

As an immediate consequence of the theory in [3] we derive

(1) Theorem

Let  $G$  be a positive self-adjoint operator in  $X$  which generates a group  $t \rightarrow e^{tG}$ ,  $t \in \mathbb{R}$ , of continuous linear mappings on  $S_{X,A}$ . Then

$$S_{X,A} \subset S_{X,G}.$$

Proof

Let  $\alpha, \beta > 0$ . Then following [3],  $e^{\alpha G} e^{-\beta A}$  is a bounded operator on  $X$ .

Let  $f \in S_{X,A}$ . Then there exists  $t > 0$  such that  $e^{tA} f \in X$ . So we have for each  $\alpha > 0$

$$f = e^{-\alpha G} (e^{\alpha G} e^{-tA}) e^{tA} f$$

and thus  $f \in S_{X,G}$ . □

(2) Theorem

Let  $G$  be an operator in  $X$  which generates a holomorphic group  $z \mapsto e^{zG}$ ,  $z \in \mathbb{C}$ , of continuous linear mappings on  $S_{X,A}$ . Then

$$S_{X,A} \subset C^\omega(G)$$

or, equivalently,  $A$  analytically dominates  $G$ .

Proof

For every  $\alpha > 0$ , the operators  $e^{zG} e^{-\alpha A}$ ,  $z \in \mathbb{C}$ , are bounded, and the operator valued function

$$z \mapsto e^{zG} e^{-\alpha A}$$

is holomorphic. So there exists  $M > 0$  such that

$$\|e^{zG} e^{-\alpha A}\| \leq M, \quad |z| = 1,$$

uniformly. Further

$$G^n e^{-\alpha A} = \frac{n!}{2\pi i} \int_{|z|=1} \frac{e^{zG} e^{-\alpha A}}{z^{n+1}} dz,$$

hence  $\|G^n e^{-\alpha A}\| \leq \frac{M}{2\pi} n!$

Let  $f \in S_{X,A}$ . Then there is  $\alpha > 0$  such that  $e^{\alpha A} f \in X$  and

$$\|G^n f\| = \|G^n e^{-\alpha A} e^{\alpha A} f\| \leq \frac{M}{2\pi} \|e^{\alpha A} f\| n!.$$

Therefore  $f \in C^\omega(G)$ . □

We note that in [4] there are given conditions on operators  $G$  in  $X$  in order that  $G$  generates a holomorphic group of continuous linear mappings on  $S_{X,A}$ .

We now start with the main part of this paper and discuss the problem in case  $B$  is obtained from  $A$  by a perturbation. The results are contained in two lemmas.

(3) Lemma

Let  $P$  be an operator in  $X$  with  $D(P) \supset S_{X,A}$  which satisfies the following conditions:

- (i) There exists a Hilbert space  $Y$  such that  $S_{X,A} \subset Y$  and  $A + P$  is a positive, essentially self-adjoint operator in  $Y$ .
- (ii) There exist  $d > 0$  and  $k > 0$  such that for all  $\alpha > 0$

$$\| e^{\alpha A} P A^{-1} e^{-\alpha A} \| \leq d e^{k\alpha} .$$

Then  $S_{X,A} \subset S_{Y,A+P}$ .

Proof

Let  $t > 0$ , and let  $0 < \tau < t$  be fixed. We want to estimate the operator norm of  $e^{\tau A} (A + P)^n e^{-tA}$ . Therefore we factor as follows

$$\begin{aligned} e^{\tau A} (A + P)^n e^{-tA} &= \\ \prod_{j=0}^{n-1} \exp\left[\left(\frac{n-j}{n}\tau + \frac{j}{n}t\right)A\right] (I + P A^{-1}) \exp\left[-\left(\frac{n-j}{n}\tau + \frac{j}{n}t\right)A\right] &\circ \\ \circ A \exp\left[-\frac{1}{n}(t-\tau)A\right] . & \end{aligned}$$

This factorization yields the following estimate

$$\begin{aligned} \| e^{\tau A} (A + P)^n e^{-tA} \| &\leq \\ \leq \| A e^{-\frac{1}{n}(t-\tau)A} \| \prod_{j=0}^{n-1} \| e^{(\frac{n-j}{n}\tau + \frac{j}{n}t)A} (I + P A^{-1}) e^{-(\frac{n-j}{n}\tau + \frac{j}{n}t)A} \| & \\ \leq n! (t - \tau)^{-n} \prod_{j=0}^{n-1} \left( 1 + d \exp\left(k\tau \frac{n-j}{n} + \frac{ktj}{n}\right) \right) & \end{aligned}$$

$$\leq n! (t - \tau)^{-n} (1 + d)^n \exp\left(\frac{1}{2} nk(t + \tau)\right).$$

Now it is obvious that for all  $\sigma$ ,  $0 < \sigma < (1+d)^{-1}(t-\tau) \exp(-kt)$ ,

$$\|e^{\tau A} e^{\sigma(A+P)} e^{-tA}\| < \infty$$

where we take

$$e^{\sigma(A+P)} = \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} (A + P)^n.$$

We remark that the operator norms are taken with respect to  $B(X)$ , the algebra of bounded operators on  $X$ .

Let  $f \in S_{X,A}$ . Then there exist  $t > 0$  and  $\tau$ ,  $0 < \tau < t$ , such that  $e^{tA} f \in X$ . Now  $f$  can be written as

$$f = e^{-\sigma(A+P)} e^{-\tau A} (e^{\tau A} e^{\sigma(A+P)} e^{-tA}) e^{tA} f.$$

Since  $e^{-\tau A} (e^{\tau A} e^{\sigma(A+P)} e^{-tA}) e^{tA} f \in S_{X,A} \subset Y$ , the result  $f \in S_{Y,A+P}$  is obtained, completing the proof.  $\square$

A corollary of Lemma (3) contains conditions on  $P$  such that  $A$  analytically dominates the perturbed operator  $A + P$ .

#### (4) Corollary

Let  $P$  be an operator in  $X$  with  $D(P) \supset S_{X,A}$  for which there are constants  $d > 0$  and  $k > 0$  such that

$$(ii) \quad \|e^{\alpha A} P A^{-1} e^{-\alpha A}\| \leq d e^{k\alpha}, \quad \alpha > 0.$$



Then  $S_{X,A} \subset C^\omega(A+P)$ . ( $\subset X$ ).

Proof

As in the proof of Lemma (3) the following estimate can be derived

$$\|(A+P)^n e^{-tA}\| \leq n! t^{-1} (1+d)^n \exp\left(\frac{1}{2}nkt\right), \quad t > 0.$$

Hence for  $f \in S_{X,A}$ ,

$$\begin{aligned} \|(A+P)^n f\| &\leq \|(A+P)^n e^{-tA}\| \|e^{tA} f\| \\ &\leq n! a^n \|e^{tA} f\| \end{aligned}$$

where  $a = (1+d) t^{-1} \exp\left(\frac{1}{2}kt\right)$ , and  $t$  is chosen sufficiently small.  $\square$

Remark: In Lemma (3) and Corollary (4) we could as well demand

$$(ii') \quad \|e^{\alpha A} A^{-1} P e^{-\alpha A}\| \leq d e^{k\alpha}, \quad \alpha > 0.$$

As a preparation to the following lemma the function space  $\hat{B}_+(\mathbb{R})$  is introduced

$$\psi \in \hat{B}_+(\mathbb{R}) : \Leftrightarrow (i) \exists_{\varepsilon>0} \forall_{x \in \mathbb{R}} : \psi(x) > 1+\varepsilon,$$

$$(ii) \forall_{t>0} \sup_{x \geq 0} (\psi(x) e^{-tx}) < \infty.$$

For  $\psi \in \hat{B}_+(\mathbb{R})$ , the operator  $\log \psi(A)$  is self-adjoint, and it has a bounded everywhere defined inverse. Here we want to estimate the norm,

$$\| [\log \psi(A)]^n e^{-tA} \|$$

for all  $n \in \mathbb{N}$  and all  $t > 0$ . To this end, let  $t > 0$ ,  $n \in \mathbb{N}$  and take  $\tau$ ,  $0 < \tau < t$ . Then there exists  $M_\tau > 1$  such that

$$\psi(x) e^{-\tau x} < M_\tau, \quad x \geq 0.$$

Hence

$$(\log \psi(x))^n \leq (\log M_\tau + x\tau)^n.$$

From the latter inequality we derive

$$\begin{aligned} \sup_{x \geq 0} [(\log \psi(x))^n e^{-tx}] &\leq \sup_{x \geq 0} [(\log M_\tau + x\tau)^n e^{-tx}] \\ &\leq n! \left(\frac{\tau}{t}\right)^n (M_\tau)^{t/\tau}. \end{aligned}$$

So the following estimate holds true

$$(5) \quad \| (\log \psi(A))^n e^{-tA} \| \leq n! \left(\frac{\tau}{t}\right)^n M_\tau^{t/\tau}$$

where  $0 < \tau < t$  and  $M_\tau = \sup_{x \geq 0} (\psi(x) e^{-\tau x})$ .

(6) Lemma

Let  $P$  be a well-defined operator in  $X$  with  $D(P) \supset S_{X,A}$  which satisfies

- (i) There exists a Hilbert space  $Y$  with  $S_{X,A} \subset Y$  such, that  $A + P$  is a positive, essentially self-adjoint operator in  $Y$ , and  $S_{Y,A+P} \subset X$ .
- (ii) There exists  $\psi \in \hat{B}_+(\mathbb{R})$ ,  $d > 0$  and  $k > 0$  such, that

$$\|e^{\alpha A} (\log \psi(A))^{-1} P e^{-\alpha A}\| \leq d e^{k|\alpha|}$$

for all  $\alpha \in \mathbb{R}$ . (Here we take the operator norm with respect to  $X$ .)

Then  $S_{Y, A+P} \subset S_{X, A}$ .

Proof

With the aid of Duhamel's iteration principle the operator  $e^{-t(A+P)}$ ,  $t > 0$ , can be expressed by a series

$$e^{-t(A+P)} = \sum_{n=0}^{\infty} P_n(t)$$

where

$$P_0(t) = e^{-tA},$$

$$P_n(t) = (-1)^n \int_0^t \dots \int_0^{t_{n-1}} e^{-tA} (e^{t_1 A} P e^{-t_1 A}) \dots (e^{t_n A} P e^{-t_n A}) dt_1 \dots dt_n$$

$$n = 1, 2, \dots$$

We shall prove that for every  $t > 0$  the series

$$\sum_{n=0}^{\infty} \|e^{\frac{1}{2}tA} P_n(t)\|$$

converges. To this end we factor the integrand as follows

$$e^{-tA} (e^{t_1 A} P e^{-t_1 A}) \dots (e^{t_n A} P e^{-t_n A}) =$$

$$= e^{-\frac{t}{2}A} \prod_{j=1}^n (e^{-\frac{t}{2n}A} \log \psi(A)) (e^{(t_j - \frac{n-j}{2n}t)A} (\log \psi(A))^{-1} P e^{-(t_j - \frac{n-j}{2n}t)A}).$$

Straightforward computation yields

$$\begin{aligned} & \| e^{-\frac{t}{2}A} (e^{\frac{t_1}{2}A} P e^{-\frac{t_1}{2}A}) \dots (e^{\frac{t_n}{2}A} P e^{-\frac{t_n}{2}A}) \| \leq \\ & \leq \| \log(\psi(A)) e^{-\frac{t}{2}A} \| \prod_{j=1}^n \| e^{(t_j - \frac{n-j}{2n}t)A} (\log \psi(A))^{-1} P e^{-(t_j - \frac{n-j}{2n}t)A} \| . \end{aligned}$$

Since (by 5) for all  $\tau$ ,  $0 < \tau < t$ ,

$$\| (\log \psi(A)) e^{-\frac{t}{2n}A} \| \leq \| \log \psi(A) \| e^{-\frac{t}{2n}A} \leq n! \left(\frac{\tau}{t}\right)^n (M_{\tau/2})^{t/\tau}$$

and since

$$\begin{aligned} & \prod_{j=0}^n \| e^{(t_j - \frac{n-j}{2n}t)A} (\log \psi(A))^{-1} P e^{-(t_j - \frac{n-j}{2n}t)A} \| \leq \\ & \leq \prod_{j=0}^n d e^{k|t_j - \frac{n-j}{2n}t|} \\ & \leq d^n e^{2nkt} , \end{aligned}$$

we derive

$$\begin{aligned} \| e^{\frac{1}{2}tA} P_n(t) \| & \leq C_{t,\tau} \left(\frac{\tau}{t}\right)^n n! e^{2nkt} \left( \int_0^t \dots \int_0^{t_{n-1}} dt_1 \dots dt_n \right) \\ & \leq C_{t,\tau} \tau^n d^n e^{2nkt} . \end{aligned}$$

If we take  $0 < \tau < d^{-1} e^{-2kt}$ , then the series satisfy

$$\sum_{n=0}^{\infty} \| e^{\frac{1}{2}tA} P_n(t) \| \leq C_{t,\tau} \sum_{n=0}^{\infty} (\tau d e^{2kt})^n < \infty ;$$

and hence

$$\|e^{\frac{1}{2}tA} e^{-t(A+P)}\| < \infty .$$

Now let  $f \in Y$ , and let  $t > 0$ . Then for  $0 < \tau < t$

$$e^{-t(A+P)} f = e^{-\frac{1}{2}\tau A} (e^{\frac{1}{2}\tau A} e^{-\tau(A+P)}) e^{-(t-\tau)(A+P)} f$$

and

$$e^{-(t-\tau)(A+P)} f \in X ,$$

by assumption. This yields the result

$$e^{-t(A+P)} f \in S_{X,A} .$$

□

A combination of Lemma (3) and Lemma (6) yields the following theorem.

(7) Theorem

Let  $P$  be a well-defined operator in  $X$ ,  $D(P) \supset S_{X,A}$ , which satisfies the following conditions

- (i) There exists a Hilbert space  $Y$  such that  $S_{X,A} \subset Y$ ,  $A + P$  is a positive, essentially self-adjoint operator in  $Y$  and  $S_{Y,A+P} \subset X$ .
- (ii) There exists  $\psi \in \hat{B}_+(\mathbb{R})$ ,  $d > 0$  and  $k > 0$  such that

$$\|e^{\rho A} (\log \psi(A))^{-1} P e^{-\rho A}\| \leq d e^{k|\rho|}$$

for all  $\rho \in \mathbb{R}$ .

Then  $S_{X,A} = S_{Y,A+P}$ .

Proof. Combine Lemma (3) and Lemma (6).

□

An application of the theory of this paper can be found in [5] where it is proved that

$$S_{X_\alpha, A_\alpha} = S_{X_\beta, A_\beta}$$

for all  $\alpha, \beta > -1$ . Here we take

$$X_\gamma = L_2(0, \infty), x^{2\gamma+1} dx)$$

$$A_\gamma = -\frac{d^2}{dx^2} + x^2 - (2\gamma + 1) \frac{1}{x} \frac{1}{dx} ,$$

where  $\gamma > -1$ .

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