

On the connection between a symmetry condition and several nice properties of the spaces $S_{\{\Phi(A)\}}$ en $T_{\{\Phi(A)\}}$

Citation for published version (APA):

Elst, ter, A. F. M. (1987). *On the connection between a symmetry condition and several nice properties of the spaces $S_{\{\Phi(A)\}}$ en $T_{\{\Phi(A)\}}$* . (Eindhoven University of Technology : Dept of Mathematics : memorandum; Vol. 8701). Technische Hogeschool Eindhoven.

Document status and date:

Published: 01/01/1987

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

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Department of Mathematics and Computing Science

Memorandum 1987-01
September 1987

ON THE CONNECTION BETWEEN
A SYMMETRY CONDITION AND SEVERAL NICE
PROPERTIES OF THE SPACES $S_{\Phi(A)}$ AND $T_{\Phi(A)}$

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Abstract.

In this paper it is proved that several topological properties of the spaces $S_{\Phi(A)}$ and $T_{\Phi(A)}$ are equivalent with a symmetry condition on the directed set Φ .

0. Introduction

This paper is based on a paper [EGK] of S.J.L. van Eijndhoven, J. de Graaf and P. Kruszyński in which the spaces $S_{\Phi(A)}$ and $T_{\Phi(A)}$ are introduced. In Chapter IV of that paper it is proved that a symmetry condition implies some topological properties of the spaces $S_{\Phi(A)}$ and $T_{\Phi(A)}$. In the underlying paper we show that a weaker symmetry condition is equivalent with all those topological properties and a lot more.

For the terminology of locally convex topological vector spaces we refer to [Wil].

1. Notations and some known theorems

Let X be a separable Hilbert space, $n \in \mathbb{N}$ and let A_1, \dots, A_n denote n self-adjoint operators whose corresponding spectral projections mutually commute. There exists a unique spectral measure E on the set $B(\mathbb{R}^n)$ of Borel sets of \mathbb{R}^n so that for every $k \in \{1, \dots, n\}$ the map $\Delta \mapsto E(\mathbb{R}^{k-1} \times \Delta \times \mathbb{R}^{n-k})$, Δ a Borel set in \mathbb{R} , equals the spectral measure of A_k . For every Borel measurable function f on \mathbb{R}^n , there can be defined the self-adjoint operator

$$f(A) = \int_{\mathbb{R}^n} f(\lambda) dE_\lambda$$

in a natural manner. (See [EGK], page 280.)

For every $\Delta \in B(\mathbb{R}^n)$ let χ_Δ be the characteristic function of Δ . For all $m \in \mathbb{Z}^n$ let

$$Q_m := \{ \lambda \in \mathbb{R}^n : \forall_{k \in \{1, \dots, n\}} [\lambda_k \in [m_k - 1, m_k]] \}.$$

Let $B_b(\mathbb{R}^n)$ be the set of all bounded Borel sets of \mathbb{R}^n and let G^+ be the set of all maps F from $B_b(\mathbb{R}^n)$ into X with the property

$$F(\Delta_1 \cap \Delta_2) = E(\Delta_2)F(\Delta_1) \quad (\Delta_1, \Delta_2 \in B_b(\mathbb{R}^n)).$$

Define $\text{emb} : X \rightarrow G^+$ by $[\text{emb } x](\Delta) := E(\Delta)x$, $x \in X$, $\Delta \in B_b(\mathbb{R}^n)$. The map emb is injective. So the Hilbert space X is embedded in G^+ .

Let ϕ be a Borel measurable function on \mathbb{R}^n which is bounded on bounded Borel sets. Denote $\phi := \{ \lambda \in \mathbb{R}^n : \phi(\lambda) \neq 0 \}$. (Remark: ϕ need not be closed.) Let $x \in X$. Define $\phi(A) \cdot x \in G^+$ by $[\phi(A) \cdot x](\Delta) = \phi(A)E(\Delta)x$, $\Delta \in B_b(\mathbb{R}^n)$. For every $F \in \phi(A) \cdot X$ there exists a unique $x \in E(\phi)(X)$ such that $F = \phi(A) \cdot x$. Hence an inner product can be defined on $\phi(A) \cdot X$ such that the map $\phi(A) \cdot : E(\phi)(X) \rightarrow \phi(A) \cdot X$ is a unitary map between two Hilbert spaces. The set $B_b(\mathbb{R}^n)$ is a directed set under inclusion, so we can define the following subspace of G^+ :

$$D_\phi := \{ F \in G^+ : \Delta \mapsto \phi(A)F(\Delta), \Delta \in B_b(\mathbb{R}^n), \text{ is a Cauchy net in } X \}.$$

For every $F \in D_\phi$ define $\phi(A) * F := \lim_{\Delta} \phi(A)F(\Delta) \in X$. Corresponding to the same function ϕ we can also define an operator $\phi(A) : G^+ \rightarrow G^+$ by

$$[\phi(A)F](\Delta) := \phi(A)F(\Delta) \quad (F \in G^+, \Delta \in B_b(\mathbb{R}^n)).$$

Let Φ be a set of Borel functions on \mathbb{R}^n . Suppose the set Φ satisfies the next axiom.

AXIOM 1.

Φ is a directed set of real valued Borel functions on \mathbb{R}^n and every element of Φ is bounded on bounded Borel sets. The set Φ has the following properties:

- AI. Each $\phi \in \Phi$ is nonnegative and the function $\lambda \mapsto \phi(\lambda)^{-1}$, $\lambda \in \phi$ is bounded on bounded Borel sets.
- AII. The sets $\phi, \phi \in \Phi$, cover the whole \mathbb{R}^n , i.e. $\mathbb{R}^n = \bigcup_{\phi \in \Phi} \phi$.
- AIII. $\forall_{\phi \in \Phi} \exists_{\psi \in \Phi} \exists_{c > 0} \forall_{m \in \mathbb{Z}^n} [(1 + |m|) \sup_{\lambda \in Q_m} \phi(\lambda) \leq c \inf_{\lambda \in Q_m} \psi(\lambda)]$.

The set Φ induces a new set Φ^+ .

DEFINITION 2.

Let Φ be a set which satisfies Axiom 1. Then Φ^+ will denote the set of all Borel functions f on \mathbb{R}^n so that

- i) f is a nonnegative Borel function and the map $\lambda \mapsto f(\lambda)^{-1}$, $\lambda \in \underline{f}$ is bounded on bounded Borel sets.
- ii) $\forall_{\phi \in \Phi} [\sup_{\lambda \in \mathbb{R}^n} f(\lambda) \phi(\lambda) < \infty]$.

LEMMA 3.

The set Φ^+ satisfies Axiom 1.

Proof. See [EGK], Lemma 1.5. □

Now we can define two subspaces $S_{\Phi(A)}$ and $T_{\Phi(A)}$ of G^+ . Let Φ be a set which satisfies Axiom 1.

DEFINITION 4.

Let $S_{\Phi(A)} := \bigcup_{\phi \in \Phi} \phi(A) \cdot X$.

The topology σ_{ind} for $S_{\Phi(A)}$ is the inductive limit topology generated by the Hilbert spaces $\phi(A) \cdot X$, $\phi \in \Phi$.

DEFINITION 5.

Let $f \in \Phi^+$. Then $S_{\Phi(A)} \subset D_f$. The seminorm s_f on $S_{\Phi(A)}$ is defined by $s_f(w) := \|f(A) * w\|$, $w \in S_{\Phi(A)}$.

THEOREM 6.

The locally convex topology for $S_{\Phi(A)}$ generated by the seminorms s_f , $f \in \Phi^+$ is equivalent to the topology σ_{ind} for $S_{\Phi(A)}$.

Proof. See [EGK], Theorem 1.8. □

Remark: It follows that the topology σ_{ind} is Hausdorff.

DEFINITION 7.

Let $T_{\Phi(A)} := \{F \in G^+ : \forall_{\phi \in \Phi} [\phi(A)F \in \text{emb}(X)]\}$.

The topology τ_{proj} is the locally convex topology generated by the seminorms t_ϕ , $\phi \in \Phi$, defined by $t_\phi(F) := \|\text{emb}^{-1}(\phi(A)F)\|$, ($F \in T_{\Phi(A)}$, $\phi \in \Phi$).

There exists a characterisation of bounded sets in $T_{\Phi(A)}$.

THEOREM 8.

Let $B \subset T_{\Phi(A)}$ be a set. Then B is bounded in $(T_{\Phi(A)}, \tau_{\text{proj}})$ iff there exist $f \in \Phi^+$ and a bounded set $B_0 \subset X$ so that $B = f(A) \cdot B_0$.

Proof. See [EGK], Theorem 2.4.II. □

It follows that $T_{\Phi(A)} = \bigcup_{f \in \Phi^+} f(A) \cdot X$.

DEFINITION 9.

The topology τ_{ind} for $T_{\Phi(A)}$ is the inductive limit topology generated by the Hilbert spaces $f(A) \cdot X$, $f \in \Phi^+$.

Further a duality between the spaces $S_{\Phi(A)}$ and $T_{\Phi(A)}$ can be introduced.

DEFINITION 10.

Define $\langle , \rangle : S_{\Phi(A)} \times T_{\Phi(A)} \rightarrow \mathbb{C}$

$\langle \phi(A) \cdot x, F \rangle = (x, \text{emb}^{-1}(\phi(A)F))$ ($\phi \in \Phi$, $x \in E(\phi)(X)$, $F \in T_{\Phi(A)}$).

(See [EGK], page 288.)

Note: For all $f \in \Phi^+$, $w \in S_{\Phi(A)}$ and $x \in X$ holds: $\langle w, f(A) \cdot x \rangle = (f(A) * w, x)$.

THEOREM 11.

$\langle S_{\Phi(A)}, T_{\Phi(A)} \rangle$ is a dual pair and the topology σ_{ind} resp. τ_{proj} is compatible with the dual pair $\langle S_{\Phi(A)}, T_{\Phi(A)} \rangle$ resp. $\langle T_{\Phi(A)}, S_{\Phi(A)} \rangle$.

Proof. See [EGK], Theorem 3.1. □

2. The weak symmetry condition

Let Φ be a set which satisfies Axiom 1. In Chapter IV of [EGK] the authors require the following strong symmetry condition on the set Φ .

$$\text{AIV.} \quad \forall_{\zeta \in \Phi^{++}} \exists_{\phi \in \Phi} \exists_{c > 0} [\zeta \leq c \phi].$$

With this condition they prove several nice properties of the topological spaces $(S_{\Phi(A)}, \sigma_{\text{ind}})$ and $(T_{\Phi(A)}, \tau_{\text{proj}})$. They note that the operator $\zeta(A) \phi(A)^{-1} \chi_{\phi}(A)$ extends to a bounded operator on X . So the set Φ satisfies the following weak symmetry condition.

$$\text{AIV'}. \quad \forall_{\zeta \in \Phi^{++}} \exists_{\phi \in \Phi} \exists_{c > 0} [\chi_{\{\lambda \in \mathbb{R}^n : \zeta(\lambda) > c \phi(\lambda)\}}(A) = 0].$$

A careful reading of their proofs shows that they use only condition AIV' to get the nice properties. The next theorem shows that the spaces $S_{\Phi(A)}$ and $T_{\Phi(A)}$ cannot have those nice properties without condition AIV'.

THEOREM 12.

Let Φ be a set of Borel functions on \mathbb{R}^n which satisfies Axiom 1. The following conditions are equivalent.

- I. Φ has property AIV'.
- II. $(T_{\Phi(A)}, \tau_{\text{proj}}) = (S_{\Phi^+(A)}, \sigma_{\text{ind}})$ as topological vector spaces.

- III. $(T_{\Phi(A)}, \tau_{\text{proj}}) = (T_{\Phi(A)}, \tau_{\text{ind}})$ as topological vector spaces.
- IV. $(T_{\Phi(A)}, \tau_{\text{proj}})$ is bornological.
- V. $(T_{\Phi(A)}, \tau_{\text{proj}})$ is barrelled.
- VI. $(T_{\Phi(A)}, \tau_{\text{proj}})$ is quasibarrelled.
- VII. $S_{\Phi(A)}$ is complete.
- VIII. $S_{\Phi(A)}$ is sequentially complete.
- IX. $S_{\Phi(A)} = \bigcap_{f \in \Phi^+} D_f$.
- X. $S_{\Phi(A)} = S_{\Phi^+(A)}$ as sets.
- XI. For every bounded set $B \subset S_{\Phi(A)}$ there exist $\phi \in \Phi$ and a bounded set $B_0 \subset X$ so that $\phi(A) \cdot |_{B_0} : B_0 \rightarrow B$ is a homeomorphism.

Proof.

I \Rightarrow II. Theorem 4.2.I of [EGK].

II \Rightarrow III. Always $(S_{\Phi^+(A)}, \sigma_{\text{ind}}) = (T_{\Phi(A)}, \tau_{\text{ind}})$ holds.

III \Rightarrow I. Let $\zeta \in \Phi^+$. Define $W : T_{\Phi(A)} \rightarrow X$ by $W(f(A) \cdot x) := (\zeta f)(A)x$, $f \in \Phi^+$, $x \in X$. Let $f \in \Phi^+$. Then $\|W(f(A) \cdot x)\| \leq \|(\zeta f)(A)\| \|x\| = \|(\zeta f)(A)\| \|f(A) \cdot x\|_{f(A)X}$ for all $x \in E(f)(X)$. By definition of τ_{ind} , the map W is continuous from $(T_{\Phi(A)}, \tau_{\text{ind}})$ into X . By assumption, the map W is continuous from $(T_{\Phi(A)}, \tau_{\text{proj}})$ into X , so there exist $\phi \in \Phi$ and $c > 0$ such that $\|W(F)\| \leq t_\phi(F)$ for all $F \in T_{\Phi(A)}$. In particular,
 $\|(\zeta \chi_{Q_m})(A)x\| = \|W(\chi_{Q_m}(A) \cdot x)\| \leq c \|(\phi \chi_{Q_m})(A)x\|$
 for all $x \in X$ and $m \in \mathbb{Z}^*$. So $\chi_{\{\lambda \in \mathbb{R}^n : \zeta(\lambda) > c \phi(\lambda)\}}(A) = 0$.

III \Rightarrow IV \Rightarrow VI and III \Rightarrow V \Rightarrow VI are trivial.

VI \Rightarrow III. Always $\tau_{\text{proj}} \subset \tau_{\text{ind}}$. Let $\Omega \subset T_{\Phi(A)}$ be a τ_{ind} -neighbourhood of 0. Because τ_{ind} is regular, there exist absolutely convex τ_{ind} -open $\Omega_1 \subset T_{\Phi(A)}$ so that $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega$. Assertion: Ω_1 is a bornivore in $(T_{\Phi(A)}, \tau_{\text{proj}})$. Let $B \subset T_{\Phi(A)}$ be a τ_{proj} -bounded set. By Theorem 8 there exist $f \in \Phi^+$ and a bounded set $B_0 \subset X$ so that $B_0 = f(A) \cdot B_0$. Let $M > 0$ be so that $\|x\| \leq M$ for all $x \in B_0$. Since Ω_1 is τ_{ind} -open, there exists $\varepsilon > 0$ so that for all $x \in X$, $\|x\| < \varepsilon$ holds $f(A) \cdot x \in \Omega_1$. Then for all $t \in \mathcal{C}$, $|t| < \varepsilon M^{-1}$ we get $tB \subset \Omega_1$. This proves the assertion. Hence $\overline{\Omega_1}$ is a bornivore barrel and by assumption a τ_{proj} -neighbourhood of 0. So $\tau_{\text{ind}} \subset \tau_{\text{proj}}$.

I \Rightarrow VII. See [EGK], Corollary 4.3.III.

VII \Rightarrow VIII. Trivial.

VIII \Rightarrow IX. Always $S_{\Phi(A)} \subset \bigcap_{f \in \Phi^+} D_f$. Let $F \in \bigcap_{f \in \Phi^+} D_f$. For $p \in \mathbb{N}$ let $\Delta_p := \{\lambda \in \mathbb{R}^n : |\lambda| \leq p\}$, $x_p := F(\Delta_p)$ and $F_p := \chi_{\Delta_p}(A) \cdot x_p$. Then $F_p \in S_{\Phi(A)}$. Assertion: $(F_p)_{p \in \mathbb{N}}$ is a Cauchy sequence in $S_{\Phi(A)}$. Let $f \in \Phi^+$ and $\varepsilon > 0$. There exists $\Delta_0 \in B_b(\mathbb{R}^n)$ so that for all $\Delta, \Delta' \in B_b(\mathbb{R}^n)$, $\Delta \supset \Delta_0$, and $\Delta' \supset \Delta_0$ holds

$\|f(A)F(\Delta) - f(A)F(\Delta')\| \leq \varepsilon$. Let $p_0 \in \mathbb{N}$ be so that $\Delta_{p_0} \supset \Delta_0$. Let $p \in \mathbb{N}$, $p \geq p_0$. For all $\Delta \in B_b(\mathbb{R}^n)$, $\Delta \supset \Delta_0$ we obtain $\|f(A)F(\Delta) - f(A)F(\Delta_p)\| \leq \varepsilon$, so $\|f(A) * F - f(A) * F_p\| = \|f(A) * F - f(A)F(\Delta_p)\| \leq \varepsilon$. So $p \mapsto f(A) * F_p$ is a Cauchy sequence in X with limit $f(A) * F$ and the assertion is proved (Theorem 6). Let $F_0 \in S_{\Phi(A)}$ be the limit of the sequence $(F_p)_{p \in \mathbb{N}}$. Let $\Delta \in B_b(\mathbb{R}^n)$. Then $\chi_\Delta \in \Phi^+$ and $F_0(\Delta) = \chi_\Delta(A) * F_0 = \lim_{p \rightarrow \infty} \chi_\Delta(A) * F_p = \chi_\Delta(A) * F = F(\Delta)$. So $F = F_0 \in S_{\Phi(A)}$.

IX \Rightarrow IV. Let $W : T_{\Phi(A)} \rightarrow \mathcal{C}$ be a linear map which is bounded on τ_{proj} -bounded sets. For all $f \in \Phi^+$ the map $x \mapsto W(f(A) \cdot x)$ from X into \mathcal{C} is bounded on bounded sets by Theorem 8, so this map is continuous. In particular: for every $\Delta \in B_b(\mathbb{R}^n)$ there exists unique $F(\Delta) \in X$ so that for all $x \in X$ holds $(x, F(\Delta)) = W(\chi_\Delta(A) \cdot x)$. Then $F \in G^+$. Assertion: $F \in \bigcap_{f \in \Phi^+} D_f$. Let $f \in \Phi^+$. There exists $y \in X$ so that for all $x \in X$ holds $W(f(A) \cdot x) = (x, y)$. Let $x \in X$. Then $\lim_{\Delta} (x, f(A)F(\Delta)) = \lim_{\Delta} (f(A)\chi_\Delta(A)x, F(\Delta)) = \lim_{\Delta} W(\chi_\Delta(A) \cdot f(A)\chi_\Delta(A)x) = \lim_{\Delta} W(f(A) \cdot \chi_\Delta(A)x) = \lim_{\Delta} (\chi_\Delta(A)x, y) = (x, y)$. So weak $\lim_{\Delta} f(A)F(\Delta) = y$. But also $\lim_{\Delta} \|f(A)F(\Delta)\| = \lim_{\Delta} \sup_{\|x\| \leq 1} |W(f(A) \cdot \chi_\Delta(A)x)| = \sup_{\|x\| \leq 1} |W(f(A) \cdot x)| = \|y\|$. So strong $\lim_{\Delta} f(A)F(\Delta) = y$. Hence $F \in D_f$ and $y = f(A) * F$. So $F \in \bigcap_{f \in \Phi^+} D_f = S_{\Phi(A)}$. Let $H \in T_{\Phi(A)}$. There are $f \in \Phi^+$ and $x \in X$ so that $H = f(A) \cdot x$. Then $W(H) = (x, f(A) * F) = \langle \overline{F}, f(A) \cdot x \rangle = \langle \overline{F}, H \rangle$. By Theorem 11 it follows that W is continuous.

IX \Leftrightarrow X. By equivalence of I and IX: $S_{\Phi(A)} \subset S_{\Phi^{++}(A)} = \bigcap_{f \in \Phi^{++}} D_f = \bigcap_{f \in \Phi^+} D_f$.

I \Rightarrow XI. See [EGK], Corollary 4.3.IV.

XI \Rightarrow VIII. Let w_1, w_2, \dots be a Cauchy sequence in $S_{\Phi(A)}$. Then $\{w_n : n \in \mathbb{N}\}$ is bounded, so there exist $\phi \in \Phi$ and a Cauchy sequence x_1, x_2, \dots in X so that $w_n = \phi(A) \cdot x_n$, $n \in \mathbb{N}$. Let $x := \lim_{n \rightarrow \infty} x_n$. Then $\lim_{n \rightarrow \infty} w_n = \phi(A) \cdot x$ in $S_{\Phi(A)}$. \square

Remark: It is trivial by now that property AIV' is equivalent with $(T_{\Phi(A)}, \tau_{\text{proj}})$ is reflexive and also with $(S_{\Phi(A)}, \sigma_{\text{ind}}) = (T_{\Phi^+(A)}, \tau_{\text{proj}})$ as topological vector spaces. If $(T_{\Phi(A)}, \tau_{\text{proj}})$ happens to be metrizable, then property AIV' holds.

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