On the connection between a symmetry condition and several nice properties of the spaces $S_{\Phi(A)}$ en $T_{\Phi(A)}$

EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computing Science

Memorandum 1987-01
September 1987

ON THE CONNECTION BETWEEN A SYMMETRY CONDITION AND SEVERAL NICE PROPERTIES OF THE SPACES $S_{\Phi(A)}$ AND $T_{\Phi(A)}$

by

A.F.M. ter Elst

Eindhoven University of Technology
Department of Mathematics and Computing Science
P.O. Box 513
5600 MB Eindhoven
The Netherlands

ON THE CONNECTION BETWEEN A SYMMETRY CONDITION AND SEVERAL NICE PROPERTIES OF THE SPACES $S_{\Phi(A)}$ AND $T_{\Phi(A)}$

A.F.M. ter Elst

Abstract.

In this paper it is proved that several topological properties of the spaces $S_{\Phi(A)}$ and $T_{\Phi(A)}$ are equivalent with a symmetry condition on the directed set Φ.

0. Introduction

This paper is based on a paper [EGK] of S.J.L. van Eijndhoven, J. de Graaf and P. Kruszyński in which the spaces $S_{\Phi(A)}$ and $T_{\Phi(A)}$ are introduced. In Chapter IV of that paper it is proved that a symmetry condition implies some topological properties of the spaces $S_{\Phi(A)}$ and $T_{\Phi(A)}$. In the underlying paper we show that a weaker symmetry condition is equivalent with all those topological properties and a lot more.

For the terminology of locally convex topological vector spaces we refer to [Wil].

1. Notations and some known theorems

Let X be a separable Hilbert space, $n \in \mathbb{N}$ and let $A_1, \cdots, A_n$ denote n self-adjoint operators whose corresponding spectral projections mutually commute. There exists a unique spectral measure $\mathbf{E}$ on the set $B(\mathbb{R}^n)$ of Borel sets of $\mathbb{R}^n$ so that for every $k \in \{1, \cdots, n\}$ the map $\Delta \mapsto \mathbf{E}(\mathbb{R}^{k-1} \times \Delta \times \mathbb{R}^{n-k})$, Δ a Borel set in $\mathbb{R}$, equals the spectral measure of A_k. For every Borel measurable function f on $\mathbb{R}^n$, there can be defined the self-adjoint operator

$$f(A) = \int_{\mathbb{R}^n} f(\lambda)\, d\mathbf{E}_\lambda$$

in a natural manner. (See [EGK], page 280.)

For every $\Delta \in B(\mathbb{R}^n)$ let χ_Δ be the characteristic function of Δ. For all $m \in \mathbb{Z}^n$ let

$$Q_m := \{\lambda \in \mathbb{R}^n : \forall_{k\in\{1,\cdots,n\}} [\lambda_k \in [m_k - 1, m_k)]\} .$$

Let $B_b(\mathbb{R}^n)$ be the set of all bounded Borel sets of $\mathbb{R}^n$ and let G^+ be the set of all maps F from $B_b(\mathbb{R}^n)$ into X with the property

$$F(\Delta_1 \cap \Delta_2) = \mathrm{E}(\Delta_2) F(\Delta_1) \qquad (\Delta_1, \Delta_2 \in B_b(\mathbb{R}^n)) .$$

Define $\mathrm{emb} : X \to G^+$ by $[\mathrm{emb}\, x](\Delta) := \mathrm{E}(\Delta)x$, $x \in X$, $\Delta \in B_b(\mathbb{R}^n)$. The map emb is injective. So the Hilbert space X is embedded in G^+.

Let ϕ be a Borel measurable function on $\mathbb{R}^n$ which is bounded on bounded Borel sets. Denote $\underline{\phi} := \{\lambda \in \mathbb{R}^n : \phi(\lambda) \neq 0\}$. (Remark: $\underline{\phi}$ need not be closed.) Let $x \in X$. Define $\phi(A)\cdot x \in G^+$ by $[\phi(A)\cdot x](\Delta) = \phi(A)\mathrm{E}(\Delta)x$, $\Delta \in B_b(\mathbb{R}^n)$. For every $F \in \phi(A)\cdot X$ there exists a unique $x \in \mathrm{E}(\underline{\phi})(X)$ such that $F = \phi(A)\cdot x$. Hence an inner product can be defined on $\phi(A)\cdot X$ such that the map $\phi(A)\cdot : \mathrm{E}(\underline{\phi})(X) \to \phi(A)\cdot X$ is a unitary map between two Hilbert spaces. The set $B_b(\mathbb{R}^n)$ is a directed set under inclusion, so we can define the following subspace of G^+:

$$D_\phi := \{F \in G^+ : \Delta \mapsto \phi(A)F(\Delta),\ \Delta \in B_b(\mathbb{R}^n), \text{is a Cauchy net in } X\} .$$

For every $F \in D_\phi$ define $\phi(A) * F := \lim_{\Delta} \phi(A)F(\Delta) \in X$. Corresponding to the same function ϕ we can also define an operator $\phi(A) : G^+ \to G^+$ by

$$[\phi(A)F](\Delta) := \phi(A)F(\Delta) \qquad (F \in G^+, \Delta \in B_b(\mathbb{R}^n)) .$$

Let Φ be a set of Borel functions on $\mathbb{R}^n$. Suppose the set Φ satisfies the next axiom.

AXIOM 1.

Φ is a directed set of real valued Borel functions on $\mathbb{R}^n$ and every element of Φ is bounded on bounded Borel sets. The set Φ has the following properties:

AI. *Each $\phi \in \Phi$ is nonnegative and the function $\lambda \mapsto \phi(\lambda)^{-1}$, $\lambda \in \underline{\phi}$ is bounded on bounded Borel sets.*

AII. *The sets $\underline{\phi}$, $\phi \in \Phi$, cover the whole $\mathbb{R}^n$, i.e. $\mathbb{R}^n = \bigcup_{\phi\in\Phi} \underline{\phi}$.*

AIII. $\forall_{\phi\in\Phi} \exists_{\psi\in\Phi} \exists_{c>0} \forall_{m\in\mathbb{Z}^n} [(1+|m|) \sup_{\lambda\in Q_m} \phi(\lambda) \leq c \inf_{\lambda\in Q_m} \psi(\lambda)]$.

The set Φ induces a new set Φ^+.

DEFINITION 2.

Let Φ be a set which satisfies Axiom 1. Then Φ^+ will denote the set of all Borel functions f on $\mathbb{R}^n$ so that

i) *f is a nonnegative Borel function and the map $\lambda \mapsto f(\lambda)^{-1}$, $\lambda \in \underline{f}$ is bounded on bounded Borel sets.*

ii) $\forall_{\phi\in\Phi} [\sup_{\lambda\in \mathbb{R}^n} f(\lambda)\phi(\lambda) < \infty]$.

LEMMA 3.
The set Φ^+ satisfies Axiom 1.

Proof. See [EGK], Lemma 1.5. □

Now we can define two subspaces $S_{\Phi(A)}$ and $T_{\Phi(A)}$ of G^+. Let Φ be a set which satisfies Axiom 1.

DEFINITION 4.
Let $S_{\Phi(A)} := \bigcup_{\phi\in\Phi} \phi(A)\cdot X$.
The topology σ_{ind} for $S_{\Phi(A)}$ is the inductive limit topology generated by the Hilbert spaces $\phi(A)\cdot X$, $\phi \in \Phi$.

DEFINITION 5.
Let $f \in \Phi^+$. Then $S_{\Phi(A)} \subset D_f$. The seminorm s_f on $S_{\Phi(A)}$ is defined by $s_f(w) := \| f(A) * w \|$, $w \in S_{\Phi(A)}$.

THEOREM 6.
The locally convex topology for $S_{\Phi(A)}$ generated by the seminorms s_f, $f \in \Phi^+$ is equivalent to the topology σ_{ind} for $S_{\Phi(A)}$.

Proof. See [EGK], Theorem 1.8. □

Remark: It follows that the topology σ_{ind} is Hausdorff.

DEFINITION 7.
Let $T_{\Phi(A)} := \{F \in G^+ : \forall_{\phi\in\Phi} [\phi(A)F \in \text{emb}(X)]\}$.
The topology τ_{proj} is the locally convex topology generated by the seminorms t_ϕ, $\phi \in \Phi$, defined by $t_\phi(F) := \| \text{emb}^{-1}(\phi(A)F) \|$, $(F \in T_{\Phi(A)}, \phi \in \Phi)$.

There exists a characterisation of bounded sets in $T_{\Phi(A)}$.

THEOREM 8.
Let $B \subset T_{\Phi(A)}$ be a set. Then B is bounded in $(T_{\Phi(A)}, \tau_{\text{proj}})$ iff there exist $f \in \Phi^+$ and a bounded set $B_0 \subset X$ so that $B = f(A)\cdot B_0$.

Proof. See [EGK], Theorem 2.4.II. □

It follows that $T_{\Phi(A)} = \bigcup_{f\in\Phi^+} f(A)\cdot X$.

DEFINITION 9.

The topology τ_{ind} *for* $T_{\Phi(A)}$ *is the inductive limit topology generated by the Hilbert spaces* $f(A)\cdot X$, $f\in\Phi^{+}$.

Further a duality between the spaces $S_{\Phi(A)}$ and $T_{\Phi(A)}$ can be introduced.

DEFINITION 10.

Define $<,>: S_{\Phi(A)}\times T_{\Phi(A)}\to \mathbb{C}$

$<\phi(A)\cdot x, F> = (x, \text{emb}^{-1}(\phi(A)F)) \quad (\phi\in\Phi, x\in \text{E}(\underline{\phi})(X), F\in T_{\Phi(A)})$.

(*See* [EGK], *page* 288.)

Note: For all $f\in\Phi^{+}$, $w\in S_{\Phi(A)}$ and $x\in X$ holds: $<w, f(A)\cdot x> = (f(A) * w, x)$.

THEOREM 11.

$<S_{\Phi(A)}, T_{\Phi(A)}>$ *is a dual pair and the topology* σ_{ind} *resp.* τ_{proj} *is compatible with the dual pair* $<S_{\Phi(A)}, T_{\Phi(A)}>$ *resp.* $<T_{\Phi(A)}, S_{\Phi(A)}>$.

Proof. See [EGK], Theorem 3.1. □

2. The weak symmetry condition

Let Φ be a set which satisfies Axiom 1. In Chapter IV of [EGK] the authors require the following strong symmetry condition on the set Φ.

AIV. $\quad \forall_{\zeta\in\Phi^{++}}\,\exists_{\phi\in\Phi}\,\exists_{c>0}\,[\zeta\le c\,\phi]$.

With this condition they prove several nice properties of the topological spaces $(S_{\Phi(A)},\sigma_{\text{ind}})$ and $(T_{\Phi(A)},\tau_{\text{proj}})$. They note that the operator $\zeta(A)\,\phi(A)^{-1}\chi_{\phi}(A)$ extends to a bounded operator on X. So the set Φ satisfies the following weak symmetry condition.

AIV'. $\quad \forall_{\zeta\in\Phi^{++}}\,\exists_{\phi\in\Phi}\,\exists_{c>0}\,[\chi_{\{\lambda\in\mathbb{R}^n:\zeta(\lambda)>c\,\phi(\lambda)\}}(A)=0]$.

A careful reading of their proofs shows that they use only condition AIV' to get the nice properties. The next theorem shows that the spaces $S_{\Phi(A)}$ and $T_{\Phi(A)}$ cannot have those nice properties without condition AIV'.

THEOREM 12.

Let Φ *be a set of Borel functions on* $\mathbb{R}^n$ *which satisfies Axiom* 1. *The following conditions are equivalent.*

I. $\quad \Phi$ *has property* AIV'.

II. $\quad (T_{\Phi(A)},\tau_{\text{proj}}) = (S_{\Phi^{+}(A)},\sigma_{\text{ind}})$ *as topological vector spaces.*

III. $(T_{\Phi(A)},\tau_{\rm proj}) = (T_{\Phi(A)},\tau_{\rm ind})$ *as topological vector spaces.*

IV. $(T_{\Phi(A)},\tau_{\rm proj})$ *is bornological.*

V. $(T_{\Phi(A)},\tau_{\rm proj})$ *is barrelled.*

VI. $(T_{\Phi(A)},\tau_{\rm proj})$ *is quasibarrelled.*

VII. $S_{\Phi(A)}$ *is complete.*

VIII. $S_{\Phi(A)}$ *is sequentially complete.*

IX. $S_{\Phi(A)} = \bigcap_{f\in\Phi^+} D_f$.

X. $S_{\Phi(A)} = S_{\Phi^{++}(A)}$ *as sets.*

XI. *For every bounded set* $B \subset S_{\Phi(A)}$ *there exist* $\phi \in \Phi$ *and a bounded set* $B_0 \subset X$ *so that* $\phi(A)\cdot|_{B_0} : B_0 \to B$ *is a homeomorphism.*

Proof.

I $\Rightarrow$ II. Theorem 4.2.I of [EGK].

II $\Rightarrow$ III. Always $(S_{\Phi^+(A)},\sigma_{\rm ind}) = (T_{\Phi(A)},\tau_{\rm ind})$ holds.

III $\Rightarrow$ I. Let $\zeta \in \Phi^{++}$. Define $W : T_{\Phi(A)} \to X$ by $W(f(A)\cdot x) := (\zeta f)(A)x$, $f \in \Phi^+$, $x \in X$. Let $f \in \Phi^+$. Then $\| W(f(A)\cdot x)\| \le \|(\zeta f)(A)\|\|x\| = \|(\zeta f)(A)\|\|f(A)\cdot x\|_{f(A)X}$ for all $x \in \underline{E}(f)(X)$. By definition of $\tau_{\rm ind}$, the map W is continuous from $(T_{\Phi(A)},\tau_{\rm ind})$ into X. By assumption, the map W is continuous from $(T_{\Phi(A)},\tau_{\rm proj})$ into X, so there exist $\phi \in \Phi$ and $c > 0$ such that $\|W(F)\| \le t_\phi(F)$ for all $F \in T_{\Phi(A)}$. In particular,

$\|(\zeta\chi_{Q_m})(A)x\| = \|W(\chi_{Q_m}(A)\cdot x)\| \le c\,\|(\phi\chi_{Q_m})(A)x\|$

for all $x \in X$ and $m \in \mathbb{Z}^n$. So $\chi_{\{\lambda\in\mathbb{R}^n : \zeta(\lambda) > c\,\phi(\lambda)\}}(A) = 0$.

III $\Rightarrow$ IV $\Rightarrow$ VI and III $\Rightarrow$ V $\Rightarrow$ VI are trivial.

VI $\Rightarrow$ III. Always $\tau_{\rm proj} \subset \tau_{\rm ind}$. Let $\Omega \subset T_{\Phi(A)}$ be a $\tau_{\rm ind}$-neighbourhood of 0. Because $\tau_{\rm ind}$ is regular, there exist absolutely convex $\tau_{\rm ind}$-open $\Omega_1 \subset T_{\Phi(A)}$ so that $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega$. Assertion: Ω_1 is a bornivore in $(T_{\Phi(A)},\tau_{\rm proj})$. Let $B \subset T_{\Phi(A)}$ be a $\tau_{\rm proj}$-bounded set. By Theorem 8 there exist $f \in \Phi^+$ and a bounded set $B_0 \subset X$ so that $B_0 = f(A)\cdot B_0$. Let $M > 0$ be so that $\|x\| \le M$ for all $x \in B_0$. Since Ω_1 is $\tau_{\rm ind}$-open, there exists $\varepsilon > 0$ so that for all $x \in X$, $\|x\| < \varepsilon$ holds $f(A)\cdot x \in \Omega_1$. Then for all $t \in \mathbb{C}$, $|t| < \varepsilon M^{-1}$ we get $tB \subset \Omega_1$. This proves the assertion. Hence $\overline{\Omega_1}$ is a bornivore barrel and by assumption a $\tau_{\rm proj}$-neighbourhood of 0. So $\tau_{\rm ind} \subset \tau_{\rm proj}$.

I $\Rightarrow$ VII. See [EGK], Corollary 4.3.III.

VII $\Rightarrow$ VIII. Trivial.

VIII $\Rightarrow$ IX. Always $S_{\Phi(A)} \subset \bigcap_{f\in\Phi^+} D_f$. Let $F \in \bigcap_{f\in\Phi^+} D_f$. For $p \in \mathbb{N}$ let $\Delta_p := \{\lambda \in \mathbb{R}^n : |\lambda| \le p\}$, $x_p := F(\Delta_p)$ and $F_p := \chi_{\Delta_p}(A)\cdot x_p$. Then $F_p \in S_{\Phi(A)}$. Assertion: $(F_p)_{p\in\mathbb{N}}$ is a Cauchy sequence in $S_{\Phi(A)}$. Let $f \in \Phi^+$ and $\varepsilon > 0$. There exists $\Delta_0 \in B_b(\mathbb{R}^n)$ so that for all $\Delta,\Delta' \in B_b(\mathbb{R}^n)$, $\Delta \supset \Delta_0$, and $\Delta' \supset \Delta_0$ holds

$\|f(A)F(\Delta) - f(A)F(\Delta')\| \le \varepsilon$. Let $p_0 \in \mathbb{N}$ be so that $\Delta_{p_0} \supset \Delta_0$. Let $p \in \mathbb{N}$, $p \ge p_0$. For all $\Delta \in B_b(\mathbb{R}^n)$, $\Delta \supset \Delta_0$ we obtain $\|f(A)F(\Delta) - f(A)F(\Delta_p)\| \le \varepsilon$, so $\|f(A) * F - f(A) * F_p\| = \|f(A) * F - f(A)F(\Delta_p)\| \le \varepsilon$. So $p \mapsto f(A) * F_p$ is a Cauchy sequence in X with limit $f(A) * F$ and the assertion is proved (Theorem 6). Let $F_0 \in S_{\Phi(A)}$ be the limit of the sequence $(F_p)_{p \in \mathbb{N}}$. Let $\Delta \in B_b(\mathbb{R}^n)$. Then $\chi_\Delta \in \Phi^+$ and $F_0(\Delta) = \chi_\Delta(A) * F_0 = \lim_{p \to \infty} \chi_\Delta(A) * F_p = \chi_\Delta(A) * F = F(\Delta)$. So $F = F_0 \in S_{\Phi(A)}$.

IX $\Rightarrow$ IV. Let $W : T_{\Phi(A)} \to \mathbb{C}$ be a linear map which is bounded on τ_{proj}-bounded sets. For all $f \in \Phi^+$ the map $x \mapsto W(f(A) \cdot x)$ from X into $\mathbb{C}$ is bounded on bounded sets by Theorem 8, so this map is continuous. In particular: for every $\Delta \in B_b(\mathbb{R}^n)$ there exists unique $F(\Delta) \in X$ so that for all $x \in X$ holds $(x, F(\Delta)) = W(\chi_\Delta(A) \cdot x)$. Then $F \in G^+$. Assertion: $F \in \bigcap_{f \in \Phi^+} D_f$. Let $f \in \Phi^+$. There exists $y \in X$ so that for all $X \in X$ holds $W(f(A) \cdot x) = (x, y)$. Let $x \in X$. Then $\lim_\Delta (x, f(A)F(\Delta)) =$
$= \lim (f(A)\chi_\Delta(A)x, F(\Delta)) = \lim W(\chi_\Delta(A) \cdot f(A)\chi_\Delta(A)x) = \lim W(f(A) \cdot \chi_\Delta(A)x) =$
$= \lim (\chi_\Delta(A)x, y) = (x, y)$. So weak $\lim_\Delta f(A)F(\Delta) = y$. But also $\lim_\Delta \|f(A)F(\Delta)\| =$
$= \lim_\Delta \sup_{\|x\| \le 1} |W(f(A) \cdot \chi_\Delta(A)x)| = \sup_{\|x\| \le 1} |W(f(A) \cdot x)| = \|y\|$. So strong $\lim f(A)F(\Delta) = y$. Hence $F \in D_f$ and $y = f(A) * F$. So $F \in \bigcap_{f \in \Phi^+} D_f = S_{\Phi(A)}$. Let $H \in T_{\Phi(A)}$. There are $f \in \Phi^+$ and $x \in X$ so that $H = f(A) \cdot x$. Then $W(H) = (x, f(A) * F) = \overline{<F, f(A) \cdot x>} = \overline{<F, H>}$. By Theorem 11 it follows that W is continuous.

IX $\Leftrightarrow$ X. By equivalence of I and IX: $S_{\Phi(A)} \subset S_{\Phi^{++}(A)} = \bigcap_{f \in \Phi^{+++}} D_f = \bigcap_{f \in \Phi^+} D_f$.

I $\Rightarrow$ XI. See [EGK], Corollary 4.3.IV.

XI $\Rightarrow$ VIII. Let $w_1, w_2, \cdots$ be a Cauchy sequence in $S_{\Phi(A)}$. Then $\{w_n : n \in \mathbb{N}\}$ is bounded, so there exist $\phi \in \Phi$ and a Cauchy sequence $x_1, x_2, \cdots$ in X so that $w_n = \phi(A) \cdot x_n$, $n \in \mathbb{N}$. Let $x := \lim_{n \to \infty} x_n$. Then $\lim_{n \to \infty} w_n = \phi(A) \cdot x$ in $S_{\Phi(A)}$. □

Remark: It is trivial by now that property AIV' is equivalent with $(T_{\Phi(A)}, \tau_{\text{proj}})$ is reflexive and also with $(S_{\Phi(A)}, \sigma_{\text{ind}}) = (T_{\Phi^+(A)}, \tau_{\text{proj}})$ as topological vector spaces. If $(T_{\Phi(A)}, \tau_{\text{proj}})$ happens to be metrizable, then property AIV' holds.

References

[EGK] Eijndhoven, S.J.L. van, J. de Graaf and P. Kruszyński, Dual systems of inductive-projective limits of Hilbert spaces originating from self-adjoint operators. Proc. Kon. Ned. Akad. van Wetensch., A88, 277-297 (1985).

[Wil] Wilansky, A., Modern methods in topological vector spaces. McGraw-Hill, New York (1978).

A.F.M. ter Elst
Eindhoven University of Technology
Department of Mathematics and Computing Science
PO Box 513
5600 MB Eindhoven
The Netherlands