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The Non-linear Flexural–Torsional Behaviour of Straight Slender Elastic Beams with Arbitrary Cross Sections

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ABSTRACT

A potential energy functional for the non-linear flexural–torsional behaviour of straight slender elastic beams with arbitrary cross sections is derived. The result is generally applicable to situations where the strains are small and the Bernoulli hypotheses are valid. It is shown that special theories can be derived from the general expressions in a consistent manner, by neglecting some specific terms.

NOTATION

\vec{a}	Vector
$ \vec{a} $	Length of \vec{a}
$\vec{a} \cdot \vec{b}$	Inner product of two vectors
$\vec{a} * \vec{b}$	Cross product of two vectors
$\vec{a} \vec{b}$	Dyadic product of two vectors
A	Undeformed cross sectional area
\mathbf{A}	Second-order tensor
\mathbf{A}^c	Conjugate of \mathbf{A}
$\mathbf{A} \cdot \vec{a}$	Inner product of a tensor and a vector
$\mathbf{A} \cdot \mathbf{B}$	Inner product of two tensors
$\vec{e}, \vec{i}, \vec{\eta}$	Unit vectors
\vec{e}_i	Unit base vector
e_{ij}	Cartesian components of the Green–Lagrange strain tensor

\mathbf{F}	Deformation gradient tensor
H	Higher-order torsion constant
I_s	Polar moment of inertia about the shear centre
I_2, I_3	Moments of inertia about the y and z axes respectively
L	Length
p_i, P_i	Cartesian component of force vector
q_i	Cartesian component of distributed load vector
\bar{r}	Radial distance of a point on the cross section to the shear centre
s	Arc length
S	Surface
u_s, v_s, w_s	Displacement components of the shear centre in the directions $\bar{e}_1, \bar{e}_2, \bar{e}_3$ respectively
x	Coordinate along beam axis
y, z	Coordinate along the principal centroidal axes
\bar{y}, \bar{z}	Coordinate along \bar{y} and \bar{z} axes through the shear centre
$\beta_2, \beta_3, \beta_\theta$	Geometric constants
Γ	Warping constant
ϵ	Strain
χ_i	Curvature components
$\bar{\chi}$	Curvature vector
$\bar{\nabla}$	Gradient operator
<i>Subscripts</i>	
s	Shear centre
0	Undeformed state

I INTRODUCTION

During the past 15 years several articles¹⁻¹³ on non-linear flexural-torsional behaviour of beams have been published. In the majority of these articles, a potential energy functional in terms of displacements and rotations is used, which is mostly derived in the following manner.

First, a displacement field containing the components of the rotation matrix is determined; then this field is used to calculate the non-linear strain components and, finally, these strains are used to determine the potential energy functional.

Deriving the potential energy functional in this manner, without introducing various approximations, is almost impossible, because both the components of the rotation matrix and the strain expressions in terms of displacements are lengthy and complicated in their exact form. Therefore most of the articles on non-linear flexural-torsional behaviour of beams

restrict themselves to a special class of deformation as a consequence of the approximations made.

In this paper, a coordinate-free dyadic notation is used to avoid approximations concerning the magnitude of the deflections and rotations. This enables the derivation of a potential energy functional and curvature expressions which are generally applicable to situations where the strains are small and the Bernoulli hypotheses are valid.

The special theories of the articles mentioned can be derived easily from these general expressions by neglecting specific terms.

Rotations in non-linear beam theory are mostly described in terms of Euler angles or modified Euler angles. This results in a 'geometric torsion' expression which is asymmetric in the transverse displacement components and therefore often leads to confusion. In this report, the rotations are described in a special way, resulting in a 'geometric torsion' expression which is skew-symmetric in these components, as would be expected.

2 THE GENERAL POTENTIAL ENERGY FUNCTIONAL

Assuming the strain energy density to be a quadratic functional of the Lagrangian strain tensor components, $e_{ij} = (1/2)(u_{i,j} + u_{j,i} + u_{h,i}u_{h,j})$ ($i, j, h = 1, 2, 3$), and taking into account the condition $e_{22} = e_{33} = -\nu e_{11}$, the potential energy functional (see Ref. 14) may be written as

$$\pi_i[\bar{u}] = \frac{1}{2} \int_{V_0} [Ee_{11}^2 + 4Ge_{12}^2 + 4Ge_{13}^2] dV - \int_{V_0} \rho_0 q_i u_i dV - \int_{S_p^0} \bar{P}_{0i} u_i dS \quad (1)$$

where \bar{u} is the displacement field, V_0 is the volume of the undeformed body, S_p^0 is the part of the boundary where the loads are prescribed, ρ_0 is the mass density, q_i represents the body forces per unit mass, \bar{P}_{0i} are the prescribed external loads, E is Young's modulus and G is the shear modulus. In this equation the summation convention is applied, which means that repetition of an index in a term denotes a summation with respect to that index over its range.

3 KINEMATICS OF STRAIGHT SLENDER ELASTIC BEAMS

A straight slender prismatic beam of length L is shown in Fig. 1(a). The beam is made of a homogeneous isotropic linear elastic material. Each material point in this beam is described by a rectangular Cartesian system of coordinates (x, \bar{y}, \bar{z}) . The coordinate x coincides with the elastic axis of the

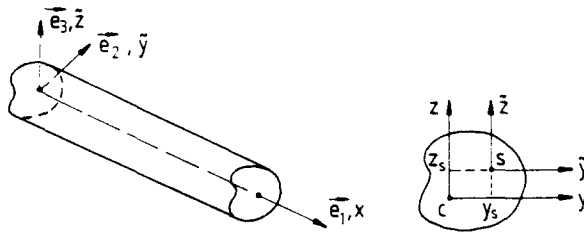


Fig. 1. (a) Beam with coordinate system. (b) Cross section.

beam, defined as the line which connects the shear centres of the cross sections of the beam. If the shear centre is not a material point of the cross section, as is often the case for thin-walled open sections, it is still considered to follow all the deformations of the cross section, as if it was a real material point of that cross section. It is assumed that before deformation the elastic axis is a straight line. With x representing length along this axis, it can be represented by

$$\vec{r}_0 = x\vec{e}_1 \ ; \ x \in [0, L] \tag{2}$$

In the undeformed state the cross section is oriented such that \vec{e}_2 and \vec{e}_3 are parallel to the principal axes. The position of an arbitrary material point, before deformation, is given by

$$\vec{x}_0(x, \bar{y}, \bar{z}) = \vec{r}_0(x) + \vec{\alpha}_0(\bar{y}, \bar{z}) \tag{3}$$

where $\vec{\alpha}_0 = \bar{y}\vec{e}_2 + \bar{z}\vec{e}_3$, while \bar{y} and \bar{z} denote length along the \vec{e}_2 and \vec{e}_3 axes. Besides the coordinate axes \bar{y}, \bar{z} a second set of coordinate axes is defined in the cross section, parallel to \bar{y}, \bar{z} but with the origin located at the centroid (Fig. 1(b)). \bar{y}, \bar{z} and y, z are related by

$$\bar{y} = (y - y_s) \text{ and } \bar{z} = (z - z_s) \tag{4}$$

y_s, z_s are the y and z coordinates of the shear centre.

After deformation, the position vector of a material point is given by $\vec{x}(x, \bar{y}, \bar{z})$. In order to determine \vec{x} the following assumptions are made:

- The total deformation of the beam can be considered as the result of two successive motions: first, a rigid translation and rotation of each cross section due to bending and warping free torsion; next, a warping displacement perpendicular to the displaced cross sections.
- The cross section does not distort in its plane during deformation.
- Shear deformation due to transverse forces can be neglected.

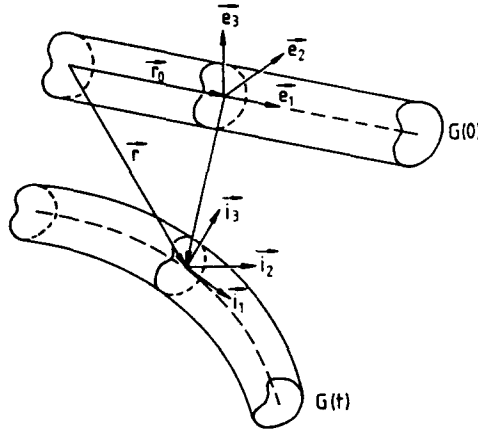


Fig. 2. Beam after warping free motion.

The position vector \vec{x} can now be expressed as

$$\vec{x}(x, \bar{y}, \bar{z}) = \vec{r}(x) + \vec{\alpha}(x, \bar{y}, \bar{z}) + f(x, \bar{y}, \bar{z})\vec{i}_1 \tag{5}$$

where

$$\vec{\alpha}(x, \bar{y}, \bar{z}) = \bar{y}\vec{i}_2(x) + \bar{z}\vec{i}_3(x);$$

$\vec{r} = \vec{r}(x)$ represents the beam axis in the deformed configuration;

\vec{i}_2, \vec{i}_3 are unit vectors parallel to the principal axes of the cross section after the warping free motion;

$f(x, \bar{y}, \bar{z})\vec{i}_1$ represents small normal warping displacements ($\vec{i}_1 = \vec{i}_2 * \vec{i}_3$).

If the displacement of a point on the elastic axis is described by its components u_s, v_s, w_s in the directions $\vec{e}_1, \vec{e}_2, \vec{e}_3$ respectively, \vec{r} can be expressed as

$$\vec{r} = (x + u_s)\vec{e}_1 + v_s\vec{e}_2 + w_s\vec{e}_3 \tag{6}$$

The unit tangent vector to the deformed elastic axis at a point Q can be obtained from

$$\frac{d\vec{r}}{ds}(Q) = \vec{n}(Q) \tag{7}$$

where s is the arc length along the deformed elastic axis.

Differentiating \vec{r} with respect to x instead of s gives

$$\frac{d\vec{r}}{dx} = \frac{d\vec{r}}{ds} \frac{ds}{dx} = (1 + \epsilon_s)\vec{n} = [(1 + u'_s)\vec{e}_1 + v'_s\vec{e}_2 + w'_s\vec{e}_3] \tag{8}$$

where the prime (') means differentiation with respect to x , and

$$\epsilon_s = \left| \frac{d\vec{r}}{dx} \right| - 1 = [(1 + u'_x)^2 + v'^2 + w'^2]^{1/2} - 1 \quad (9)$$

If shear deformation due to transverse forces is neglected, the unit tangent vector $\vec{n}(Q)$ will be perpendicular to the cross section in every point Q .

$$\vec{n} = \vec{i}_1 = \vec{i}_2 * \vec{i}_3 \quad (10)$$

The triad $\vec{e}_1, \vec{e}_2, \vec{e}_3$ can be transformed into the triad $\vec{i}_1, \vec{i}_2, \vec{i}_3$ by means of a rigid translation and rotation.

The rigid rotation can be described by an orthogonal tensor \mathbf{R} :

$$\mathbf{R} \cdot \vec{e}_k = \vec{i}_k \quad (k = 1, 2, 3) \quad (11)$$

$$\mathbf{R} = \mathbf{R}(x) ; \mathbf{R}^c \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{R}^c = \mathbf{I} ; \det \mathbf{R} = 1$$

where \mathbf{I} is the unit tensor and \mathbf{R}^c is the conjugate of \mathbf{R} .

To analyse the beam deformation, that is to define the flexural and torsional curvatures of the beam axis, the derivatives of \vec{i}_k with respect to s are studied:

$$\frac{d\vec{i}_k}{ds} = \frac{d\mathbf{R}}{ds} \cdot \vec{e}_k = \frac{d\mathbf{R}}{ds} \cdot \mathbf{R}^c \cdot \vec{i}_k \quad (12)$$

The orthogonality property of \mathbf{R} implies that $(d\mathbf{R}/ds) \cdot \mathbf{R}^c$ is a skew tensor, and therefore eqn (12) may also be written as

$$\frac{d\vec{i}_k}{ds} = \frac{d\mathbf{R}}{ds} \cdot \mathbf{R}^c \cdot \vec{i}_k = \vec{\rho} \cdot \vec{i}_k \quad (13)$$

where $\vec{\rho}$ is the axial vector of $(d\mathbf{R}/ds) \cdot \mathbf{R}^c$.

According to the classical definition,¹⁵ the torsional and flexural beam curvatures are defined as the components of the vector $\vec{\rho}$ with respect to the local basis $\vec{i}_1, \vec{i}_2, \vec{i}_3$:

$$\vec{\rho} = \chi_1 \vec{i}_1 + \chi_2 \vec{i}_2 + \chi_3 \vec{i}_3 \quad (14)$$

where χ_1 is the torsional curvature and χ_2 and χ_3 are the flexural curvatures.

Differentiating \vec{i}_k ($k = 1, 2, 3$) with respect to x instead of s yields

$$\frac{d\vec{i}_k}{dx} = \frac{ds}{dx} \frac{d\vec{i}_k}{ds} = (1 + \epsilon_s) \vec{\rho} \cdot \vec{i}_k = \vec{\chi} \cdot \vec{i}_k \quad (15)$$

where $\vec{\chi} = (1 + \epsilon_s) \vec{\rho}$.

Combination of eqns (14) and (15) yields

$$\vec{\chi} = (1 + \epsilon_s)[\chi_1 \vec{i}_1 + \chi_2 \vec{i}_2 + \chi_3 \vec{i}_3] \tag{16}$$

The deformation process can be described completely in terms of the deformation gradient tensor \mathbf{F} , which is given by

$$\mathbf{F} = (\vec{\nabla}_0 \vec{x})^C \tag{17}$$

The gradient operator with respect to the undeformed configuration can be written as

$$\vec{\nabla}_0 = \vec{e}_1 \frac{\partial}{\partial x} + \vec{e}_2 \frac{\partial}{\partial y} + \vec{e}_3 \frac{\partial}{\partial z} \tag{18}$$

Substitution of eqns (4) and (5) into (17) yields

$$\begin{aligned} \mathbf{F} = & \left[\epsilon_s \vec{i}_1 + (\vec{\chi} \cdot \vec{\alpha}) + \frac{\partial f}{\partial x} \vec{i}_1 + f(\vec{\chi} \cdot \vec{i}_1) \vec{e}_1 \right] \\ & + \frac{\partial f}{\partial y} \vec{i}_1 \vec{e}_2 + \frac{\partial f}{\partial z} \vec{i}_1 \vec{e}_3 + \mathbf{R} \end{aligned} \tag{19}$$

where $\mathbf{R} = (\vec{i}_1 \vec{e}_1 + \vec{i}_2 \vec{e}_2 + \vec{i}_3 \vec{e}_3)$.

The Green-Lagrange strain tensor is defined by

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^C \cdot \mathbf{F} - \mathbf{I}) \tag{20}$$

The components of \mathbf{E} which are relevant in beam theory are (see Section 2)

$$\begin{aligned} e_{11} &= \vec{e}_1 \cdot \mathbf{E} \cdot \vec{e}_1 = \frac{1}{2}[(\mathbf{F} \cdot \vec{e}_1) \cdot (\mathbf{F} \cdot \vec{e}_1) - 1] \\ e_{12} &= \vec{e}_1 \cdot \mathbf{E} \cdot \vec{e}_2 = \frac{1}{2}[(\mathbf{F} \cdot \vec{e}_1) \cdot (\mathbf{F} \cdot \vec{e}_2)] \\ e_{13} &= \vec{e}_1 \cdot \mathbf{E} \cdot \vec{e}_3 = \frac{1}{2}[(\mathbf{F} \cdot \vec{e}_1) \cdot (\mathbf{F} \cdot \vec{e}_3)] \end{aligned} \tag{21}$$

Substitution of eqn (19) into (21) yields the expressions for e_{11} , e_{12} and e_{13} .¹⁴

Before proceeding, attention is focused on the warping function f . In the case of thin-walled open sections, the normalized warping displacements are mostly described in terms of the so-called sectorial area $\omega(n, p)$. In this paper, however, preference is given to the equivalent, but more general, Saint Venant warping function $\psi(y, z)$. If χ_1 is chosen as the warping amplitude, the function f can be written as

$$f(x, y, z) = \chi_1(x)\psi(y, z) \tag{22}$$

(The consequences of choosing χ_1 as the warping amplitude are discussed in Section 10 where an alternative approach is proposed.)

Besides the Bernoulli hypotheses, no assumptions have been introduced so far. In the case of small strains, however, strains may be neglected compared with unity. Applying this type of approximation to eqn (21) and substituting eqn (22) yields¹⁴

$$e_{11} = \epsilon_x + (-\bar{y}\chi_3 + \bar{z}\chi_2) + \frac{1}{2}(\bar{y}^2 + \bar{z}^2)\chi_1^2 + \frac{\partial\chi_1}{\partial x}\psi \quad (23)$$

$$e_{12} = \frac{1}{2}\left(-\bar{z}\chi_1 + \chi_1\frac{\partial\psi}{\partial y}\right) \quad (24)$$

$$e_{13} = \frac{1}{2}\left(\bar{y}\chi_1 + \chi_1\frac{\partial\psi}{\partial z}\right) \quad (25)$$

Using relation (4), e_{11} may also be written as

$$e_{11} = \bar{\epsilon} - y\chi_3 + z\chi_2 + \frac{1}{2}\bar{r}^2\chi_1^2 + \frac{\partial\chi_1}{\partial x}\psi \quad (26)$$

where

$$\bar{\epsilon} = \epsilon_x + y_3\chi_3 - z_2\chi_2 \quad \text{and} \quad \bar{r}^2 = (\bar{y}^2 + \bar{z}^2)$$

4 STRAIN ENERGY

In beam theory the strain energy is given by (see eqn (1))

$$U = \int_{V_0} \frac{1}{2}[Ee_{11}^2 + 4Ge_{12}^2 + 4Ge_{13}^2]dV \quad (27)$$

Since y and z are coordinates along the principal axes, the following identities hold:

$$\int_A y dA = \int_A z dA = \int_A yz dA = \int_A z\psi dA = \int_A y\psi dA = \int_A \psi dA = 0 \quad (28)$$

where A is the area of the cross section.

Substituting eqns (23)–(26) into eqn (27) yields

$$U = \frac{1}{2} \int_0^L \left[EA\bar{\epsilon}^2 + EI_2\chi_2^2 + EI_3\chi_3^2 + GJ\chi_1^2 + E\Gamma \left(\frac{\partial\chi_1}{\partial x}\right)^2 + EH\chi_1^4 \right. \\ \left. + \chi_1^2 \left(EI_1\bar{\epsilon} + EI_2\beta_2\chi_2 - EI_3\beta_3\chi_3 + E\Gamma\beta_w \left(\frac{\partial\chi_1}{\partial x}\right) \right) \right] dx \quad (29)$$

where

$$I_2 = \int_A z^2 dA \quad ; \quad I_3 = \int_A y^2 dA \quad ; \quad \Gamma = \int_A \psi^2 dA \quad ;$$

$$I_5 = \int_A \bar{r}^2 dA \quad ; \quad J = \int_A \left[\left(\frac{d\psi}{dy} - \bar{z} \right)^2 + \left(\frac{\partial\psi}{\partial z} + \bar{y} \right)^2 \right] dA \quad ;$$

$$H = \frac{1}{4} \int_A \bar{r}^4 dA \quad ; \quad \beta_2 = \frac{1}{I_2} \int_A (y^2 + z^2)z dA - 2z_s = \frac{1}{I_2} \int_A z \bar{r}^2 dA \quad ;$$

$$\beta_3 = \frac{1}{I_3} \int_A (y^2 + z^2)y dA - 2y_s = \frac{1}{I_3} \int_A y \bar{r}^2 dA \quad ; \quad \beta_\psi = \frac{1}{\Gamma} \int_A (y^2 + z^2)\psi dA$$

$$\bar{\epsilon} = \epsilon_s + y_s \chi_3 - z_s \chi_2 \quad ; \quad \epsilon_s = [(1 + u_s')^2 + v_s'^2 + w_s'^2]^{1/2} - 1$$

In Ref. 13 a general non-linear beam theory is also derived. The strain energy expression obtained is the same as in this paper, except for the last term of eqn (29) which is missing. This term however has to be taken into account in the case of asymmetric sections.

5 THE POTENTIAL ENERGY FUNCTIONAL

The total potential energy is the sum of the strain energy U and the potential of the loads Ω

$$\pi = U + \Omega \tag{30}$$

It is assumed that the beam is subjected to dead weight surface tractions $p_1 \bar{e}_1 + p_2 \bar{e}_2 + p_3 \bar{e}_3$ at both ends ($x = 0$ and $x = L$) and $q_2 \bar{e}_2, q_3 \bar{e}_3$ per unit length.

Besides these 'external' tractions the beam is loaded by a force $\bar{q} \bar{e}_3$ acting at the centroid, representing the weight of the beam.

The potential of the loads may be written as

$$\Omega = - \sum_{i=1}^3 \left[\int_A p_i u_i dA \right]_{x=0; x=L} - \int_0^L [q_2 u_2 + q_3 u_3 + \bar{q} u_3(c)] dx \tag{31}$$

where $u_3(c)$ is the displacement component of the centroid in the \bar{e}_3 direction.

The displacement vector \bar{u} of a material point B is given by

$$\bar{u}(B) = \bar{x}(B) - \bar{x}_0(B) \quad (32)$$

The components of $\bar{u}(B)$ in the directions $\bar{e}_1, \bar{e}_2, \bar{e}_3$ can be expressed (see Ref. 14) as

$$\begin{aligned} u_1 &= \bar{u} + yR_{12} + zR_{13} + \chi_1\psi R_{11} \\ u_2 &= v_s + \bar{y}(R_{22} - 1) + \bar{z}R_{23} + \chi_1\psi R_{21} \\ u_3 &= w_s + \bar{y}R_{32} + \bar{z}(R_{33} - 1) + \chi_1\psi R_{31} \end{aligned} \quad (33)$$

where $\bar{u} = u_s - y_s R_{12} - z_s R_{13}$ and R_{ij} are the components of \mathbf{R} with respect to $\bar{e}_1, \bar{e}_2, \bar{e}_3$ ($R_{ij} = \bar{e}_i \cdot \bar{i}_j$). The terms $\chi_1\psi R_{21}$ and $\chi_1\psi R_{31}$ in eqn (35) may be neglected according to the assumption of small strains.¹⁴

Combination of eqns (29) and (31) yields

$$\begin{aligned} \pi &= \frac{1}{2} \int_0^L \left[EA\bar{\epsilon}^2 + EI_2\chi_2^2 + EI_3\chi_3^2 + GJ\chi_1^2 + E\Gamma \left(\frac{\partial\chi_1}{\partial x} \right)^2 + EH\chi_1^4 \right. \\ &\quad \left. + \chi_1^2 \left(EI_s\bar{\epsilon} + EI_2\beta_2\chi_2 - EI_3\beta_3\chi_3 + E\Gamma\beta_w \left(\frac{\partial\chi_1}{\partial x} \right) \right) \right] dx \\ &\quad - [P_1(u_s - y_s R_{12} - z_s R_{13}) + P_2 v_s + P_3 w_s + M_2 R_{13} + (-M_3) R_{12} \\ &\quad + B\chi_1 R_{11} + (-M_{12}) R_{23} + M_{13} R_{32} + (R_{22} - 1) \int_A p_2 \bar{y} dA \\ &\quad + (R_{33} - 1) \int_A p_3 \bar{z} dA]_{x=0, x=L} \\ &\quad - \int_0^L [q_2 v_s + q_2 \bar{y}(R_{22} - 1) + (-m_{12}) R_{23} + q_3 w_s + m_{13} R_{32} \\ &\quad + q_3 \bar{z}(R_{33} - 1) + \bar{q}(w_s - y_s R_{32} - (R_{33} - 1)z_s)] dx \end{aligned} \quad (34)$$

where

$$\begin{aligned} P_1 &= \int_A p_1 dA & ; & & P_2 &= \int_A p_2 dA & ; & & P_3 &= \int_A p_3 dA \\ M_2 &= \int_A (p_1 z) dA & ; & & M_3 &= \int_A (-p_1 y) dA & ; & & B &= \int_A (p_1 \psi) dA \\ M_{12} &= \int_A (-p_2 \bar{z}) dA & ; & & M_{13} &= \int_A (p_3 \bar{y}) dA \\ m_{12} &= & (-q_2 \bar{z}) & ; & m_{13} &= & q_3 \bar{y} \end{aligned}$$

Equation (34) includes the non-linear contributions resulting from the movement of the points of application of the loads. This contribution, which can have a significant influence on the behaviour of the beam, has to be taken into account.^{11,12}

In the case of thin-walled open sections, the average displacement of the cross section is normally written as^{2,11}

$$\bar{u} = u_D - y_D R_{12} - z_D R_{13} \tag{35}$$

where D indicates the sectorial origin. In this paper, however, the following expression is preferred

$$\bar{u} = u_s - y_s R_{12} - z_s R_{13} \tag{36}$$

It can readily be seen that both eqns (35) and (36) represent the displacement of the centroid as no warping occurs. Since eqn (36) results in simpler expressions, preference is given to it.

Constitutive equations for the normal force N and the bending moments \bar{M}_2 and \bar{M}_3 about the centroidal axes in the deformed state are obtained by integration over the cross section of the normal stress and its moments. The result is

$$N = EA \left(\bar{\epsilon} + \frac{1}{2} \frac{I_s}{A} \chi_1^2 \right)$$

$$\bar{M}_2 = EI_2 (\chi_2 + \frac{1}{2} \beta_2 \chi_1^2) \tag{37}$$

$$\bar{M}_3 = EI_3 (\chi_3 - \frac{1}{2} \beta_3 \chi_1^2)$$

The bimoment acting on the cross section in the deformed state is

$$B = \int_A (\psi \sigma_1) dA \tag{38}$$

where σ_1 is the normal stress in the deformed state. Integration of this expression yields

$$B = E\Gamma \chi_{1,x} + \frac{1}{2} E\Gamma \beta_u \chi_1^2 \tag{39}$$

where $,x$ means differentiation with respect to x .

Considering the integrand of the strain energy U (29) as a function of $\bar{\epsilon}$, χ_2 , χ_3 and $\chi_{1,x}$, it is readily verified that the constitutive expressions of N , \bar{M}_2 , \bar{M}_3 and B are the partial derivatives of the integrand with respect to $\bar{\epsilon}$, χ_2 , χ_3 and $\chi_{1,x}$ respectively.

Let M_{1x} be the partial derivative of the integrand of U with respect to χ_1 . The constitutive equation for M_{1s} may then be written as

$$M_{1s} = M_{1x} - B_{,x} \quad (40)$$

where M_{1s} is the torsional moment with respect to the shear centre. Calculating M_{1x} and $B_{,x}$ and substituting the result in eqn (40) yields

$$M_{1s} = GJ\chi_1 - E\Gamma\chi_{1,xx} + \chi_1(EI_s\bar{\epsilon} + EI_2\beta_2\chi_2 - EI_3\beta_3\chi_3) + 2EH\chi_1^3 \quad (41)$$

An alternative expression for M_{1s} which is often used^{1,12,16} is given in eqn (42) and is obtained by solving eqn (37) for $\bar{\epsilon}$, χ_2 and χ_3 and substituting into eqn (41):

$$M_{1s} = GJ\chi_1 - E\Gamma\chi_{1,xx} + \chi_1 \left(\frac{I_s}{A}N + \beta_2\bar{M}_2 - \beta_3\bar{M}_3 \right) + \frac{1}{2}E\chi_1^3 \left(4H - \frac{I_s^2}{A} - I_2\beta_2^2 - I_3\beta_3^2 \right) \quad (42)$$

Equation (42) is the same as the general non-linear differential equation for torsion obtained in Refs 11–13. The differential equations for bending which are used in Ref. 12 do not contain the terms with β_2 and β_3 (eqn (37)), but these terms should be taken into account if the cross section is asymmetric.

6 THE ROTATION TENSOR \mathbf{R}

To express the strain energy and load potential in terms of displacements, the rotation tensor \mathbf{R} is studied more closely.

The rigid rotation which transforms the triad $\bar{e}_1, \bar{e}_2, \bar{e}_3$ at a material point Q of the undeformed axis into the triad $\vec{i}_1, \vec{i}_2, \vec{i}_3$ at Q of the deformed axis is represented by a rotation tensor \mathbf{R} . In beam theory this rotation is often described in terms of so-called modified Euler angles. The rotation tensor is then broken down as

$$\mathbf{R} = \mathbf{R}_\gamma \cdot \mathbf{R}_\beta \cdot \mathbf{R}_\alpha \quad (43)$$

where $\mathbf{R}_\alpha, \mathbf{R}_\beta, \mathbf{R}_\gamma$ are respectively rotations of magnitude α, β, γ about the reference axes $\bar{e}_1, \bar{e}_2, \bar{e}_3$. This method, however, has some disadvantages (to be discussed at the end of this section) and therefore an alternative method is used in this paper.

An axis with unit vector $\vec{\eta}$, perpendicular to \vec{e}_1 and \vec{i}_1 , is chosen as rotation axis. When \vec{e}_1 and \vec{i}_1 are inclined at an angle ϕ , $\vec{\eta}$ can be written as

$$\vec{\eta} = (\sin \phi)^{-1} \vec{e}_1 \cdot \vec{i}_1 \tag{44}$$

A rotation tensor \mathbf{Q} , representing a rotation, of an arbitrary vector \vec{a} , over an angle γ , about an axis with unit vector \vec{e} can be expressed as¹⁴

$$\mathbf{Q} \cdot \vec{a} = (\cos \gamma) \vec{a} + (1 - \cos \gamma) \vec{e} \vec{e} \cdot \vec{a} + (\sin \gamma) \vec{e} \cdot \vec{a} \tag{45}$$

The rotation tensor \mathbf{R}_1 , which maps \vec{e}_1 on \vec{i}_1 , can thus be written as

$$\mathbf{R}_1 \cdot \vec{a} = (\cos \phi) \vec{a} + (1 - \cos \phi) \vec{\eta} \vec{\eta} \cdot \vec{a} + (\sin \phi) \vec{\eta} \cdot \vec{a} \tag{46}$$

where $\cos \phi = \vec{e}_1 \cdot \vec{i}_1$ and $(\sin \phi) \vec{\eta} = \vec{e}_1 \cdot \vec{i}_1$.

When \vec{e}_1 is mapped on \vec{i}_1 by \mathbf{R}_1 , $\mathbf{R}_1 \cdot \vec{e}_2$ and $\mathbf{R}_1 \cdot \vec{e}_3$ will generally not coincide with \vec{i}_2 and \vec{i}_3 respectively. Coincidence of $\mathbf{R}_1 \cdot \vec{e}_2$ and $\mathbf{R}_1 \cdot \vec{e}_3$ with \vec{i}_2 and \vec{i}_3 can be achieved by an additional rotation α about \vec{i}_1 . It is, however, also possible to rotate \vec{e}_2 and \vec{e}_3 over an angle β about \vec{e}_1 first such that, if \mathbf{R}_1 is applied to the rotated basis, $\mathbf{R}_1 \cdot \vec{e}_2^*$ and $\mathbf{R}_1 \cdot \vec{e}_3^*$ coincide with \vec{i}_2 and \vec{i}_3 . From a kinematical point of view both approaches are equivalent but, since the second approach leads to simpler expressions, preference is given to it. The rotation about \vec{e}_1 , over an angle β , is represented by a rotation tensor \mathbf{R}_2 . The rotation tensor $\mathbf{R} = \mathbf{R}_1 \cdot \mathbf{R}_2$ may be written (see Ref. 14) as

$$\mathbf{R} \cdot \vec{a} = [(\cos \phi) \mathbf{I} - \vec{e}_1 \vec{i}_1 + (1 - \cos \phi) \vec{\eta} \vec{\eta}] \cdot [(\cos \beta) \vec{a} + (\sin \beta) \vec{e}_1 \cdot \vec{a}] + \vec{i}_1 \vec{e}_1 \cdot \vec{a} \tag{47}$$

When strains are neglected compared with unity, the following components R_{ij} of \mathbf{R} with respect to $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are obtained:

$$\left[\begin{array}{cc} 1 + u'_s & -v'_s \cos \beta - w'_s \sin \beta & -w'_s \cos \beta + v'_s \sin \beta \\ v'_s & (1 + u'_s) \cos \beta + b(w'_s{}^2 \cos \beta - w'_s v'_s \sin \beta) & -(1 + u'_s) \sin \beta + b(-w'_s v'_s \cos \beta - w'_s{}^2 \sin \beta) \\ w'_s & (1 + u'_s) \sin \beta + b(-w'_s v'_s \cos \beta + v'_s{}^2 \sin \beta) & (1 + u'_s) \cos \beta + b(v'_s{}^2 \cos \beta + w'_s v'_s \sin \beta) \end{array} \right] \tag{48}$$

where

$$(1 + u'_s) = \cos \phi = \sqrt{1 - v'_s{}^2 - w'_s{}^2}$$

$$\text{and } b = \frac{1 - \cos \phi}{\sin^2 \phi} = \frac{1 - \sqrt{1 - v'_s{}^2 - w'_s{}^2}}{v'_s{}^2 + w'_s{}^2}$$

7 CURVATURES

The rotation tensor \mathbf{R} consists of two successive rotations ($\mathbf{R} = \mathbf{R}_1 \cdot \mathbf{R}_2$). $\mathbf{R}^C \cdot (d\mathbf{R}/dx) \cdot \vec{a}$ may therefore also be written as

$$\begin{aligned} \mathbf{R}^C \cdot \frac{d\mathbf{R}}{dx} \cdot \vec{a} &= \mathbf{R}_2^C \cdot \mathbf{R}_1^C \cdot \left(\frac{d\mathbf{R}_1}{dx} \cdot \mathbf{R}_2 + \mathbf{R}_1 \cdot \frac{d\mathbf{R}_2}{dx} \right) \cdot \vec{a} \\ &= (\mathbf{R}_2^C \cdot \vec{\mu} + \vec{\gamma}) \cdot \vec{a} \end{aligned} \quad (49)$$

where $\vec{\mu}$ and $\vec{\gamma}$ are the axial vectors of $\mathbf{R}_1^C \cdot (d\mathbf{R}_1/dx)$ and $\mathbf{R}_2^C \cdot (d\mathbf{R}_2/dx)$ respectively.

The axial vector \vec{k} of a skew tensor $\mathbf{Q}^C \cdot (d\mathbf{Q}/dx)$, with \mathbf{Q} given as

$$\mathbf{Q} \cdot \vec{a} = (\cos \gamma) \vec{a} + (1 - \cos \gamma) \vec{e} \vec{e} \cdot \vec{a} + (\sin \gamma) \vec{e} \cdot \vec{a}$$

can be written (see Ref. 14) as

$$\vec{k} = \left[\frac{d\gamma}{dx} \vec{e} + (1 - \cos \gamma) \left(\frac{d\vec{e}}{dx} \cdot \vec{e} \right) + \sin \gamma \frac{d\vec{e}}{dx} \right] \quad (50)$$

With eqn (50) the following curvature expressions (see Ref. 14) can be derived

$$\begin{aligned} \chi_1 &= \left[\beta' + \frac{1 - \cos \phi}{\sin^2 \phi} (w_s' v_s'' - w_s'' v_s') \right] \\ \chi_2 &= \cos \beta \left[-w_s' - \frac{1 - \cos \phi}{\sin^2 \phi \cos \phi} (v_s' v_s'' w_s' + w_s'^2 w_s'') \right] \\ &\quad + \sin \beta \left[v_s'' + \frac{1 - \cos \phi}{\sin^2 \phi \cos \phi} (v_s'^2 v_s'' + w_s' w_s'' v_s') \right] \\ \chi_3 &= \cos \beta \left[v_s'' + \frac{1 - \cos \phi}{\sin^2 \phi \cos \phi} (v_s'^2 v_s'' + w_s' w_s'' v_s') \right] \\ &\quad + \sin \beta \left[w_s'' + \frac{1 - \cos \phi}{\sin^2 \gamma \cos \phi} (v_s' v_s'' w_s' + w_s'^2 w_s'') \right] \end{aligned} \quad (51)$$

where

$$\cos \phi = (1 - v_s'^2 - w_s'^2)^{1/2} \quad \text{and} \quad \sin^2 \phi = (v_s'^2 + w_s'^2)$$

As already mentioned, the rotations in beam theory are often described in

terms of modified Euler angles. If these angles are expressed in terms of transverse displacements, χ_1 may be written (see Ref. 7) as

$$\chi_1 = \varphi' + \left(v_s'' w_s' + \frac{v_s' w_s''}{1 - w_s'^2} \right) \frac{1}{(1 - v_s'^2 - w_s'^2)^{1/2}} \tag{52}$$

where φ is an Euler angle. Equation (52) is asymmetric in v_s and w_s . That this may lead to confusion is shown in Ref. 11.

If the method of this paper is used to describe the rotations, χ_1 is given by

$$\chi_1 = \beta' + \frac{1 - (1 - v_s'^2 - w_s'^2)^{1/2}}{v_s'^2 + w_s'^2} [v_s'' w_s' - w_s'' v_s'] \tag{53}$$

This equation is skew-symmetric in v_s and w_s as would be expected.

In Ref. 13 a second-order beam theory is derived in a totally different manner from that in this paper. The expression derived for χ_1 is

$$\chi_1 = \beta' + \frac{1}{2}(v_s'' w_s' - w_s'' v_s') \tag{54}$$

If in eqn (53) $(1 - v_s'^2 - w_s'^2)^{1/2}$ is approximated by $1 - \frac{1}{2}v_s'^2 - \frac{1}{2}w_s'^2$, the same expression is obtained.

Substitution of the curvature expressions eqn (51) and rotation matrix components eqn (48) into the potential energy functional eqn (34) renders a functional in terms of the displacements u_s , v_s , w_s and the rotation β . The only approximation applied so far is the replacement of terms of order $(1 + \epsilon)$ by unity, in accordance with the assumption of small strains. The expressions are therefore valid for beams exhibiting arbitrary large deflections and rotations.

8 SPECIAL THEORIES

In many practical problems the magnitude of the deflections and rotations is limited and expressions as accurate as eqns (50) and (53) are not needed. Therefore simplified expressions are derived in the following by neglecting higher order terms.

8.1 Beams exhibiting moderate deflections and large rotations

To determine the order of magnitude of the terms that should be retained in the case of moderate deflections, a side-step is made at this point, and the classical case of a beam loaded by a bending moment in its plane is studied.

$$\mathbf{R} = \begin{bmatrix} \sqrt{1-v'^2} & -v' & 0 \\ v' & \sqrt{1-v'^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{aligned} \vec{i}_1 &= (1+u')\vec{e}_1 + v'\vec{e}_2 \\ \cos\theta &= \vec{e}_1 \cdot \vec{i}_1 = (1-v'^2)^{1/2} \rightarrow \sin\theta = v' \\ \cos\theta \frac{d\theta}{dx} &= v'' \end{aligned} \quad (55)$$

$$\chi = \frac{d\theta}{dx} = v''(1-v'^2)^{-1/2}$$

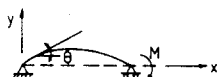


Fig. 3. Beam loaded by a bending normal.

Equations (55), which can be found in many standard text books on applied mechanics, represent the exact expressions when strains are neglected compared with unity. In the linear theory, the term $\cos\theta$ is put equal to unity; this results in

$$\mathbf{R} = \begin{bmatrix} 1 & -v' & 0 \\ v' & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \chi = v'' \quad (56)$$

To distinguish the theory of moderate deflections from the linear theory, at least one extra term should be taken into account in the approximation of $\cos\theta$. This results in

$$\cos\theta = 1 - \frac{1}{2}v'^2 \quad (57)$$

Substitution of this expression into (55) yields

$$\mathbf{R} = \begin{bmatrix} 1 - \frac{1}{2}v'^2 & -v' & 0 \\ v' & 1 - \frac{1}{2}v'^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \chi = v'' + \frac{1}{2}v''v'^2 \quad (58)$$

From this it can be concluded that, in the case of moderate deflections, terms quadratic in the derivatives of the deflections should be retained compared with unity.

Applying this type of approximation to the rotation matrix and curvature expressions derived in this paper yields

$$\mathbf{R} = \begin{bmatrix} 1 - \frac{1}{2}v_s'^2 - \frac{1}{2}w_s'^2 & -v_s' \cos \beta - w_s' \sin \beta & -w_s' \cos \beta + v_s' \sin \beta \\ v_s' & (1 - \frac{1}{2}v_s'^2) \cos \beta - \frac{1}{2}w_s' v_s' \sin \beta & -(1 - \frac{1}{2}v_s'^2) \sin \beta - \frac{1}{2}w_s' v_s' \cos \beta \\ w_s' & (1 - \frac{1}{2}w_s'^2) \sin \beta - \frac{1}{2}w_s' v_s' \cos \beta & (1 - \frac{1}{2}w_s'^2) \cos \beta + \frac{1}{2}w_s' v_s' \sin \beta \end{bmatrix} \quad (59)$$

$$\begin{aligned} \chi_1 &= \beta' + \frac{1}{2}(w_s' v_s'' - w_s'' v_s') \\ \chi_2 &= \cos \beta [-w_s'' - \frac{1}{2}(w_s'^2 w_s'' + v_s' v_s'' w_s')] + \sin \beta [v_s'' + \frac{1}{2}(v_s'^2 v_s'' + w_s' w_s'' v_s')] \\ \chi_3 &= \cos \beta [v_s'' + \frac{1}{2}(v_s'^2 v_s'' + w_s' w_s'' v_s')] + \sin \beta [w_s'' + \frac{1}{2}(v_s' v_s'' w_s' + w_s'^2 w_s'')] \end{aligned} \quad (60)$$

The strain ϵ_s is now given by

$$\epsilon_s = u_s' + \frac{1}{2}v_s'^2 + \frac{1}{2}w_s'^2$$

The rotation matrix and curvature expressions for a beam loaded by bending in its plane in the case of moderate deflections eqn (58) can be obtained from eqns (59) and (60) by putting β and w_s equal to zero.

The rotation matrix and curvatures for problems with moderate deflections and moderate rotations can be obtained from eqns (59) and (60) when in these expressions $\cos \beta$ and $\sin \beta$ are replaced by

$$\cos \beta = 1 - \frac{1}{2}\beta^2 \text{ and } \sin \beta = \beta - \frac{1}{6}\beta^3 \quad (61)$$

For problems with moderate deflections and small rotations $\cos \beta$ and $\sin \beta$ may be replaced by

$$\cos \beta = 1 \text{ and } \sin \beta = \beta \quad (62)$$

8.2 Beams exhibiting small deflections and large rotations

For this specific class of problems only terms linear in the derivatives of v_s and w_s have to be retained in the rotation matrix and curvatures. This implies

$$\mathbf{R} = \begin{bmatrix} 1 & -v_s' \cos \beta - w_s' \sin \beta & -w_s' \cos \beta + v_s' \sin \beta \\ v_s' & \cos \beta & -\sin \beta \\ w_s' & \sin \beta & \cos \beta \end{bmatrix} \quad (63)$$

$$\begin{aligned} \chi_1 &= \beta' \\ \chi_2 &= -\cos \beta w_s'' + \sin \beta v_s'' \\ \chi_3 &= \cos \beta v_s'' + \sin \beta w_s'' \\ \epsilon_s &= u_s' \end{aligned} \quad (64)$$

Comparison of eqns (61), (62), (65), (66) with the results of Ref. 11 shows that, due to the neglect of the geometric torsion in that paper, the results are only valid for small deflections and large rotations, according to the terminology of this paper.

The rotation matrix and curvatures for problems with small deflections and moderate rotations can be obtained from eqns (63) and (64) when $\cos \beta$ and $\sin \beta$ are replaced by

$$\cos \beta = 1 - \frac{1}{2}\beta^2 \quad \text{and} \quad \sin \beta = \beta - \frac{1}{6}\beta^3 \quad (65)$$

If small deflections and small rotations are considered, only first-order terms in β , β' and the derivatives of v_s and w_s have to be taken into account. Equations (48) and (51) then reduce to the following well known forms:

$$\mathbf{R} = \begin{bmatrix} 1 & -v'_s & -w'_s \\ v'_s & 1 & -\beta \\ w'_s & \beta & 1 \end{bmatrix} \quad \text{and} \quad \begin{aligned} \chi_1 &= \beta' \\ \chi_2 &= -w''_s \\ \chi_3 &= v''_s \\ \epsilon_s &= u'_s \end{aligned} \quad (66)$$

9 COMPARISON WITH OTHER RESULTS

During the last 15 years several articles dealing with the non-linear flexural-torsional behaviour of beams have appeared in the literature. Almost all these articles are limited to a special class of deformations and/or beams. In Ref. 14 it is shown that these special theories can be derived from the general expressions of this paper in a consistent manner, by neglecting specific terms.

10 WARPING

In the case of slender beams, the warping displacements are mostly written as

$$f(x, y, z) = \chi_1(x)\psi(y, z) \quad (67)$$

However, by postulating that the amplitude of the warping displacement equals χ_1 , it is also postulated that, at a section where warping is prevented, the strains e_{12} and e_{13} have to vanish too (eqns (24) and (25)). This is certainly not the case in reality and should in general not be assumed, since these

strains are proportional to the stresses σ_{12} and σ_{13} . To overcome this problem, an expression similar to the one proposed by Reissner¹⁷ for the case of non-uniform linear torsion can be used:

$$f(x, y, z) = g(x)\psi(y, z) \quad (68)$$

where $g(x)$ is a function yet to be determined.

Reissner¹⁷⁻¹⁹ showed that, in the case of non-uniform linear torsion of thin-walled beams with open cross sections, the practical improvement gained by working with eqn (68) instead of eqn (67) will in general be negligible. For non-uniform linear torsion of thin-walled beams with closed or partly closed cross sections, however, the more accurate eqn (68) leads to results which are quite different from what would follow from a use of eqn (67). In the case of beams with an arbitrary cross section undergoing both bending and non-uniform torsion, eqn (68) must also be expected to lead to more accurate results than eqn (67). The influence of this alternative warping approach is at present being investigated at the Eindhoven University of Technology.

11 CONCLUSIONS

A potential energy functional for non-linear flexural-torsional behaviour of straight elastic beams with arbitrary cross sections has been presented. The functional is generally applicable to situations where the strains are small and the Bernoulli hypotheses are valid. It has been shown that the potential energy expressions for special theories can be derived from this general expression in a consistent manner, by neglecting specific terms.

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