Old geometry theorems, such as those of Pappos, Pascal and Desargues, turned alive by time

OLD GEOMETRY THEOREMS, SUCH AS THOSE OF PAPPOS, PASCAL AND DESARGUES, TURNED ALIVE BY TIME

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Summary

Old and even very old configurations may be turned to "life" by adjoining the dimension of time. Under particular circumstances, however, they are allowed to move only very slightly from their original positions. Others may move more abundantly if the conditions are right: they then turn into a mechanism.

It is extraordinary to find that the configurations of Pappos, Pascal and Desargues appear to have such direct links with the theory of mechanisms. The theorems are particularly handy when applied on linkages, containing bars each supposed to be moving as an elliptic motion. In this context, four-bar- and hexagonal-linkages are observed in which the successive turning-joints of the loop are alternately moving along two fixed axes. In case these axes are perpendicular, the configurations turn into over-constrained mechanisms, leading to interesting applications. In case they are not perpendicular, conditions have been found to turn the configurations into vacillating structures.

Keywords:
Perspectivity, Pole-Configurations, Vacillating Structures, Over-constrained-Linkages, Remote Controlled Parallel Motion.

1. Historical Introduction

After Apollonius of Perga, before Christ the last greek geometrist of some importance, further development of geometry almost died down at the start of our christian era.
It took almost 300 years to revive, initiated this time by Pappos of Alexandria, who specifically wanted to regain that ancient greek domain in order to bring it back to its old glory.

The nearness of the egyptian pyramids, built with their astonishing exactitude no doubt has cathalized his scope.
Also, the influence of the greek pharaos in Alexandria, like Ptolemeus I till VI with the greek philosophers in the background among whom were men like Plato, Socrates, Archimedes, Hyppocrates and not to forget Euclides, who founded geometry as a basic science, must not be underestimated.

The proposition, still named after Pappos, and which we further on intend to set "in motion" may be formulated like:

> *Opposite sides of a hexagon, the vertices of which alternately join two straight-lines, always intersect at three points of a straight-line, to be named the Pappos-Line* (figure 1).

Centuries later, namely in 1640 (A.D.), Blaise Pascal generalized this proposition, which may then be brought into the form:

> *Opposite sides of a hexagon, circumscribed by a conic section, intersect at three points of a straight-line, to be named the Pascal-Line* (figure 2).

Girard Desargues' Proposition, discovered in the same century, - somewhere in the period of (1596 - 1650) -, though geometrically quite different from the other two, proves to have a strong kinematic bond with the earlier named propositions of Pappos and Pascal. The said Desargues' Proposition reads as follows:

> *If corresponding sides of two triangles intersect at three points of a straight-line, the three lines connecting the corresponding vertices of these triangles, intersect at one point, and vice versa* (figure 3).

The straight-line, appearing in this proposition, is named the line of Desargues, whereas the coincident point listens to the name of Desargues' Point.
It is said that the two triangles, are perspective to one another, with Desargues' point named as the center of perspectivity.
There are at least three ways to prove Desargues' Configuration. The oldest way perhaps would be the one by which a triangle ABC, set in 3D-space, is projected onto the plane by two different points of the space. The two projected triangles of the plane then appear to be the perspective triangles of Desargues (figure 4).
Another proof would be by observing Desargues' Configuration as a three-dimensional figure, to wit as a tetra-hedron $P_{15}P_{14}P_{13}P_{12}$ intersected by a plane $P_{23}P_{34}P_{45}P_{52}$ (figure 5).
The third proof would be the one based on the Aronhold (1872) - Kennedy (1886) - Rule, which says that *at any point of time any three coplanar moving links have three (relative) velocity-poles joining a straight-line*. A kinematic chain having five links, all of them allowed to move in a constrained way, however small, have ten (relative) velocity-poles of which each time three have to join a straight-line, according to the Aronhold-Kennedy-Rule.
Then, totally ten of these Aronhold-Kennedy-lines appear, making up for Desargues' Configuration (figures 3 and 5).
All three proofs of Desargues' Configuration were based on an extra dimension. With the first two, a length-dimension was added, with the one based on Aronhold-Kennedy-Rule, however, the dimension of "time" was adjoined.

2. Kinematic Introduction

Clearly, the addition of the dimension "time" may revive static configurations, by which we enter the domain of *motion geometry* or kinematics. The addition of time may be done in different ways: with Desargues' Configuration we did it by observing all occurring points as "velocity-poles" giving rise to a complete pole configuration. The configuration is named complete then as it includes all ten velocity-poles that belong to a constrained motion concerning five movable links. Thus, a mechanism moving in a constrained way and having five links, possesses a pole configuration of Desargues. The same will be true for a mechanism that is allowed to move only instantaneously. In other words, a non-rigid (i.e. a wobbly or vacillating) structure with five links may similarly have such a configuration. Also the reverse is true: To a complete "pole configuration" of Desargues an (over-) constrained one-degree-of-freedom mechanism or a vacillating structure may be coordinated that is at least allowed to move instantaneously. How many of such mechanisms or structures exist, it is difficult to say. The permanency of the motion only depends on the permanency of Desargues' configuration: if such a configuration exists in any position, the motion is to be continued and is said to be permanent. This leads to a second more natural way to introduce "time": namely, by observing a configuration and then moving it in such a way that it possibly retains its crucial property. For instance, we may try to move Pappos' Configuration by moving the turning-joints of a hexagonal chain of bars along the two fixed lines, they were meant to join in the first place. Or, we may try to move them along a conic section to revive Pascal's Theorem.

As said before, Desargues' Configuration will simply be brought to life through the instantaneous (or sometimes permanent) association to singular overconstrained five-link mechanisms or to vacillating structures meeting the complete but unique pole configuration for five links.

3. Overconstrained Linkages

Generally, a mechanism having an odd number of links and an even number of sliding-pairs, can't possibly have the mobility one. However, the example shown in figure 6, containing a four-bar in which each bar is allowed to move according to a different elliptic motion, shows to be an exemption. Here, the turning-joints of the four-bar alternately move along two perpendicular axes. Leaving out bar four, an ordinary mechanism ensues, having mobility one. One then proves that $\overline{AD}^2 = \ell_3^2 - \ell_2^2 + \ell_1^2$, showing that $\overline{AD}$ is of constant length in such a mechanism. Hence, bar 4 may be reincorporated without hindering the motion. Thus, Desargues' pole configuration appears, since five links are involved in this case. The figure demonstrates the perspectivity of the triangles $BP_{35}C$ and $AP_{45}D$. This leads to Desargues' line 13-35-51 and Desargues' point of perspectivity P_{42}. Desargues' pole configuration is a permanent one in this case. So, in fact, the linkage mechanism is an overconstrained one, still having mobility one. The figures 6C, 6D, 6E, and 6F further demonstrate the extreme positions of this particular type of mechanism.

Naturally, the four mid-points of the elliptically moving rods may all be connected to the origin O of the cartesian system. The mechanism still won't loose its mobility. From this, it takes only a few steps to embarge on the mechanism demonstrated in figure 7. Here, the right hand side of the configuration was magnified, such that the initial four bar turned into a hexagonal chain with two bars moving along the y-axis. Exchanging frames and omitting the two axes then resulted into the mechanism, to be named the "Doubling of Sylvester's Plagiograph." In fact, two of Sylvester's Plagiographs are seen to be interconnected. It is easily seen then, that bar AB remains parallel to the fixed bar. Clearly, a two degree of motion mechanism is at hand with or without bar AB. In fact, bar AB is a redundant one, and has to be disregarded in case it is necessary to apply Grübler's Criterium. Combining the properties of Sylvester's Plagiograph and those of parallel motion then results into saying that *any input-curve enforced on the triple-turning joint O, will be magnified (by a factor 2 in this case) to the curve traced by any point of the plane attached to AB.* In practice, this may be a very handy mechanism indeed if a remote control is required for the parallel motion of an entire plane.

4. Vacillating Structures

An example of this type is demonstrated in figure 8 in which the turning-joints of all six hexagons, to be observed in the chain, join the circumference of a conic section. This results into six lines of Pascal, similarly representing the six Aronhold-Kennedy-lines, not running along a bar of the configuration. Thus, the structure, generally being a statically determined one, here appears to have a unique pole configuration and therefore (first order), instantaneous mobility.

Figure 9 demonstrates another example of a vacillating structure. In here, the two pairs of opposite turning-joints of a cyclic four-bar are allowed to move along two non-perpendicular axes. Indeed, the structure may be associated with Desargues' Configuration as there are just five links; the four bars and the frame. In fact, this case shows to be a particular case of the foregoing by taking two of the hexagons' opposite double-joints at infinity. Naturally, the conic section, joining the six double-joints, then turns into a hyperbola. It remains to prove that a hyperbola may be drawn through the four vertices of the cyclic four-bar as well as through the two points at infinity of the normals at the diagonals of the four-bar (figure 10). Thus, relabeling these points 1, 2, 3, 4, 5^*, and 6^*, it then suffices to prove, that opposite sides of the hexagon $1-2-3-4-5^*-6^*$ intersect at a line of Pascal; or symbolically: the three points $(5^*-6^*) \times (2-3) = (2-3)^*$, $(5^*-4) \times (1-2) = H_{12}$, and $(4-3) \times (6^*-1) = H_{34}$ should prove to join a straight-line (figure 10).

Proof: Thus, assuming that □1234 is cyclic, it follows that $\angle 124 = \angle 134$. Further, $\angle H_{12}24 = 90° = \angle H_{34}13$. Hence, $\triangle H_{12}24 \sim \triangle H_{34}31$, deducing that $\angle 4H_{34}1 = \angle 4H_{12}1$, whence $□4H_{34}H_{12}1$ has to be cyclic, yielding that $\angle 4H_{34}H_{12} = \angle 412 = 180° - \angle 234$, the latter on account of □1234 being cyclic. We conclude, that the line $H_{12}H_{34}$ indeed runs parallel to the side 2-3 of the quadrilateral 1-2-3-4.

Thus, according to the inverse of Pascal's Theorem, the six vertices of the hexagon, $(1-2-3-4-5^*-6^*)$, join a conic section, in this case a hyperbola.

The six vertices may be connected differently to form other hexagon-linkages, such as the hexagon-linkages $1-2-5^*-4-3-6^*$ and $1-4-5^*-2-3-6^*$. Applying Pascal's Theorem in these cases, then leads to the respective lines of Pascal $P_{20}-P_{04}-P_{42}$ and $P_{13}-P_{30}-P_{01}$ by which

$$P_{20} = (2-5^*) \times (3-6^*) \qquad P_{13} = (2-3) \times (1-4)$$
$$P_{04} = (5^*-4) \times (6^*-1) \quad \text{and} \quad P_{30} = (3-6^*) \times (4-5^*)$$
$$P_{42} = (4-3) \times (1-2) \qquad P_{01} = (6^*-1) \times (5^*-2)$$

Whence these two Pascal-lines are, in fact, Aronhold-Kennedy-lines, forming a part of Desargues' pole-configuration; as, for instance the triangles $2P_{20}3$ and $1P_{04}4$ are perspective since $P_{13}-P_{30}-P_{01}$ indeed forms a D-line. Hence, a unique (Desargues') Pole Configuration is at hand, with the result that the five-link structure observed, allows (first order) instantaneous mobility. Normally, the chain would have represented a statically determinate structure. Here, however, where the four bar is made to be cyclic, the structure becomes vacillating since (first order) instantaneous motion is admitted.

2^{nd} order instantaneous mobility may be attained if additionally:

$$1 = \frac{d \angle (1,4)}{d \angle (2,3)} = \frac{\omega_{14}}{\omega_{23}} = \frac{\omega_{14}}{\omega_{34}} \cdot \frac{\omega_{43}}{\omega_{23}} =$$

$$= \frac{P_{13}P_{34}}{P_{13}P_{14}} \cdot \frac{P_{42}P_{23}}{P_{42}P_{43}} \quad \text{or} \quad \frac{PC}{PD} = -\frac{QB}{QC}$$

Two cyclic four-bars, meeting this ratio-equality, are simultaneously demonstrated in figure 11. Each of them is obtained by trial and error, starting from the circumcircle and the points C, D, and P. Point B (or B') is then moved along this circle until the ratio-equality was met. Apparently two solutions arise. Both four-bars sustain 1^{st} and 2^{nd} order instantaneous mobility, meaning that their turning-joints may move along their sliding axes during three infinitesimally near positions of the two vacillating structures.

(The cyclic four-bar, for which the sliding axes $B'D$ and $A'C$ are perpendicular, may be found by varying O along a circle having DC for diameter. However, the solution for which A B then joins P, doesn't generally meet the mentioned ratio-equality. The resulting mechanism though, has an infinite order of mobility, not only two).

5. Pappos' Configuration brought to life

In order to mechanize Pappos' Theorem, it is natural to turn the hexagon, appearing in this theorem, into a hexagon-linkage with turning-joints at her vertices. The vertices should then possibly remain on the two (then fixed) lines, each meeting either three even-, or otherwise three odd-numbered vertices of the hexagon (figure 12). Now, the chain consists of seven links. Again one condition is necessary to obtain instantaneous motion. It is possible to induce such a tiny motion by meeting Desargues' Generalized Configuration [12] involving seven links. This is done, by confining ourselves to the six trivial poles (such as 12, 23, 34, 45, 56, and 61) and to the velocity-poles of the cardan-motions of, for instance, the odd-numbered bars. We then obtain the necessary equations:

35 = (34-45) x (30-05), 31 = (32-21) x (30-01), and
51 = (56-61) x (50-01)

in which (34-45) represents bar four, (32-21) bar two, and (56-61) bar six. Hence, $\triangle(10,30,50)$ is perspective to $\triangle(2,4,6)$. As a consequence, Desargues' Theorem, applied on these triangles, leads us to the fact that the three lines, to wit 10 - (2x6), 30 - (2x4), and 50 - (4x6) all have to join a singular point, Desargues' Point. If this happens, then the structure will be vacillating, meaning that at least only first order instantaneous motion is allowed.

A special case appears, when Pappos' line turns to infinity (figure 13). If that happens, each pair of opposite sides in the hexagon-linkage will represent a set of parallel bars. As a consequence, the triangles (2,4,6) and (5,1,3) are similar and in this case also perspective. As the triangles (10,30,50) and (20,40,60) are perspective anyway, like they were in the general case, one may prove that also the triangles (2,4,6) and (10,30,50) are perspective, as required for instantaneous motion. Clearly, a vacillating structure ensues when the line of Pappos vanishes towards infinity. The configuration shows some resemblance with the Neurenberg Scissors, that appear in the work of Leonardo Da Vinci (1452-1519).

The motion becomes permanent when the angle α between the fixed lines, along which the vertices slide, turns to 90° (figure 14). This is easily explained if we adjoin the bar A'B, splitting the hexagon into two four-bars, to wit into A'CC'B and into BB'AA'. Naturally, other bars, connecting opposite vertices of the hexagon, such as the bars B'C and AC' are also adjoinable. Like the adjoinment of A'B, they do not hinder the overall motion. In this case, the straight-line appearing in Pappos' Theorem, coincides with the Aronhold-Kennedy-line of the bars A'B, B'C and AC':

$1 \times 4 = P_{A'B,AC'}$; $2 \times 5 = P_{A'B,CB'}$; and $3 \times 6 = P_{AC',CB'}$.

Thus, opposite sides of our hexagon, now intersect at the velocity-poles between the three adjoinable bars.

Naturally, we may continue this way, by observing an octagonal linkage of which the odd-numbered vertices move along an X-axis and the even-numbered ones along the Y-axis of an orthogonal coordinate system.
As long as orthogonality is preserved, each bar is allowed to produce the cardan-motion. The bars may obstruct one another but otherwise a permanent motion is to be allowed in this case.

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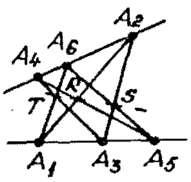
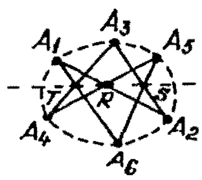


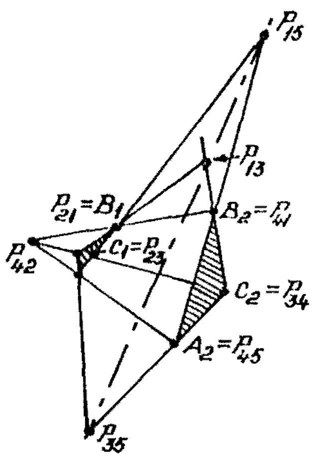
Pappos' Theorem (300 A.D.)

Fig. 1



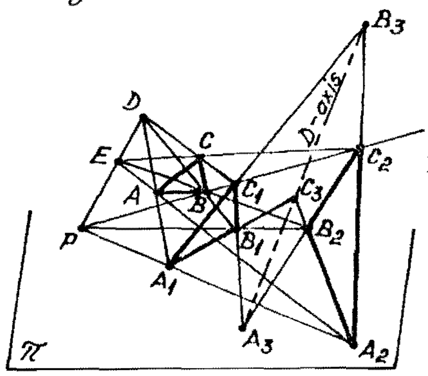
Blaise Pascal's Theorem (1640)

Fig. 2



Desargues' Theorem (1596-1650)

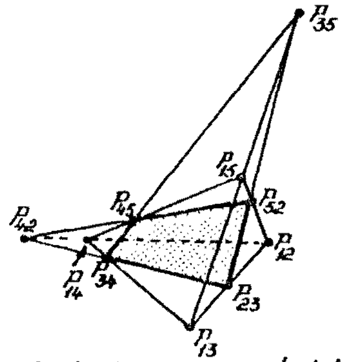
Fig. 3



ΔABC, set in space, projected onto π from space-points D, E, giving Desargues' Configuration in plane π.

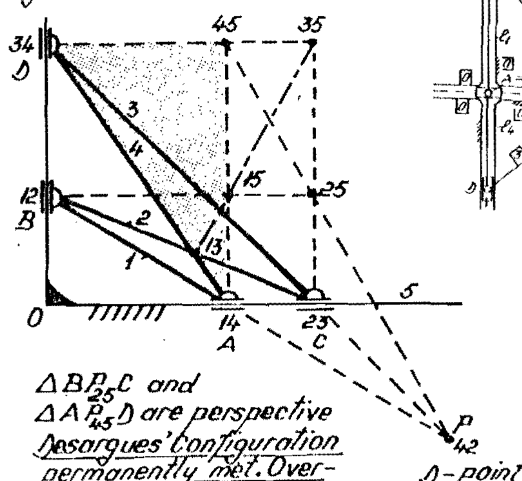
3D - Proof of Desargues' Configuration.

Fig. 4



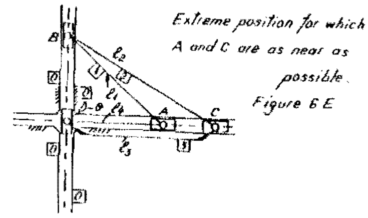
A tetrahedron intersected by a plane, represents Desargues' planar Configuration.

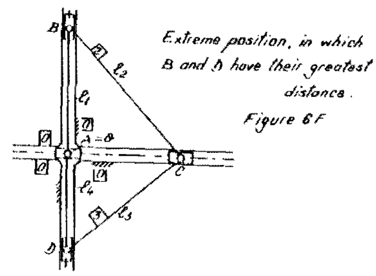
Fig. 5

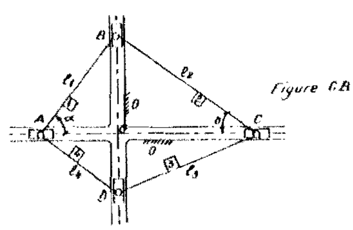


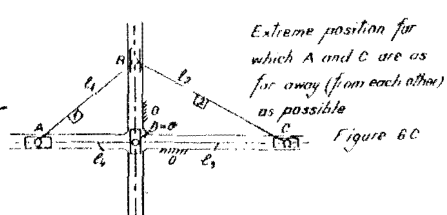
$\Delta B P_{25} C$ and $\Delta A P_{45} D$ are perspective Desargues' Configuration permanently met. Over-constrained Linkage Mechanism.

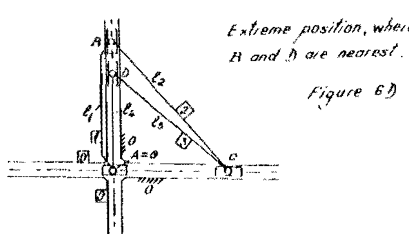
Fig. 6A

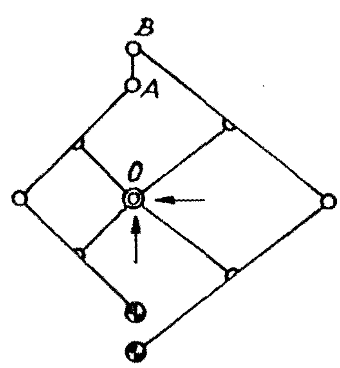








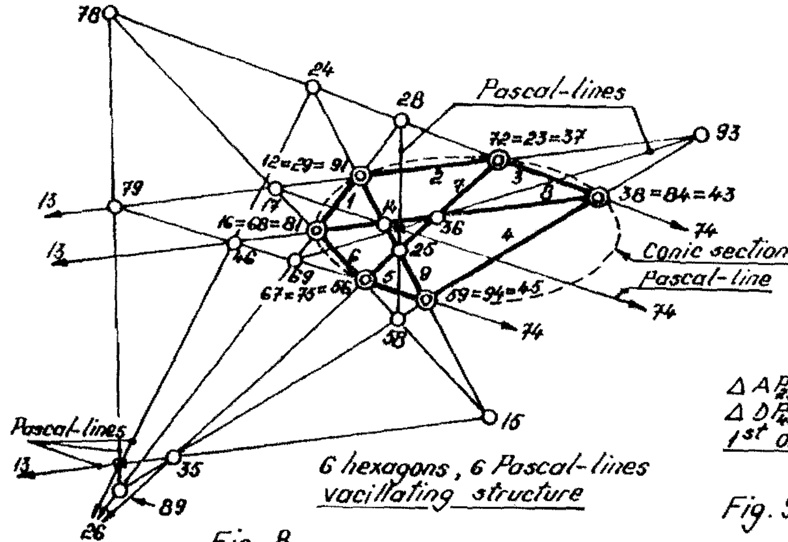




Doubling of Sylvester's Plagiograph.

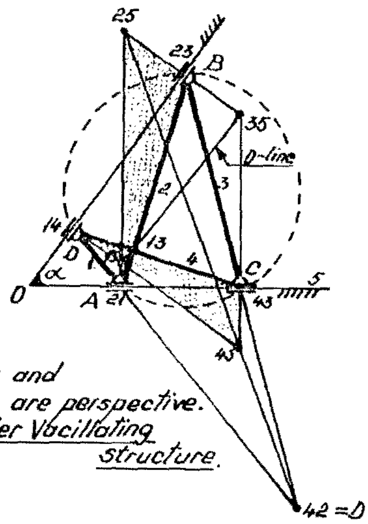
An overconstrained two degrees of freedom linkage mechanism, producing parallel motion for bar AB.

Fig. 7



6 hexagons, 6 Pascal-lines vacillating structure

Fig. 8



$\Delta AP_{25}B$ and $\Delta DP_{45}C$ are perspective. 1st order Vacillating structure.

Fig. 9

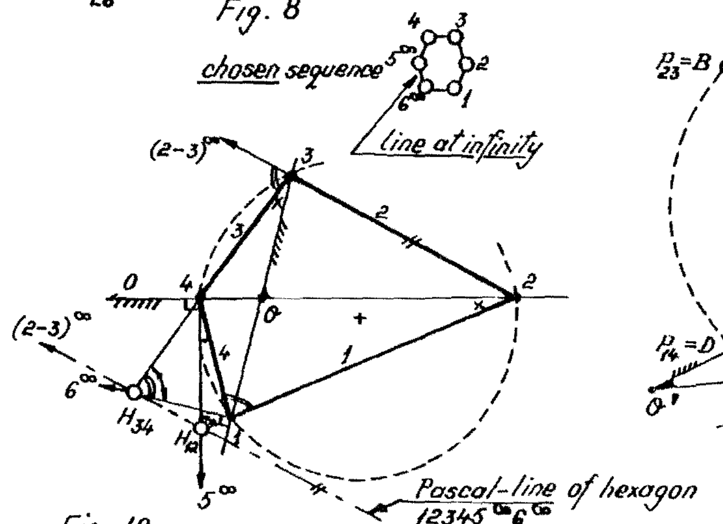
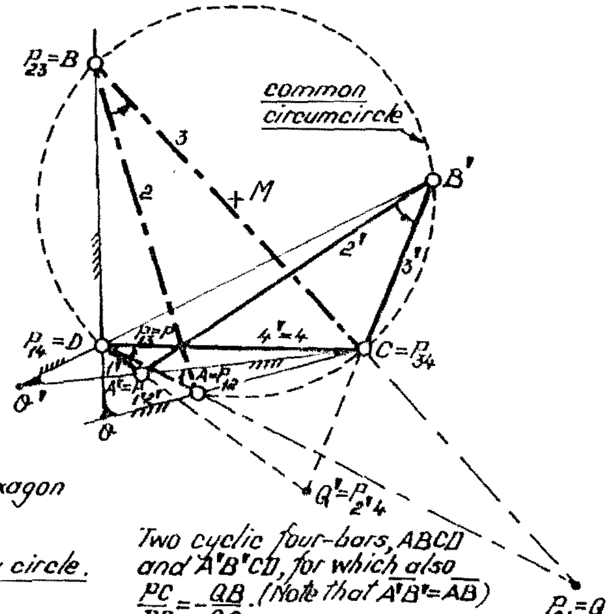
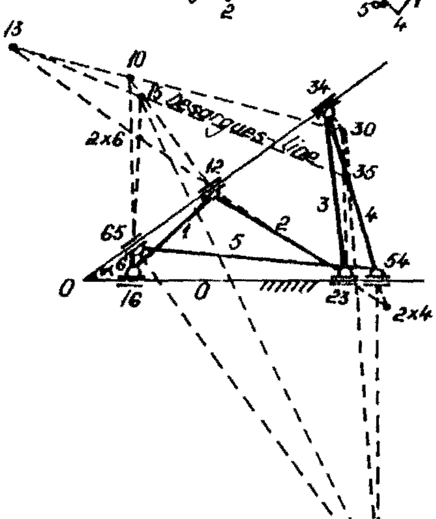


Fig. 10

Vacillating structure if 4 joints join a circle.



Two cyclic four-bars, ABCD and A'B'CD, for which also $\frac{PC}{PD} = -\frac{QB}{QC}$. (Note that $\overline{A'B'} = \overline{AB}$)
Only 2nd order instantaneous mobility when the turning-joints of one of these four-bars try to move along their sliding-axes.
Two separate vacillating structures, drawn in one figure 11.



Vacillating Structure

Fig. 12

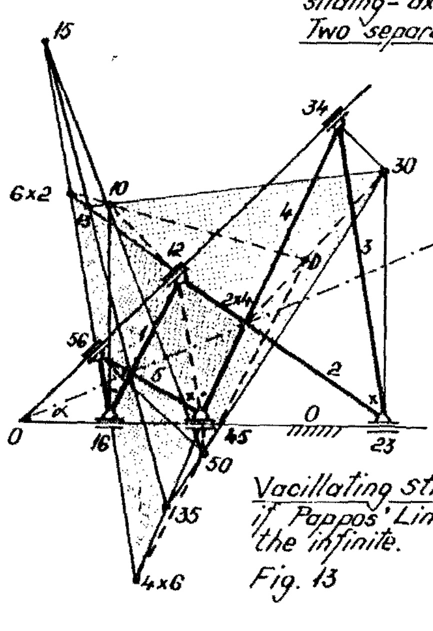

Vacillating structure if Pappos' Line joins the infinite.

Fig. 13

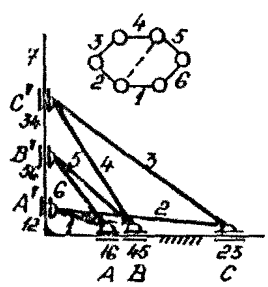

Overconstrained linkage mechanism.

Fig. 14