Telescope mappings

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~~LP notation~~

Abstractor string height k $[x_1, A_1][x_2, A_2(x_1)] \cdots [x_k, A_k(x_1, \ldots, x_{k-1})]$

An LP is an abstractor string mod α-reduction on the bound variables

A vector of height k is a column $\begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}$, where the v_i are AUT-expressions

We use three lambda-like ~~operators~~ quantifiers:

$\lambda_{x \in Q}\, v(x)$ will be a vector-valued function

(Q is an LP, $v(x)$ is a vector)

$\pi_{x \in Q}\, R(x)$ is the concatenation of the LP's Q and R

$\mu_{x \in Q}\, R(x)$ will be the LP of the mappings of Q into R

If v and Q have height k, then

$v \in Q$ (v is a folder for Q)

says that $v_1 \mathrm{E}\, A_1$, $v_2 \mathrm{E}\, A_2(v_1)$, $v_3 \mathrm{E}\, A_3(v_1, v_2)$, ...

(where $v = \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}$, $Q = [x_1, A_1] \cdots [x_k, A_k(x_1, \ldots, x_{k-1})]$)

Later notation: E repl. by :

v is folder for Q	v fits into Q
LP	Telescope

Folders on higher level: If in context $x \in R$ we have $v(x) \in Q(x)$, ~~for all $x \in R$~~ we write

$$\forall_{x \in R}\left(v(x) \in Q(x)\right) \qquad \cancel{\text{or}} \qquad {}^{*)}$$

Here it is assumed that $\cancel{\emptyset}_{x \in R}\, Q(x)$ is an acceptable LP. (π)

Book lines: ~~R~~ In context R we write a line as

$$\forall_{x \in R}\left(\quad f := v(x) \in Q(x)\right)$$

or

$$\forall_{x \in R}\left(\quad f := PN \in Q(x)\right)$$

We need not write block opening lines.

We can write <u>LP-defining lines</u>:

$$\cancel{\forall_{x \in R}}\left(\quad S := \pi_{x \in Q}\, R(x)\right)$$

$$\cancel{\forall_{x \in R}}\left(\quad T := \mu_{x \in Q}\, R(x)\right)$$

<u>Folding</u>. if $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix}$ then

$\langle v \rangle w$ denotes $\begin{pmatrix} \langle v_n \rangle \cdots \langle v_1 \rangle w_1 \\ \vdots \\ \langle v_n \rangle \cdots \langle v_1 \rangle w_k \end{pmatrix}$

wat is dat?

*) Remark (1983): If $\begin{array}{|l} \boxed{x \in R} \\ v(x) \in Q(x) \end{array}$ then $\begin{array}{|l} \boxed{x \in R} \\ x \circ v(x) \in \\ \quad \in \pi_{y \in R}\, Q(y) \end{array}$

If $Q = [x_1, A_1] \cdots [x_k, A_k(x_1, \ldots, x_{k-1})] \mod \alpha_{x_1 \cdots x_k}$

~~then~~ $v(x) = \begin{pmatrix} f_1(x_1, \ldots, x_k) \\ \vdots \\ f_n(x_1, \ldots, x_k) \end{pmatrix}$

then

$$\lambda_{x \in Q} v(x) \quad \text{denotes} \quad \begin{pmatrix} [x_1, A_1] \cdots [x_k, A_k(x_1, \ldots, x_{k-1})]\, f_1(x_1, \ldots, x_k) \\ \vdots \\ [x_1, A_1] \cdots [x_k, A_k(x_1, \ldots, x_{k-1})]\, f_n(x_1, \ldots, x_k) \end{pmatrix}$$

If $Q = [x_1, A_1] \cdots [x_k, A_k(x_1, \ldots, x_{k-1})] \mod \alpha_x$

$R(x) = [y_1, B_1(x)] \cdots [y_m, B_m(x, y_1, \ldots, y_{m-1})] \mod \alpha_y$

then

$$\pi_{x \in Q} R(x) = [x_1, A_1] \cdots [x_k, A_k(\cdots)]\,[y_1, B_1(x)] \cdots [y_m, B_m(\cdots)] \mod \alpha_{x,y}$$

$$\mu_{x \in Q} R(x) = [s_1, \lambda_{x \in Q} B_1(x)]\,[s_2, \lambda_{x \in Q} B_2(x, \langle x \rangle s_1)] \cdots$$
$$\cdots [s_m, \lambda_{x \in Q} B_m(x, \langle x \rangle s_1, \ldots, \langle x \rangle s_{m-1})]$$

If R has length 1, $R(x) = ([y, \Lambda(x)] \mod \alpha_y)$, then

$$\mu_{x \in Q} R(x) = ([t, \lambda_{x \in Q} \Lambda(x)] \mod \alpha_t)$$

<u>Composition</u> $\quad Q \to R_1 \otimes R_2$

(wh var [illegible])

$$\mu_{x \in Q} \left(\pi_{y \in R_1(x)} \; R_2(x,y) \right) =$$

$$= \pi_{f \in \mu_{x \in Q} R_1(x)} \left(\mu_{x \in Q} \; R_2(x, \langle x \rangle f) \right)$$

Compare this to

$$\forall_{x \in A} \, \exists_{y \in B} \, T(x,y) \iff \exists_{f \in (A \to B)} \, \forall_{x \in A} \, T(x, f(x))$$

under Ax. of Choice

<u>Rules</u>

$$\frac{f \leq \mu_{x \in M} R(x) \quad , \quad v \in M}{\langle v \rangle f \leq R(v)}$$

$$\langle a \rangle \left(\lambda_{x \in M} f(x) \right) \stackrel{D}{=} f(a)$$

$$\frac{\forall_{x \in M} \quad f(x) \leq R(x)}{\lambda_{x \in M} \; f(x) \leq \mu_{x \in M} R(x)}$$

Cantor

These rules also hold with extra $\forall$'s in front, e.g.

$$\forall_{y \in Q} \left(f(y) \leq \mu_{x \in M(y)} R(y,x) \; , \; v(y) \in M(y) \right)$$

$$\boxed{\underline{\text{Compositie}} \qquad R_1 \otimes R_2 \rightarrow w \qquad \lambda_{u \in \pi_{x \in R_1} R_2(x)} \; w(u) = \lambda_{x \in R_1} \left(\lambda_{y \in R_2(x)} \; w\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) \right)}$$

Compositie $R_1 \otimes R_2 \rightarrow Q$

(Π ver door Σ)

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} =: f_1 \otimes f_2$$

pr $\forall_{x \in R} \ldots$

$\ldots$

$\mathcal{B} \vdash_{x \in R} \ldots$

schrifte aan afgekeerde Q's en v's.

p 4: vgl met som - prod.

$$\Sigma \cdot \{ \qquad \prod_{i \in S} \prod_{j \in T(i)} \ldots$$

Controleer $\lambda_{x \in M} f(x) \Leftarrow \mu_{x \in M} R(x)$

<u>Compositie</u>

$$\boxed{\mu_{u \in \pi_{x \in Q_1} Q_2(x)} R(u) = \mu_{x \in Q} \; \mu_{y \in Q_2(x)}} \; \boxed{R\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)}$$

Zie compositieregel van $Q \to R_1 \otimes R_2$ met R_1 ge hoogte 1,

$$R_1 = ([X, \Lambda(x)])_{\mathrm{mod}\ \alpha_x}$$

$$\mu_{x \in Q} \left(\pi_{y \in R_1(x)} R_2(x,y) \right) = [f, \lambda_{t \in Q} \Lambda(t)] \left(\mu_{x \in Q} R_2(x, \langle x \rangle f) \right)$$

We kunnen inderdaad de regel $\forall_{x} f \in R'$ ———— bewijzen
$\lambda_{x \in M} f(x) \in \mu_{x \in M} R(x)$.

$\Leftarrow$ uit $\left\{ \begin{array}{ll} f \in \mu_{x \in M} R(x), & v \in M \\ \langle v \rangle f \in R(v) & \end{array} \right.$

Voorbeeld v.e. mappings LP

$$\mu_{[\xi, \mathrm{type}]} \binom{\xi}{p} \in ([\eta, \mathrm{type}][Q, [t, \eta]\, \mathrm{prop}])_{\mathrm{mod}\ \alpha(\eta, Q)} \qquad ([y, \xi][X, \langle y \rangle P])_{\mathrm{mod}\ \alpha(y,x)}$$

Deze voegt aan de parameters ξ, P van een set $S(\xi, P)$ het set LP toe, d.i. ~~[x, ξ]~~ het ding met de eigenschap

dat $\binom{a}{b} \in S(\xi, P)$ betekent $\left\{ \begin{array}{l} a \in \xi \\ b \in \langle a \rangle P \end{array} \right.$

<u>Notatie</u> $\boxed{\chi(P) := ([t, P])_{\mathrm{mod}\ t}}$

of iets anders — [illegible]!

ook $\chi \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = ([t_1, v_1] \ldots [t_n, v_n])_{\mathrm{mod}\ t_n, \ldots, t_1}$

Nu als $a_1 \in v_1, \ldots$ dan ~~$\langle a \rangle \in$~~ $a \in \chi(v)$

Basis-type $\quad \tau_0 := ([t, \underline{type}])_{mod\ \alpha(l)}$

Nu type $\quad \xi := \underline{\qquad} \tau$

schrijven $\quad \xi \in \tau_0$

prop. $\quad prop \in \tau_0$

$PROP := \mathcal{X}(prop)$

$\xi := \underline{\qquad} \in \tau_0$

$P :=$ ...

Zij S een deel of ξ. Dan

Definieer, als $\quad \forall_{x \in S}\ (P(x) \in PROP)$,

$$ALL_{x \in S}\ P(x) := \prod_{x \in S} P(x)$$

Nu geldt (zie regel $\ell\ell_4$)

$$\frac{f \in ALL_{x \in S}\ P(x)\ , \quad v \in S}{\langle v \rangle f \in P(v)}$$

<u>Nu is eens ai ... het algemeend</u> voor slechts op S gedef. pred. P

Als P op de hele ξ gedef. is, dan eerst

$\forall_{x \in S} \quad P^*(x) := P\ (1^e$ proj. van $x)$

en nu $\quad ALL_{x \in S}\ P^*(x)$

Composition (v. 2C's order)

$$R\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left[y_1 , B_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right] \left[y_2 , \quad \cdot \quad \right] \left[y_3 , B_3\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} , y_1 , y_2 \right) \right]$$

$$\mu_{\cancel{a} \in Q_2(x_1)} R\begin{pmatrix} x_1 \\ u \end{pmatrix} = \left[s_1 , \lambda_{a \in Q_2(x_1)} B_1\begin{pmatrix} x_1 \\ a \end{pmatrix} \right] \cdots \left[s_3 , \lambda_{a \in Q_2(x_1)} B_3\left(\begin{pmatrix} x_1 \\ a \end{pmatrix} , \langle a \rangle s_1 , \langle a \rangle s_2 \right) \right]$$

$$\mu_{v \in Q_1} \mu_{u \in Q_2(x_1)} R\begin{pmatrix} v \\ u \end{pmatrix} =$$

$$= \left[w_1 , \underbrace{\lambda_{v \in Q_1} \lambda_{a \in Q_2(x_1)}}_{\text{compiles}} B_1\begin{pmatrix} v \\ a \end{pmatrix} \right] \cdots \left[w_3 , \lambda_{v \in Q_1} \lambda_{a \in Q_2(x_1)} B_3\left(\begin{pmatrix} v \\ a \end{pmatrix} , \langle a \rangle \langle v \rangle w_1 \right) \right]$$

$$= \left[w_1 , \lambda_{\begin{pmatrix} v \\ a \end{pmatrix} \in Q} B_1\begin{pmatrix} v \\ a \end{pmatrix} \right] \cdots \left[w_3 , \lambda_{\begin{pmatrix} v \\ a \end{pmatrix} \in Q} B_3\left(\begin{pmatrix} v \\ a \end{pmatrix} , \left\langle \begin{pmatrix} v \\ a \end{pmatrix} \right\rangle w_1 \right) \right]$$

where $Q = \cancel{\mu_{v \in Q_1} \lambda_{a \in Q_2(x_1)}} \sum_{v \in Q_1} Q_2(v)$; Note: $\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle s = \langle x_2 \rangle \langle x_1 \rangle s$

OK. Klopt.

Υ_u : hele lambda

Nieuwe notaties als in NF lambda

$(Q\circ f)$	$\lambda_{u\in Q}\ f(u)$	;	$L(Q,\ \Upsilon_u f(u))$.
$Q \to R$	$\mu_{u\in Q}\ R(u)$	:	$M(Q,\ \Upsilon_u R(u))$
$Q \otimes R$	$\sum_{u\in Q} R(u)$	:	$S(Q,\ \Upsilon_u R(u))$.

Compositieregels

$(Q\to(R_1\otimes R_2)) = (Q\to R_1)\otimes(Q\to R_2)$:

ll 4 top

$$\boxed{M(Q,\ \Upsilon S(R_1, \Upsilon R_2)) = S(M(Q,\Upsilon R_1),\ \Upsilon M(Q,\Upsilon R_2))}$$

$(Q_1\otimes Q_2)\to f = Q_1\to(Q_2\to f)$

ll 5 top.

$$L(S(Q_1,\Upsilon Q_2),\Upsilon f) = L(Q_1,\Upsilon L(Q_2,\Upsilon f))$$

$(Q_1\otimes Q_2)\to R = Q_1\to(Q_2\to R)$

ll 5 bottom, ll 6

$$M(S(Q_1,\Upsilon Q_2),\Upsilon R) = M(Q_1,\Upsilon M(Q_2,\Upsilon R))$$

$(Q_1\otimes Q_2)\otimes Q_3 \quad Q_1\otimes(Q_2\otimes Q_3)$

$$S(S(Q_1,\Upsilon Q_2),\Upsilon Q_3) = S(Q_1,\ \Upsilon S(Q_2,\Upsilon Q_3))$$

Steeds volgt elke komma een Υ

$\Rightarrow$ Verander de notatie zó dat de Υ's verdwijnen.

$L(Q,\ \Upsilon_u f(u))$ noteren als $L(Q, F)$ en eis dat F iets is als wat $\Upsilon_x f(x)$

En dit direct in infix notatie zetten?.

$Q\circ F$ betekent $\lambda_{u\in Q}\ F(u)$ $\quad$ $\boxed{F \text{ is nu een } \lambda_x F(x)}$

$Q\to R$ betekent $\mu_{u\in Q}\ R$ $\quad$ R ...... $\lambda_x R(x)$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_\ell \end{pmatrix} \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix}$$

(Nw)

$$f\left(\begin{pmatrix} x_1 \\ \vdots \\ x_\ell \end{pmatrix}\right) = f(x_1, \ldots x_\ell)$$

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_\ell \end{pmatrix} \right\rangle \varphi : \langle x_\ell \rangle \ldots \langle x_1 \rangle \varphi$$

$$[y_1, A_1(x)] \ldots \quad [y_k, A_k(x, y_1, \ldots, y_{k-1})]$$

beschr. als een $\mathcal{B}$ met ~~van~~ free var list x, bd var list y.

$(\mathcal{B}, x)$ is het LP — $\boxed{\text{hee NFLP}}$ uit NF-calculus

$(\mathcal{B}, x, y)$ is een abstraction strip

$\mathcal{B}_j$ $\mathcal{B}$ ~~heeft~~ zij kan men de hoogte van de bound var. lists aangeven.

Geef met ${}_i\mathcal{B}_{j+j}$ hetzelfde soort aan als daarvoor $\mathcal{B}_j$ met ~~x lengte i~~, x lengte i, y lengte j

$$\boxed{\begin{pmatrix} x_1 \\ \vdots \\ x_j \end{pmatrix} \oplus \begin{pmatrix} x_{j+1} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}$$

v : ~~vector~~ NF vector

(v, x) is een expr. vector.

free var list

${}_i\mathcal{V}$: vector met hoogte i

Folding $\left\langle \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \right\rangle \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix}$ dan is $\begin{pmatrix} \langle v_n \rangle \ldots \langle v_1 \rangle w_1 \\ \vdots \\ \langle v_n \rangle \ldots \langle v_1 \rangle w_k \end{pmatrix}$

liever met ${}_i\mathcal{B}_j$ noteren of $\mathcal{B}_{ij}$

Zij $\mathcal{B}$ een NFLP, v een NF-vector.

Nu $\mathcal{B} \cdot v$ een vector. $\boxed{\mathcal{B} \text{ is een } {}_i\mathcal{B}_j \qquad v \text{ een } {}_{i+j}\mathcal{V}}$

Als x, y var. lists zijn van goede lengte voor $\mathcal{B}$, dan is

$$(\mathcal{B})(x, y) = [y_1, A_1(x)] \ldots \quad [y_k, A_k(\ldots)]$$

$$(v, x) = \begin{pmatrix} v_1(x) \\ \vdots \\ v_k(x) \end{pmatrix}$$

dan is $\boxed{\mathcal{B} \cdot v}$ ~~bestaing~~ ~~xxx~~ de NF vector die by var. list $x \oplus y$ is

$$\begin{pmatrix} [y_1, A_1(x)] \ldots [y_k, A_k(x, y_1 \ldots y_{k-1})]\, v_1(x \oplus y) \\ \vdots \\ v_k(x \oplus y) \end{pmatrix}$$