On the distribution of [theta] modulo 1

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On the distribution of Θ modulo 1

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In this note we report the following results, joint work with R. L. Graham, which will be published in the Canadian Journal of Mathematics.

If Θ is an irrational number and $0 = a_0 < a_1 < \cdots < a_n < a_{n+1} = 1$ is the sequence of points $\{l\Theta\}$, $1 \le l \le n$ we define

$$d_\Theta(n) = \max_{1 \le i \le n+1} (a_i - a_{i-1}).$$

THEOREM 1: $\sup_\Theta \lim_{n\to\infty} \inf n d_\Theta(n) = \frac{1+\sqrt{2}}{2}$

(this sup is a maximum taken if $\Theta = 1+\sqrt{2}$).

THEOREM 2: $\inf_\Theta \lim_{n\to\infty} \sup n d_\Theta(n) = 1 + \frac{2}{5}\sqrt{5}$

(this inf is a minimum taken if $\Theta = \frac{1+\sqrt{5}}{2}$.

THEOREM 3: $\lim_{n\to\infty} \sup n d_\Theta(n) = \infty$ iff $\lim_{n\to\infty} \sup b_n = \infty$

where $[b_1, b_2, \ldots]$ is the regular continued fraction for Θ.

REMARK: It was remarked by V. Turán-Sós that she had proved theorem 3 in one of her papers.

The proofs depend upon the somewhat surprising and apparently little-known fact that the set of numbers $\{a_{i+1}-a_i : 0 \leq i \leq n\}$ always consists of at most 3 numbers. This enables one to find an explicit expression for $d_{\otimes}(n)$.