The logarithmic solutions of linear differential equations

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The Logarithmic Solutions of Linear Differential Equations.

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SOME years ago Balth. van der Pol † found the following expression for $x^{\frac{1}{2}n}$ K_n $(2\sqrt{x})$, where n is an integer $(n \geqslant 0)$, and K_n is the Bessel function of the second kind ‡ with imaginary argument

$$x^{\frac{1}{2}n}K_n(2\sqrt{x}) = -\tfrac{1}{2}\sum_{m=0}^{\infty}\frac{d}{dm}\left(\frac{x^m}{\Pi(m)\Pi(m-n)}\right)\quad (x>0), \quad . \quad . \quad (1)$$

which is in striking resemblance to the following formula for the function of the first kind :

$$x^{\frac{1}{2}n}I_n(2\sqrt{x}) = \sum_{m=0}^{\infty}\frac{x^m}{\Pi(m)\Pi(m-n)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Formula (1) is a very short notation for the expansion usually given :

$$K_n(2\sqrt{x}) = \tfrac{1}{2}\sum_{\nu=0}^{n-1}\frac{(-1)^{\nu+n}}{\nu\,!}(n-\nu+1)\,!\;x^{\nu}$$

$$-\tfrac{1}{2}\sum_{\nu=0}^{\infty}\frac{x^{\nu}}{\nu\,!\;(n+\nu)\,!}\,[\log x-\psi(\nu+1)-\psi(\nu+n+1)]$$

where $$\psi(u)=\Gamma'(u)/\Gamma(u).$$

Prof. van der Pol also gave a formula for $e^x \operatorname{li}(e^{-x})$ (see formula (16) below), in similar analogy with the power series for e^x. He obtained these results with the aid of operational calculus, but since K_m and I_m satisfy a single second-order linear differential equation, and the same can be said concerning $e^x \operatorname{li}(e^{-x})$ and e^x, we tried to give proofs for his formulæ, starting from the corresponding differential equations. This turned out to be possible, leading to a more general process for finding logarithmic solutions of linear differential equations, which shows some resemblance to a well-known method of Frobenius §. The latter, however, does not lead directly to simple expressions like (1).

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* Communicated by Prof. B. van der Pol.

† *Wiskundige Opgaven met de oplossingen v. h. Wiskundig Genootschap*, Amsterdam, xviii. p. 96 (1943), problem 34.

‡ See for its definition : Whittaker and Watson, 'Modern Analysis,' 4th ed. 374. We notice that the definition of $K_n(x)$ in Watson, 'Bessel Functions,' ffers from this one by a factor $(-1)^n$.

§ *Journ. reine angew. Math.* lxxvi. pp. 214–235 (1873). Frobenius' method discussed in most textbooks on differential equations.

Frobenius' method furnishes all solutions of a linear differential equation

$$A_k(x)y^{(k)}+A_{k-1}(x)y^{(k-1)}+\ldots+A_1(x)y'+A_0(x)y=0, \quad . \quad . \quad (3)$$

provided that $x=0$ be a regular point for this equation, *i.e.*, $x^{k-\nu}A_\nu(x)A_k^{-1}(x)$ regular at $x=0(\nu=0, 1, \ldots\ldots, k)$. The solutions are obtained as series of the type

$$\sum_{m=0}^{\infty} P_m(\log x)x^{m+\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where ρ (the *exponent*), is a real or complex number, and the P_m are polynomials (of degree $<k$). The solutions, for which all the P_m are of zero degree, are always easy to construct, but the difficulties lie in the logarithmic solutions.

In connection with van der Pol's formulæ we introduce another process for finding logarithmic solutions. This process, however, does not necessarily give *all* solutions of a given equation, but it has the advantage of being easy to explain and to handle, and to furnish simple expressions for the logarithmic solutions in many applications. But it is of no use for the construction of a complete system of linearly independent solutions in the general case, which is the primary aim of Frobenius' method.

We have to restrict ourselves by the assumption that all coefficients $A_\nu(x)(\nu=0, 1, \ldots, k)$ be polynomials, but on the other hand we may omit the conditions about the regularity of

$$x^{k-\nu}A_\nu(x)A_k^{-1}(x) \text{ for } x=0, \nu=0, 1, \ldots, k.$$

We call a series

$$F \sim \sum_{m=-\infty}^{\infty} a(m)x^{m+\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

a *formal solution* of (3), when, on substituting F into (3), taking for F′, F″, ... the formal successive derivatives

$$F^{(\nu)} \sim \sum_{m=-\infty}^{\infty} a(m)(m+\rho)(m+\rho-1)\ldots(m+\rho-\nu+1)x^{m+\rho-\nu},$$

of F, the left-hand side of (3) becomes identically zero. That is to say, on taking together terms with equal powers of x, after formal multiplication by the polynomial coefficients $A_\nu(x)$, every power of x obtains the coefficient zero. For any integer $m=\mu$, the coefficient of $x^{\mu+\rho}$ becomes a linear combination of a finite number of $a(m)$'s. The condition that this expression vanishes forms the so-called *recurrence relation*.

Now suppose that a function $a(t)$ be defined and differentiable for all real values of t, and that

$$F \sim \sum_{m=-\infty}^{\infty} a(m+t)x^{m+t+\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

be a formal solution of (3) for any value of t. Then it will be clear that the derivative with respect to t for $t=t_0$,

$$\left(\frac{\partial F}{\partial t}\right)_{t=t_0} \sim \sum_{m=-\infty}^{\infty} \frac{\partial}{\partial t}\{a(m+t)x^{m+t+\rho}\}_{t=t_0}, \quad . \quad . \quad . \quad . \quad (7)$$

or in abbreviated notation

$$\sum_{-\infty}^{\infty} \frac{\partial}{\partial m}\{a(m+t_0)x^{m+t_0+\rho}\},$$

is also a formal solution of (3). For, if we substitute (7) into (3), and take together all terms showing the same power $x^{\mu+t_0+\rho}$, we get an expression, identical with the partial derivative of the expression, obtained in the same way for the series (6). The latter *, being equal to zero, identically in t and containing only a finite number of terms, the former vanishes as well.

If a formal solution (5) is convergent for all x of any open region of the complex x-plane, it yields a solution in the ordinary sense. The same holds true for formal solutions of the type $\sum_{-\infty}^{\infty} \mathrm{P}_m(\log x)x^{m+\rho}$, where the P_m are polynomials of degree $<k$ †. In both cases, namely, convergence in any open region includes absolute convergence of all formal derivatives in the same region. Henceforth we will use the term *actual solution* for any solution in the ordinary sense.

Now the importance of the expression (7) arises from the possibility that, although (6) cannot furnish actual solutions for all values of t (the equation (3) admitting only a finite number of exponents), nevertheless (7) may form an actual solution for some value of t_0.

The series (7) can be split into two parts (without loss of generality we take $t_0=0$) :

$$\left(\frac{\partial \mathrm{F}}{\partial t}\right)_{t=0} \sim \sum_{-\infty}^{\infty} a'(m)x^{m+\rho}+\log x \sum_{-\infty}^{\infty} a(m)x^{m+\rho}, \quad . \quad . \quad . \quad (8)$$

and, if we suppose the series (7) to be convergent for all x in a certain region, it is easy to prove that both series in (8) are convergent in that region. The two series in (8) being Laurent series, the region of convergence of the series (7), if existing at all, will be a range ‡ $\alpha<|x|<\beta$, where $0\leqslant\alpha<\beta\leqslant\infty$.

We notice that the second series in (8) is the same as the series (6) for $t=0$, whose range of convergence thus includes that of (7) for $t_0=0$.

The proposed process can thus be described as follows :—construct a formal solution with a parameter t, of the type (6), and examine whether its formal derivative with respect to t, for $t=0$, has a region of convergence. If that is the case, we have obtained an actual solution. This is a logarithmic one, unless $a(m) = 0$ for all integers m (which by no means implies the vanishing of the $a'(m)$).

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* This is the left-hand side of the recurrence relation, multiplied by the corresponding power of x.

† For log x we may take any branch of the multi-valued logarithmic function.

‡ Not mentioning possible convergence in boundary points.

Sometimes it may be useful to repeat this process, and thus take the second formal derivative of (6), and so on. Another generalization arises from the remark, that it is only necessary for (6) to be a formal solution for $-\epsilon<t<\epsilon(\epsilon>0)$. In that case, we need not define $a(t)$ outside the set of intervals $|t-m|<\epsilon(m=0, \pm 1, \pm 2, \ldots)$. Moreover, $a(t)$ only needs to be differentiable at integral values of t.

In applications, the following trivial remark may be useful. If $f(t)$ is periodic with period 1, and (6) forms a formal solution, then the series

$$\sum_{-\infty}^{\infty} a(m+t)f(m+t)x^{m+t+\rho}$$

will be a formal solution as well.

Examples.

1. *Bessel's equation.*

$$x^2y''+xy'+(x^2-n^2)y=0, \qquad \ldots \ldots \quad (9)$$

where n is an integer ($n \geqslant 0$). Substituting (6) with $\rho=0$ into (9), we obtain the recurrence relation

$$(t^2-n^2)a(t)+a(t-2)=0,$$

a solution of which is $$a(t)=\frac{2^{-t}\cos\frac{\pi}{2}(t-n)}{\Pi\left(\frac{t+n}{2}\right)\Pi\left(\frac{t-n}{2}\right)}.$$

For $t=0$ the series (6) becomes

$$\sum_{-\infty}^{\infty} a(m)x^m=\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!\,(\nu+n)!}\left(\frac{x}{2}\right)^{n+2\nu},$$

the well-known expansion of $J_n(x)$.

Partial differentiation of $\sum_{-\infty}^{\infty} a(m+t)x^{m+t}$ with respect to t does not lead, for $t=0$, to a convergent expression, owing to the appearance of terms of the type

$$-\frac{\pi}{2}\frac{2^{-m}\sin\frac{\pi}{2}(m-n)\,x^m}{\Pi\left(\frac{m+n}{2}\right)\Pi\left(\frac{m-n}{2}\right)},$$

the absolute value of which tends very rapidly to infinity when m runs to $-\infty$ through negative integers $\equiv n+1$ (mod. 2). Therefore we first multiply $a(t)$ by $\cos^2\frac{\pi}{2}(t-n)$. Multiplication by a function of period 2 is allowed here since the difference of the arguments occurring in the recurrence relation is even. The derivative of $\cos^3 \frac{\pi}{2}(t-n)$ vanishing for all integral values of t, the series (7) now becomes ($t=0$) :

$$\sum_{m=-\infty}^{\infty} \cos^3 \frac{\pi}{2}(m-n)\frac{d}{dm}\left\{\frac{(\frac{1}{2}x)^m}{\Pi\left(\frac{m+n}{2}\right)\Pi\left(\frac{m-n}{2}\right)}\right\}$$
$$=\tfrac{1}{2}\sum_{\nu=-\infty}^{\infty}(-1)^\nu\frac{d}{d\nu}\left\{\frac{(\frac{1}{2}x)^{n+2\nu}}{\Pi(\nu)\Pi(\nu+n)}\right\}.$$

Because this series contains only powers of x with exponents $\geqslant -n$, its convergence for all $x\neq 0$ is easy to establish. Thus we have found an actual solution of (9). Evaluation of a few coefficients shows that it is $\frac{\pi}{2}\mathrm{Y}_n(x)=\frac{1}{2}\mathbf{Y}(x)$, where Y_n and $\mathbf{Y}_n$ denote the Bessel functions of the second kind with the notation of Hankel and Weber, respectively *.

Our formulæ for $\mathrm{J}_n(x)$ and $\mathbf{Y}_n(x)$ show a similar resemblance as van der Pol's formulæ (1) and (2) :

$$\mathrm{J}_n(x)=\sum_{\nu=-\infty}^{\infty}(-1)^\nu\frac{(\frac{1}{2}x)^{n+2\nu}}{\Pi(\nu)\Pi(\nu+n)}$$
$$\mathbf{Y}_n(x)=\sum_{\nu=-\infty}^{\infty}(-1)^\nu\frac{d}{d\nu}\left\{\frac{(\frac{1}{2}x)^{n+2\nu}}{\Pi(\nu)\Pi(\nu+n)}\right\}. \quad . \quad . \quad . \quad . \quad (10)$$

The expression for $\mathbf{Y}_n(x)$, usually given †, is

$$\mathbf{Y}_n(x)=-\sum_{\nu=0}^{n-1}\frac{(n-\nu-1)\,!}{\nu\,!}\left(\frac{x}{2}\right)^{-n+2\nu}+$$
$$+\sum_{\nu=0}^{\infty}\frac{(-1)^\nu(\frac{1}{2}x)^{n+2\nu}}{\nu\,!\,(\nu+n)\,!}\{2\log(\tfrac{1}{2}x)-\psi(\nu+1)-\psi(\nu+n+1)\},$$

where $\psi(u)=\Gamma'(u)/\Gamma(u)$. The agreement with (10) is easily verified. Van der Pol's result (1) can be derived from (10), but of course it can also be proved independently in a similar manner.

2. *Legendre's equation.*

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{du}{d\mu}\right\}+n(n+1)u=0, \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where n is a non-negative integer. We transform it by

$$\mu=\frac{1+x}{1-x}, \quad u=(1-x)^{-n}y$$

into
$$1-x)\left(x\frac{d}{dx}\right)^2 y+2nx\left(x\frac{d}{dx}\right)y-xn^2y=0,$$

whose recurrence relation reads ($\rho=0$) :

$$t^2a(t)-(n-t+1)^2a(t-1)=0.$$

* See Whittaker and Watson, 'Modern Analysis,' 4th ed. p. 372 ; Watson, 'Bessel Functions,' 2nd ed. pp. 58, 64 (1944).

† Watson, 'Bessel Functions,' p. 62.

A solution is $$a(t)=\left\{\frac{\Pi(n)}{\Pi(n-t)\Pi(t)}\right\}^2=\binom{n}{t}^2.$$

We have

$a(t)=a'(t)=0$ for $t=n+1,\ n+2,\ n+3,\dots$ and for $t=-1,\ -2,\dots$. Hence

$$\left(\frac{1+\mu}{2}\right)^n\sum_{m=0}^{n}\binom{n}{m}^2\left(\frac{\mu-1}{\mu+1}\right)^m$$

and

$$\left(\frac{1+\mu}{2}\right)^n\sum_{m=0}^{n}\frac{d}{dm}\left\{\binom{n}{m}^2\left(\frac{\mu-1}{\mu+1}\right)^m\right\}$$

are solutions of (11), and so they are linear combinations of $P_n(\mu)$ and $Q_n(\mu)$ *. Simple considerations concerning the behaviour at $\mu=1$, -1 and ∞ show them to be equal to $P_n(\mu)$ and $-2Q_n(\mu)$, respectively. We have thus obtained

$$Q_n(\mu)=-\tfrac{1}{2}\left(\frac{1+\mu}{2}\right)^n\sum_{m=0}^{n}\frac{d}{dm}\left\{\binom{n}{m}^2\left(\frac{\mu-1}{\mu+1}\right)^m\right\}.$$

3. The equation $$xy''+(1-x)y'-y=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

has the solutions

$$y=e^x \quad\text{and}\quad y=e^{-x}\,\mathrm{li}\,(e^{-x}) \quad \left(-\mathrm{li}(e^{-x})=\int_x^\infty e^{-s}\frac{ds}{s}\quad\text{for}\quad x>0\right).$$

The recurrence relation reads ($\rho=0$)

$$t^2a(t)-ta(t-1)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

a solution of which is $$a(t)=\frac{1}{\Pi(t)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

Thus for $t=0$ the series (6) becomes the power series for e^x. Our differentiation process now yields the formal solution

$$\sum_{-\infty}^{\infty}\frac{d}{dm}\left(\frac{x^m}{\Pi(m)}\right)=\log x\ .\ \sum_{0}^{\infty}\frac{x^m}{m!}+\sum_{-\infty}^{\infty}x^m\frac{d}{dm}\left(\frac{1}{\Pi(m)}\right). \quad . \quad . \qquad (15)$$

This result is nowhere convergent, and we cannot make it convergent by multiplying $a(t)$ by suitable functions. Thus our method fails. But in this case we are able to show that (15) remains a formal solution of (12) after omitting all terms with $m<0$. For, (13) gives

$$2ta(t)-a(t-1)+t^2a'(t)-ta'(t-1)=0,$$

hence it follows from (14) that

$$t^2a'(t)-ta'(t-1)=0 \quad\text{for}\quad t=-1,\ -2,\dots$$

Taking $b(m)=a'(m)$ for $m=-1,\ -2,\ -3,\ \dots$, and $b(m)=0$ for $m=0,\ 1,\ 2,\ \dots$, our $b(m)$ now satisfies $t^2b(t)-tb(t-1)=0$ for all integers t.

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* See for its definition : Hobson, 'Spherical Harmonics,' p. 51 (1931).

Hence $\sum\limits_{\infty}^{\infty} b(m)x^m$ is a formal solution of (12). Subtracting the latter from (15), we obtain the formal solution

$$\sum_0^{\infty} \frac{d}{dm}\left(\frac{x^m}{\Pi(m)}\right).$$

This series being convergent for all $x \neq 0$, an actual solution of (12) is found. Evaluation of a few coefficients shows that

$$\sum_0^{\infty} \frac{d}{dm}\left(\frac{x^m}{\Pi(m)}\right) = e^x \operatorname{li}(e^{-x}), \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

which is van der Pol's result *.

Eindhoven, August 1946.

* *Wiskundige Opgaven met de Oplossingen v. h. Wiskundig Genootschap*, Amsterdam, xviii. (1) p. 90 (1943), problem 33.