Modelling and testing shocks

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Abstract. Suppose a device is subjected to a sequence of shocks. This paper deals with the relationship between the life distribution of the device and the probability of surviving the first k ($k=0, 1, 2, \ldots$) shocks, using cumulative damage models as introduced by Esary *et al.* (1973).

Some attention is devoted to a classification of life distributions, tests of exponentiality versus IFR and NBU, and ML estimation of the survival distribution.

1 Introduction

Suppose a device is subjected to a sequence of shocks, and eventually some shock will cause failure. Let $N(t)$ denote the number of shocks which have arrived by time t, and let $\bar{P}_k$ be the probability of surviving the first k shocks, $k=0, 1, 2 \ldots$, where $\bar{P}_k$ is a deterministic function of k. Then the probability $\bar{F}(t)$ that the device survives beyond time t can be represented in the form

$$\bar{F}(t)=\sum_{k=0}^{\infty} \bar{P}_k P\{N(t)=k\},\ t \geqq 0 \tag{1}$$

Shock models of this kind have been considered by a number of authors under all kinds of assumptions. The results centre around proving that subject to suitable assumptions on the point process $\{N(t)\}$ of shocks, various discrete reliability characteristics of the $\{\bar{P}_k\}$ sequence which arise naturally out of physical considerations are inherited by the continuous survival probability $\bar{F}(t)$, in the sense that if the shock survival probabilities $\{\bar{P}_k\}$ belong to a discrete version of one of the life distribution classes then, under appropriate assumptions the continuous time survival probability $\bar{F}(t)$ belongs to the continuous version of that class. In Section 2 the most important classes of life distributions based on notions of aging will be defined, and some related theorems will be stated.

The first systematic treatment of shock models was given by Esary *et al.* in 1973. Since then their ideas and results have been extended by many authors. The original Esary *et al.* (1973) models, and some generalisations will be dealt with in Section 3.

In Section 4 we shall discuss some tests meant to assess whether a life distribution belongs to a particular class of distributions. The benefits of determining that the survival probability $\bar{F}(t)$ belongs to a certain class are:

(i) the usual Markov bounds on the survival probability can be improved on. We shall not go into this subject, we refer to the key papers by Barlow & Marshall (1964);
(ii) more precise estimators come available (*cf* Section 5).

Other shock models can be found in Nakagawa (1976), Feldman (1976), Zuckerman (1978) and Laud & Saunders (1981). For applications we refer to Taylor (1975), Feldman (1977), Lu & Saeks (1979) and Yamada (1980). For generalisations to multivariate models we refer to Marshall & Olkin (1967), Esary & Marshall (1974), Marshall & Shaked (1979) and Block & Savits (1980).

We adapt ourselves to the usage in this field of writing increasing (↑) for nondecreasing, and decreasing (↓) for nonincreasing. Furthermore, we assume throughout that a life distribution F satisfies $F(0)=0$.

We do not give any proofs, we just present some models and results to give a global introduction to the subject.

2 Classes of life distributions

In this section we shall give a classification of life distributions F, and state some important consequences. First we give two definitions.

Definition 1. A function g(x) defined on $[0,\infty)$ is called *starshaped* if
—$g(x)/x \uparrow$ in $x \geqq 0$, or equivalently
—$g(ax) \leqq a g(x)$ for $0 \leqq a \leqq 1$, $x \geqq 0$

Definition 2. A function $g(x)$ defined on $[0,\infty)$ such that
$g(x+y) \geqq g(x)+g(y)$ for all $x \geqq 0$, $y \geqq 0$ is called *superadditive.*

Now we are in the position to give the classification. In that definition we tacitly assume that the variables are restricted to avoid zero denominators.

Definition 3. A distribution F or a survival function $\bar{F}$, $\bar{F}(t)=1-F(t)$, is said to be or to have

(i) an *increasing failure rate* (IFR) if
—$\bar{F}(x|t) \overset{D}{=} \bar{F}(t+x)/\bar{F}(t) \downarrow$ in $t \geqq 0$ whenever $x>0$, or equivalently
—the hazard function $H(t) \overset{D}{=} -\log \bar{F}(t)$ is convex
If F has a density f, this is equivalent to the condition
—the failure rate $r(t) \overset{D}{=} f(t)/\bar{F}(t) \uparrow$ in $t \geqq 0$

(ii) an *increasing failure rate average* (IFRA) if
—the hazard function $H(t)$ is starshaped, or equivalently
—$\bar{F}^{1/t}(t) \downarrow$ in $t>0$, or equivalently
—$\bar{F}(bx) \geqq \bar{F}^{b}(t)$ for all $0<b<1$ and $t \geqq 0$.
When the failure rate r exists, then $H(t)={}_0\int^t r(u)du$, whence the condition is equivalent with ${}_0\int^t r(u)\, du/t \uparrow$ in $t>0$.

(iii) a *decreasing mean residual life* (DMRL) if

${}_0\int^\infty \bar{F}(x|t)\, dx \downarrow$ in $t \geqq 0$

(iv) *new better than used* (NBU) if
—$\bar{F}(x) \geqq \bar{F}(x|t)$ for all x, $t \geqq 0$, or equivalently
—the hazard function H(t) is superadditive

(v) *new better than used in expectation* (NBUE) if

${}_0\int^\infty \bar{F}(x)\, dx \geqq {}_0\int^\infty \bar{F}(x|t)\, dx$ for all $t \geqq 0$

(vi) *harmonic new better than used in expectation* (HNBUE) if

F has finite expectation $\mu={}_0\int^\infty \bar{F}(x)\, dx$ and

${}_t\int^\infty \bar{F}(x)\, dx \leqq \mu \exp(-t/\mu)$ for all $t \geqq 0$.

The following implications hold, and no additional implications exist among these classes.

```
        IFRA  →  NBU
       ↗             ↘
IFR                    NBUE→HNBUE
       ↘             ↗
          DMRL
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Discrete versions of these classes may be formulated as follows.

Definition 4. A discrete survival probability

$$\bar{P}_k = \sum_{j=k+1}^{\infty} p_j$$

with support on $\{0, 1, 2, \ldots\}$ and with frequency function $p_k = \bar{P}_{k-1} - \bar{P}_k$ for $k=1, 2, 3, \ldots$, and $p_0 = 1 - \bar{P}_0$ is said to be or to have

(i) IFR if $\bar{P}_{k+1}/\bar{P}_k \downarrow$ in k

(ii) IFRA if $\bar{P}_k^{-1/k} \downarrow$ in k

(iii) DMRL if $\sum_{j=k}^{\infty} \bar{P}_j/\bar{P}_k \downarrow$ in k

(iv) NBU if $\bar{P}_{k+l} \leqq \bar{P}_k \bar{P}_l$

(v) NBUE if $\sum_{j=k}^{\infty} \bar{P}_j \leqq \bar{P}_k \sum_{j=0}^{\infty} \bar{P}_j$

(vi) HNBUE if $\mu = \sum_{k=0}^{\infty} \bar{P}_k$ is finite and

$$\sum_{j=k}^{\infty} \bar{P}_j \leqq \mu(1-\mu^{-1})^k, \quad k=0, 1, 2, \ldots$$

Dual classes of distributions may be defined by reversing the direction of monotonicity or inequality in the definitions 3 and 4. This leads to

(i) decreasing failure rate (DFR),
(ii) decreasing failure rate average (DFRA),
(iii) increasing mean residual life (IMRL),
(iv) new worse than used (NWU),
(v) new worse than used in expectation (NWUE),
(vi) harmonic new worse than used in expectation (HNWUE).

As we will see in the following section, these classes of distributions arise in a natural way in shock models. They also arise in a reliability context. In reliability analysis one may calculate the reliability of a complex system starting with the reliability of the components. If all components have life distributions belonging to a certain class, then one would like to conclude that the life distribution of the entire system belongs to the same, or a similar, class, because then statistical inference procedures specially developed for that class become available. From this point of view the following theorem (*cf* Barlow & Proschan (1975) and Block & Savits (1976)) indicates the central role of the IFRA class.

Theorem (IFRA Closure Theorem)

The IFRA class of distributions is closed under convolution, and it is the smallest class containing the exponential distributions, closed under formation of coherent systems and taking limits in distribution. (For a definition of a coherent system, see Barlow & Proschan (1975, p. 6); for now it suffices to say that in a coherent system every component is relevant.)

We wind up this section with the remark that there is an extensive literature about probabilistic properties of the diverse classes of distributions characterised by the wearout type, but statistical inference for these classes has received considerably less treatment. Some of the statistical procedures are discussed in later sections.

3 Shock models

In the Esary *et al.* (1973) paper for the first time the relation between properties of the survival function of a single device and the stochastic mechanism leading to failure was studied in a systematic way. Since, the ideas and results set forth have been extended by many authors. All results are of the type: given a specified stochastic damage or wear process, the survival function of the device is IFR (IFRA, NBU, etc.).

In this section we shall first present the Esary *et al.* (1973) model in some detail, and then we shall mention several generalisations.

3.1 The Esary et al. *(1973) model*

A device is subjected to a sequence of shocks occurring randomly in time as events in a Poisson process. Let the device have a probability $\bar{P}_k$ of surviving the first k shocks, $k=0, 1, 2, \ldots$ Then the probability $\bar{F}(t)$ that the device survives beyond time t can be represented in the form

$$\bar{F}(t)=\sum_{k=0}^{\infty}\bar{P}_k\, e^{-\lambda t}(\lambda t)^k/k!,\ t\geqq 0 \tag{1}$$

for some $\lambda>0$.
This formulation leads to the assumption

$$1=\bar{P}_0\geqq\bar{P}_1\geqq\bar{P}_2\geqq\ldots \tag{2}$$

The probability of 'failure on the kth shock' is given by

$$p_k=\bar{P}_{k-1}-\bar{P}_k,\ k=1, 2, \ldots \tag{3}$$

$$p_0=0$$

Since we want $\bar{F}(t)$ to be a survival function, we assume

$$\bar{P}_k\to 0 \text{ as } k\to\infty.$$

The relation between the reliability characteristics of the $\{\bar{P}_k\}$ sequence and the survival probability $\bar{F}(t)$ is given in the following theorem.

Theorem 1

Suppose that:

$$\bar{F}(t)=\sum_{k=0}^{\infty}\bar{P}_k\, e^{-\lambda t}(\lambda t)^k/k!,\ t\geqq 0$$

where $1=\bar{P}_0\geqq\bar{P}_1\geqq\bar{P}_2\geqq\ldots$ Then

(a) F is IFR if $\{\bar{P}_k\}$ is discrete IFR
(b) F is IFRA if $\{\bar{P}_k\}$ is discrete IFRA
(c) F is DMRL if $\{\bar{P}_k\}$ is discrete DMRL
(d) F is NBU if $\{\bar{P}_k\}$ is discrete NBU
(e) F is NBUE if $\{\bar{P}_k\}$ is discrete NBUE
(f) F is HNBUE if $\{\bar{P}_k\}$ is discrete HNBUE (Klefsjö, 1981b)

The proof depends heavily on the variation diminishing property of the totally positive kernel $K(k,t)=e^{-\lambda t}(\lambda t)^k/k!$

Now we shall give some physically motivated models and obtain properties of the $\{\bar{P}_k\}$ from these models. Using theorem 1, these properties are translated to properties of the survival function $\bar{F}(t)$.

Suppose that the kth shock to a device causes a nonnegative random damage X_k, and the kth shock is survived by the device if $X_1+\ldots+X_k$ does not exceed the threshold x of the device. Note that in this model damages accumulate additively. We consider four cases (A–D).

Case A. Let $X_1, X_2,\ldots$ be independent with common distribution G. Then

$$\bar{P}_k=G^{(k)}(x),\ k=0,1,\ldots \tag{4}$$

where $G^{(0)}$ is degenerate at 0, and $G^{(k)}$ denotes the kth convolution of G, $k=1, 2,\ldots$

Case B. Let successive shocks be independent, and let they become increasingly effective in causing damage. This means that for each z the distribution function $G_i(z)$ of the ith damage is decreasing in $i=1, 2,\ldots$ Then

$$\bar{P}_0=1, \text{ and } \bar{P}_k=G_1 * G_2 * \ldots * G_k(x),\ k=1,2,\ldots \tag{5}$$

where * denotes convolution.

Case C. Accumulation of damage may result in a loss of resistance to further damage. In this case damages are neither independent nor identically distributed. Let us now assume

$$P\{X_k>u|X_1,\ldots X_{k-1}\} \text{ depends on } X_1,\ldots,X_{k-1} \text{ only via } Z_{k-1}=X_1+\ldots+X_{k-1} \tag{6}$$

$$P\{X_k>u|Z_{k-1}=z\}\uparrow \text{ in } z\geqq 0 \tag{7}$$

$$P\{X_k>u|Z_{k-1}=z\}\uparrow \text{ in } k=1,2,\ldots, \text{ for every } z\geqq 0, \text{ where } Z_0=0 \tag{8}$$

Then

$$\bar{P}_k=P\{X_1+\ldots+X_k\leqq x\} \tag{9}$$

Case D. If there is a significant individual variation in ability to withstand shocks, and if there is no practical way to determine the threshold of an individual device, then the threshold must be regarded as a random variable. Let the distribution of the threshold be G_0, with G_0 IFRA. Assume that the damages $X_1, X_2,\ldots$ from successive shocks are mutually independent with common distribution G, and that $X_1, X_2, \ldots$ are independent of the threshold. The shock survival probabilities are given by

$$\bar{P}_k = {}_0\int^{\infty} G^{(k)}(x)\, d\, G_0(x),\ k=0,\ 1, \ldots \tag{10}$$

The results for these four additive damage shock models are given in the following theorem.

Theorem 2

The survival function $\bar{F}$ given by (1) is IFRA for each of the models A, B, C, and D.

In proving this theorem it suffices to prove that in all four cases

$$\bar{P}_k^{1/k} \downarrow \text{in } k=1, 2, \ldots \tag{11}$$

3.2 Generalisations of the Esary et al. *(1973) model*

It is not surprising that the first generalisations of the Esary *et al.* (1973) paper concentrate on the counting process $\{N(t)\}$ which governs the arrival of shocks, while adopting the cumulative damage principle. In this section we highlight some of these extensions.

In the first generalisation, A-Hameed & Proschan (1973), shocks occur according to a nonhomogeneous Poisson process with intensity function $\lambda(t)$, and mean value function $\Lambda(t) = {}_0\int^{\infty} \lambda(x)\, dx$, both defined on $[0,\infty)$. The survival probability $\bar{F}(t)$ is expressed in the form

$$\bar{F}(t) = \sum_{k=0}^{\infty} \bar{P}_k e^{-\Lambda(t)} \Lambda^k(t)/k!,\ t \geqq 0 \tag{12}$$

Theorem 3

Let $\bar{F}(t)$ be given by (12), where $1 = \bar{P}_0 \geqq \bar{P}_1 \geqq P_2 \geqq \ldots$

Then

(a) F is IFR if $\{\bar{P}_k\}$ is discrete IFR and $\Lambda(t)$ is convex
(b) F is IFRA if $\{\bar{P}_k\}$ is discrete IFRA and $\Lambda(t)$ is convex
(c) F is DMRL if $\{\bar{P}_k\}$ is discrete DMRL and $\Lambda(t)$ is starshaped
(d) F is NBU if $\{\bar{P}_k\}$ is discrete NBU and $\Lambda(t)$ is superadditive
(e) F is NBUE if $\{\bar{P}_k\}$ is discrete NBUE and $\Lambda(t)$ is starshaped
(f) F is HNBUE if $\{\bar{P}_k\}$ is discrete HNBUE and $\Lambda(t)$ is starshaped (Klefsjö, 1981b)

Ross (1981) generalises the IFRA result to models in which the total damage is not necessarily ΣX_i.

A-Hameed & Proschan also discuss a situation analogous to theorem 2, model D. Again, damages accumulate additively until total damage exceeds a random critical threshold, at which point the device fails. Suppose the jth shock causes damage X_j having distribution G_j with gamma density

$$g_j(x) = b^{a_j} x^{a_j - 1} e^{-bx} / \Gamma(a_j),\ x \geqq 0,\ a_j > 0$$

for $j=1, 2, \ldots$ Let the critical threshold have distribution G_0, then

$$\bar{P}_k = {}_0\int^{\infty} G_1 * \ldots * G_k(x)\, d\, G_0(x) \tag{13}$$

Define

$$A_k = \sum_{j=1}^{k} a_j,\ k=1, 2, \ldots$$

Theorem 4

Let $\bar{F}(t)$ be given by (12) and $\bar{P}_k$ by (13). Then

(a) F is IFR if G is IFR, and $\{A_k\}$ and $\Lambda(t)$ are convex
(b) F is IFRA if G is IFRA, and $\{A_k\}$ and $\Lambda(t)$ are starshaped
(c) F is DMRL if G is DMRL, $A_k \equiv k$, and $\Lambda(t)$ is convex
(d) F is NBU if G is NBU, $\{A_k\}$ and $\Lambda(t)$ are superadditive
(e) F is NBUE if G is NBUE, $A_k \equiv k$, and $\Lambda(t)$ is starshaped

In their 1975 paper, A-Hameed & Proschan extend their generalisation to the following pure birth process:

(i) shocks occur according to a Markov process
(ii) given that k shocks have occurred in $[0,t]$, the probability of a shock occurring in $(t, t+\Delta]$ is $\lambda_k \lambda(t)\Delta + o(\Delta)$, while the probability of more than one shock occurring in $(t, t+\Delta]$ is $o(\Delta)$. (If $\lambda(t) \equiv 1$ then this describes a stationary birth process for which the interarrival times between the shocks number k and $k+1$ are independent and exponentially distributed with expectation $1/\lambda_k$.) Klefsjö (1981a) slightly extends the IFRA and DMRL results given in A-Hameed & Proschan (1975). We combine the results in theorem 5 (although not in the most general formulation).

Theorem 5

Let

$$\bar{F}(t) = \sum_{k=0}^{\infty} \bar{P}_k P\{N(t)=k\} \tag{14}$$

with $\{N(t)\}$ a pure birth process. Then

(a) F is IFR if $\{\bar{P}_k\}$ is IFR, $\lambda_k\uparrow$, and $\lambda(t)\uparrow$
(b) F is IFRA if $\{\bar{P}_k\}$ is IFRA, $\lambda_k\uparrow$, and $\Lambda(t)$ starshaped
(c) F is DMRL if $\{\bar{P}_k\}$ is DMRL, $\lambda_k\uparrow$, and $\lambda(t)\uparrow$
(d) F is NBU if $\{\bar{P}_k\}$ is NBU, $\lambda_k\uparrow$, and $\Lambda(t)$ superadditive
(e) F is NBUE if

$$\bar{P}_k \sum_{j=0}^{\infty} (\bar{P}_j/\lambda_j) \geqq \sum_{j=k}^{\infty} (\bar{P}_j/\lambda_j),$$

$k=0, 1, \ldots$ and $\Lambda(t)$ starshaped

Block and Savits (1978) generalise the last two statements in theorem 5, giving sufficient conditions on the point process $\{N(t)\}$ of shocks, such that F is NBU (NBUE) whenever $\{\bar{P}_k\}$ is discrete NBU (NBUE); they show that a pure birth process satisfies these conditions.

Gottlieb (1980) gives sufficient conditions on $\{N(t)\}$ such that F is IFR (IFRA) under the assumption that as the cumulative damage increases the probability that any additional damage will cause failure increases.

Ghosh & Ebrahimi (1982) prove that F is IFR under suitable assumptions, starting with a generalised Poisson process $\{N(t)\}$.

4 Tests for IFR(A) and NBU

Consider the Esary *et al.* (1973) model. If the damage caused by shocks cannot be observed, it is not easy to determine whether the shocks already experienced by the device make it more likely to fail under the impact of future shocks or not. It may be reasonable to believe, e.g. because of the physical model, that either

(i) $\bar{P}_{k+1}/\bar{P}_k$ is constant, or
(ii) $\bar{P}_{k+1}/\bar{P}_k \downarrow$ in k, not constant

This means that the lifelength $F(t)$ is exponential, or IFR and not exponential. Therefore we would like to have a test for the exponential distribution that is particularly sensitive to IFR-alternatives or, more generally, sensitive to one of the life length classes given in section 2. In literature most papers concentrate on IFR(A) and NBU(E) alternatives. Klefsjö (1983a) discusses tests against HNBUE alternatives.

At the end of this section we shall discuss a goodness of fit test for the IFR family.

4.1 Exponentiality versus IFR(A)

An extensively discussed and recommended test for exponentiality versus IFR(A) was introduced by Epstein (1960) following Laplace (*cf* Barlow, 1968; Barlow & Proschan, 1969; S_1^* in Bickel & Doksum, 1969; and $\dot{G}_n$ in Gail & Gastwirth, 1978). Since it does not introduce any complications, we shall give the test in a form suitable for type II censored observations; the complete sample is just a special case.

Let $Z_{(1)} \leqq Z_{(2)} \leqq \ldots \leqq Z_{(n)}$ denote the order statistics from a random sample $Z_1, \ldots, Z_n$ (failure times of n devices) based on a distribution F. Then the normalised spacings are $D_1 = n\, Z_{(1)}$, $D_2 = (n-1)\,(Z_{(2)} - Z_{(1)}), \ldots, D_n = Z_{(n)} - Z_{(n-1)}$. The total time on test up to the kth order statistic is

$$T_k = \sum_{i=1}^{k} D_i \tag{1}$$

If the experiment is censored after observing the first r $(1 \leqq r \leqq n)$ failures then the null hypothesis H_0:F is exponential, is rejected in favour of the alternative hypothesis H_1:F is IFRA and not exponential, for large values of the so-called (cumulative) total time on test statistic.

$$T = \sum_{i=1}^{r-1} T_i/T_r \tag{2}$$

Barlow & Proschan (1969) proved that this test is unbiased against IFRA alternatives, and Klefsjö (1983a) showed that it is consistent against HNBUE alternatives. Under the null hypothesis T is, even for relatively small r, approximately

$N([3(r-1)]^{\frac{1}{2}},(r-1)/12)$, because $T_1,\ldots,T_{r-1}$ can be considered as the order statistics from a uniform distribution on $(0,T_r]$.

Although many alternative tests have been proposed (some recent references are Lee *et al.*, 1980; Deshpande, 1983; Klefsjö, 1983b; Kochar, 1985), there is no evidence that any of these tests is to be preferred to Epstein's tests.

4.2 Exponentiality versus NBU

Hollander & Proschan (1972) give a test for H_0: F is exponential, versus H'_1: F is NBU and not exponential. Their test is based on the following measure of the deviation of F from exponentiality

$$\gamma(F)=\iint\{\bar{F}(x)\,\bar{F}(y)-\bar{F}(x+y)\}\,d\,F(x)\,d\,F(y) \qquad (3)$$

Their test statistic is

$$J_n=2[n(n-1)\,(n-2)]^{-1}\,\Sigma\,\Psi\,(X_i,\,X_j+X_k) \qquad (4)$$

where

$$\Psi(a,b)=1,\ a>b \qquad (5)$$
$$=0,\ a\leqq b$$

and the summation is over all triples (i, j, k) of three integers such that $1\leqq i, j, k\leqq n$, $i\neq j$, $i\neq k$, and $j<k$. The test rejects for small J_n values, it is unbiased and consistent. Because of power considerations, it is not advised to use this test against IFRA alternatives.

For more tests against NBU(E), we refer to Hollander & Proschan (1975), who also give a test for H_0 versus DMRL alternatives, Koul (1977 and 1978), and Souza Borges *et al.* (1984).

4.3 Goodness of fit to the IFR family

Testing goodness of fit to one of the life distribution classes has received little attention. Graphical tests are of some help (*cf* Nelson (1972) for hazard plots, and Barlow & Campo (1975) for total time on test plots), but the only analytical contributions seem to be the papers Tenga & Santner (1984) and Santner & Tenga (1984), dealing with the IFR family. These authors study Kolmogorov-Smirnov tests and area tests based on the distance between the cumulative hazard function and its greatest convex minorant, as well as tests based on the total time on test statistic. They discuss not only the complete sample case but also several types of censoring. For the complete sample they recommend the test: reject F is IFR for large values of

$$\max_{1<j<n}\,[j/n-G_n(T_j/T_n)] \qquad (6)$$

where G_n is the greatest convex minorant, and T_k is given by (1). The test is consistent.

5 ML estimation of IFR distributions

Let $Z_{(1)}\leqq Z_{(2)}\leqq\ldots\leqq Z_{(n)}$ be the order statistics from a random sample from an unknown IFR distribution F. Since F is known to be IFR, it seems worthwhile to try to find a better estimator of F than the usual empirical distribution function. It is not possible to find a maximum likelihood estimator for F within the class of IFR distributions, because $f(Z_i)$ can be arbitrary large. Marshall & Proschan (1965) give the following solution.

Consider the class of IFR distributions with failure rates bounded by a constant m. Within the class there is a unique ML solution. It can be proved that these ML estimators $\hat{F}_n^m$ converge in distribution as $m \to \infty$ to an IFR distribution $\hat{F}_n$. This $\hat{F}_n$ is called the ML estimator of F. It works out that the density $\hat{f}_n^m$ and the failure rate $\hat{r}_n^m$ of $\hat{F}_n^m$ converge to the density $\hat{f}_n$ and failure rate $\hat{r}_n$ of $\hat{F}_n$. Furthermore, $\hat{F}_n$, $\hat{f}_n$ and $\hat{r}_n$ are consistent estimators of F, f and r, respectively. Prakasa Rao (1970, theorem 6.1) gives the asymptotic distribution of $\hat{r}$. Padgett & Wei (1980) give an analogous ML estimator for IFR distributions in the case of arbitrary right censored data. Contrary to the Kaplan-Meier product-limit estimator, the ML estimator $\hat{F}_n$ is continuous and well-defined for all $t \geqq 0$.

Mykytyn & Santner (1981) discuss several types of censorship via the Kiefer & Wolfowitz (1956) approach, i.e. let C be a class of distribution functions, $F_1, F_2 \varepsilon C$, and $f(x; F_1, F_2) = d\,F_1(x)/d(F_1+F_2)$ the Radon-Nikodym derivative of F_1 with respect to F_1+F_2; then $\hat{F}$ is called a generalised ML estimate if $f(x; \hat{F}, F) \geqq f(x; F, \hat{F})$ for all $F \varepsilon C$, where x is a sample result. It is easily seen that this generalised ML estimate coincides with the usual one when the class C is dominated by a sigma-finite measure. Furthermore, we note that the Marshall & Proschan ML-estimator coincides with the Kiefer & Wolfowitz one in the case of a complete sample.

A remarkable fact is proven by Boyles, Marshall & Proschan (1985). They show that, in general, the ML estimator for an IFRA distribution converges a.s. to a function that is not the true distribution. Using isotonic regression, Barlow, Bartholomew, Bremner & Brunk (1972) provide a consistent estimator of IFRA distributions.

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