The Banach-Steinhaus theorem in a Hilbert space

EINDHOVEN UNIVERSITY OF TECHNOLOGY

Department of Mathematics

Memorandum no. 1974-14

Issued December, 1974

The Banach-Steinhaus Theorem in a Hilbert space

by

P. van der Steen

Department of Mathematics
University of Technology
PO Box 513, Eindhoven,
The Netherlands.

The Banach-Steinhaus Theorem in a Hilbert Space

by

P. van der Steen

The Banach-Steinhaus Theorem, specialized for a Hilbert space, reads as follows.

Theorem. Let H be a Hilbert space, and F a family in H with the property that for every $g \in H$ there is an $M(g)$ such that $|(g,f)| \leq M(g)$ for all $f \in F$. Then there exists M such that $\|f\| \leq M$ for all $f \in F$.

Even in the Banach space case there are two kinds of proofs. The most elementary of these, sometimes called the gliding hump method, uses the completeness of the space directly. The other method exploits this fact via the Baire Category Theorem, and is therefore more generally applicable.

The basic idea of the gliding hump method is that the theorem in our setting is trivial for an orthogonal family. On the assumption that the assertion of the theorem is not true, one constructs a large enough subfamily of F which is "nearly orthogonal", and then imitates the proof for the orthogonal case. In the version presented below, we construct instead an orthogonal family which is near enough to a subfamily of F to yield a contradiction. This results in a slight reduction of the computation compared to other proofs, cf. [1, Ch. XI, Problem 4], [2, Solution 20], [3].

First we prove the special case that H has finite dimension, k say. Choose an orthonormal basis $e_1, e_2, \dots, e_k$ for H. If $M = \max_n \sup_f |(e_n, f)|$, then

$$\|f\|^2 = \sum_{n=1}^{k} |(e_n, f)|^2 \le kM^2 \quad \text{for every } f \in F .$$

<u>Proof of the general case</u>. Suppose no such M exists. Construct sequences $(f_n)_n$ and $(h_n)_n$ as follows. Let $f_1 = h_1$ be any non zero member of F, and proceed by induction. Assume that the f_j, h_j with $j < k$ have been chosen. Let L_k be the linear space of the h_j with $j < k$. For every $f \in F$, write $f = f' + f''$ where $f' \perp L_k$, $f'' \in L_k$. Since

$$\sup_f |(g, f'')| = \sup_f |(g, f)| < \infty \qquad \text{for every } g \in L_k ,$$

the finite dimensional case of the theorem gives $\sup_f \|f''\| < \infty$. Since $\sup_f \|f\| = \infty$, there exists $f_k \in F$ such that $\|f_k\| > k$ and $\|f_k''\| < k^{-2}\|f_k\|$. Put $h_k = f_k'$, then $\|h_k\| > (1 - k^{-2})\|f_k\| \ge \frac{3}{4}\|f_k\|$ and $\|h_k - f_k\| < k^{-2}\|f_k\|$.

Now note that $(h_n)_n$ is an orthogonal family, and put $g = \sum_{k=1}^{\infty} (k\|h_k\|)^{-1} h_k$.

Then

$$|(g,f_k)| \geq |(g,h_k)| - |(g,f_k - h_k)| \geq k^{-1}\|h_k\| - \|g\|.\|f_k - h_k\| >$$

$$> (\tfrac{3}{4} - \|g\|k^{-1})k^2 ,$$

which tends to infinity with k. Since $(f_k)_k$ is a subfamily of F, this is impossible.

References

1. J. Dieudonné, Foundations of Modern Analysis, Academic Press, New York, 1960.

2. P.R. Halmos, A Hilbert Space Problem Book, Van Nostrand, Princeton, 1967.

3. S.S. Holland, Jr., A Hilbert Space Proof of the Banach-Steinhaus Theorem, Amer. Math. Monthly 76 (1969), 40-41.