Solution to problem 350

Citation for published version (APA):
van Lint, J. H. (1974). Solution to problem 350. *Nieuw Archief voor Wiskunde*, *22*, 94-111.

Document status and date:
Gepubliceerd: 01/01/1974

Document Version:
Uitgevers PDF, ook bekend als Version of Record

SOLUTION by J.H. VAN LINT.

The reduced residue system $\mathrm{mod}(a^n-1)$ forms a multiplicative group of order $\phi(a^n-1)$. Since the set $\{a^i \mid i=0,1,2,\ldots,n-1\}$ forms a subgroup, it follows that n divides $\phi(a^n-1)$.

350. Let $A := \{1,2,\ldots,2n\}$. Let $C_1,C_2,\ldots,C_\ell$ be subsets of A with the property that for any partition of A into n disjoint pairs

$$A = \{x_1,y_1\} \cup \{x_2,y_2\} \cup \ldots \cup \{x_n,y_n\} ,$$

there are n distinct indices $i_1,i_2,\ldots,i_n$ such that

$$\{x_k,y_k\} \subset C_{i_k} \qquad (k = 1,2,\ldots,n) .$$

(This implies that $\ell \geq n$.) Show that for $n > 3$ it is possible to choose the sets $C_1,C_2,\ldots,C_\ell$ in such a way that none of the numbers $1,2,\ldots,2n$ is an element of more than $[\frac{3n+2}{4}]$ of the sets C_i.

Give better upper and lower bounds for

$$\max_{1\leq i\leq 2n} |\{j \mid i \in C_j\}| .$$

(J.H. VAN LINT)

Solutions by D. BOLTON, P. ERDÖS, J.H. VAN LINT, P. SEYMOUR, J. SPENCER and R. TIJDEMAN. All solutions use P. HALL's theorem which we shall state first (cf. M. HALL, *Combinatorial theory*, Blaisdell, Waltham, 1967).

If $S_1,S_2,\ldots,S_\ell$ are subsets of a set S and $x_1,x_2,\ldots,x_\ell$ are distinct elements of S such that $x_i \in S_i$ $(i=1,2,\ldots,\ell)$, then the set $\{x_1,x_2,\ldots,x_\ell\}$ is called a *system of distinct representatives* or a *transversal* of the collection $S_1,S_2,\ldots,S_\ell$.

THEOREM (P. HALL). *Let* I *be a finite set of indices,* $I = \{1,2,\ldots,\ell\}$. *For each* $i \in I$ *let* S_i *be a subset of a set* S. *A necessary and sufficient condition for the existence of a system of distinct representatives of* $S_1,S_2,\ldots,S_\ell$ *is that for* $1 \leq k \leq \ell$ *and for every choice of* k *distinct indices* $i_1,i_2,\ldots,i_k$

$$|S_{i_1} \cup S_{i_2} \cup \dots \cup S_{i_k}| \geq k .$$

Consider the problem as stated above. Take $\ell = n$ and for $i = 1,2,\dots,n$ let $|C_i| = n+i$. Let P be a partition of A into n disjoint pairs and denote by S_i the set of pairs of P which are contained in the set C_i. Clearly, $|S_i| \geq i$ and hence HALL's condition is trivially satisfied. So these sets C_i satisfy the conditions of the problem. It is easy to see that the C_i can be chosen in such a way that the numbers $|\{j \mid i \in C_j\}|$ $(i=1,2,\dots,2n)$ are all equal to $[\frac{3n+1}{4}]$ or $[\frac{3n+1}{4}]+1$. This is just slightly poorer than the statement of the problem.

Before discussing the different solutions we mention a trivial lower bound for the case $\ell = n$. It is obvious that if $|C_i| < n$ for some i, then there exists a partition of A into n disjoint pairs such that none of these pairs is in C_i. Hence, $[\frac{n+1}{2}]$ is a lower bound for $\max_{1\leq i\leq 2n} |\{j \mid i \in C_j\}|$.

In the solution first submitted by J.H. VAN LINT (see below) the sets C_i were chosen to be $\{2s+2i-1,2s+2i,\dots,2n+2i-2\}$ for $i=1,2,\dots,n$, where the elements of C_i are interpreted mod 2n. Then s is chosen in such a way that HALL's condition is trivially satisfied and the result follows. By a complicated argument D. BOLTON and P. SEYMOUR (see below) showed that for this same choice of the sets C_i (with a different value of s) one even has

$$\max_{1\leq i\leq n} |\{j \mid i \in C_j\}| = [\tfrac{2}{3} n] + 1 .$$

A much simpler proof of this same result, using a much more complicated construction of the sets C_i, was submitted by R. TIJDEMAN. The construction uses a block design on 4 points. The idea was generalized by J.H. VAN LINT using a block design on 8 points. The result was that the constant $\frac{2}{3}$ could be lowered to $\frac{9}{14}$ (see below). By an even more complicated argument due to R. TIJDEMAN, further improvements are possible. The details are too tedious to state here.

All solutions mentioned up to now made the extra restriction $\ell = n$. Subsequently, it was shown by P. ERDÖS and independently by J. SPENCER that, if ℓ is arbitrary, the upper bound can be replaced by $O(n^{\frac{1}{2}} \log n)$. Finally, N.G. DE BRUIJN proved that $n^{\frac{1}{2}}$ is a lower bound and improved the upper bound to $O((n \log n)^{\frac{1}{2}})$ (see below).

For arbitrary ℓ and large n there is a small gap left. If we make the restriction $\ell = n$ one can show that $\frac{1}{2}n+o(n)$ is an upper bound (see below).

SOLUTION by J.H. VAN LINT.

Let $s > 0$ be an integer. For $i=1,2,\dots,n$ we define

$$C_i := \{2s+2i-1, 2s+2i, \dots, 2n+2i-2\} ,$$

where the elements are to be considered mod 2n. Clearly, each element of A is in n-s of the subsets C_i. Hence, for every pair $\{a,b\} \subset A$, there are at least n-2s subsets C_i such that $\{a,b\} \subset C_i$. If the condition of the problem is not satisfied then by P. HALL's theorem there is a t $(1 \le t \le n-1)$ and a set of t+1 disjoint pairs of A

(1) $$\{x_1,y_1\} , \{x_2,y_2\} , \dots , \{x_{t+1},y_{t+1}\}$$

such that at most t of the subsets C_i contain at least one of these pairs. This implies

(2) $$t \ge n - 2s .$$

Any subset C_j which contains none of the pairs of (1) must satisfy

(3) $$|C_j| \le 2n - (t+1) .$$

From (3) we find

(4) $$t \le 2s - 1 .$$

Then from (4) and (2) we have

(5) $$4s \ge n + 1 .$$

From (5) we see that the condition of the problem is satisfied if $s < \frac{n+1}{4}$. In fact, even $s = \frac{n+1}{4}$ is sufficient unless equality holds in (2) and this can only happen if $t = 1$, which implies that $n = 3$. This proves the assertion of the problem.

In the following solution by D. BOLTON and P. SEYMOUR the terminology is slightly different from the one used above. Again, $\ell = n$. The C_i are represented by the vertices of the circuit graph D_n on n points. If we do this, then in the solution printed above each set $\{C_j \mid i \in C_j\}$ is a connected subgraph of D_n.

SOLUTION by D. BOLTON and P. SEYMOUR.

Let S be the set of n (> 0) vertices of the directed circuit graph D_n. An *interval* in S is a nonempty subset of S for which the induced graph is connected. Each interval, not S, induces a natural ordering on its elements and we write $\text{first}_L(X)$ for the first point of $X \cap L$ with respect to the ordering of the interval L. When X = L we just write first(L). Similarly, for last points we write $\text{last}_L(X)$.

For $J \subset S$, a *component* of J is a maximal interval subset of J. An *endpoint* of J is the first or last point of some component of J. We write k(J) for the number of components of J.

Let

$$\alpha := [\tfrac{2n}{3}] + 1 = \{\tfrac{2n+1}{3}\} . \qquad (1)$$

For $x \in S$, A_x, B_x, denote the intervals satisfying

$$|A_x| = |B_x| = \alpha \qquad (2)$$

and

$$\text{first}(A_x) = \text{last}(B_x) = x . \qquad (3)$$

There are n intervals of size α, but we take H to be a family of 2n such by having each interval exactly twice. For $X \in H$, $\hat{X}$ denotes the other copy of X in H.

A *pairing* of H is a partition of H into sets of size two. A *partial pairing* R of H *is a pairing of an even subfamily of* H. The subfamily is denoted $\cup R$.

THEOREM. *For any pairing* $(\{X_1,Y_1\},\ldots,\{X_n,Y_n\})$ *of* H, *the family* $(X_1 \cap Y_1,\ldots,X_n \cap Y_n)$ *has a transversal.*

PROOF. If the theorem is false, then by HALL's theorem there is a subset J of S and a partial pairing R such that

$$(4) \qquad \begin{cases} X \cap Y \subseteq J & \text{for all } \{X,Y\} \in R \\ \\ |R| > |J| \ . \end{cases}$$

If (J,R) satisfies (4) we say it is a c.e. (counter-example). For $n \le 3$, $\alpha = n$ and the theorem is trivially true. □

We make a number of observations concerning H, when $n > 3$.

(01) For $X,Y \in H$, $k(X \cap Y) = 1$ or 2.

(02) If $k(X \cap Y) = 2$, then each endpoint of X is an endpoint of $X \cap Y$.

(03) If $k(X \cap Y) = 1$, then $\text{first}(X \cap Y) = \text{first}(X(\text{or } Y))$ and $\text{last}(X \cap Y) = \text{last}(Y(\text{or } X))$.

(04) $|X \cap Y| \ge 2\alpha-n$.

(05) If $n > 4$, then $2\alpha-n \ge 3$, so at least one component of $X \cap Y$ has cardinality at least two.

(06) If $k(X \cap Y) = 2$, then we have equality in (04).

LEMMA 1 (Augmentation). *If* (J,R) *is a c.e. and* J *has a component* L *satisfying*

$$(5) \qquad 2\alpha-n \le |L|+2 < \alpha,$$

then there exists a c.e. (J',R') *where* J' *has a component strictly containing* L *and* $k(J') \le k(J)$.

PROOF. Let $x = \text{last}(S-L)$ and $y = \text{first}(S-L)$.

(i) Suppose $A_x \in \cup R$. Let $\{A_x,C\} \in R$, so $A_x \cap C \subseteq J$. Since $x \notin J$, $A_x \cap C$ is an interval (02) and so is in some component of J. It is not in L, for $\text{last}(A_x \cap C) = \text{last}(A_x)$ (03), so $A_x \cap C \subseteq L$ implies $A_x \subseteq L \cup \{x\}$, contradicting (5).

So $A_x \cap C \subseteq S-L-\{x,y\}$. But $A_x \supseteq L \cup \{x,y\}$, so $A_x - A_x \cap C \supseteq L \cup \{x,y\}$. By (04) and (5) we have

$$|A_x - A_x \cap C| \le \alpha - (2\alpha-n) = n-\alpha$$

so

$$n-\alpha \ge |L|+2 \ge 2\alpha-n \ ,$$

which contradicts (1).

(ii) $A_x \notin \cup R$. Similarly $\hat{A}_x, B_y, \hat{B}_y \notin \cup R$. Set $J' = J \cup \{x,y\}$ and $R' = R \cup \{\{A_x,B_y\},\{\hat{A}_x,\hat{B}_y\}\}$.

One component of $A_x \cap B_y$ is $L \cup \{x,y\}$. Let M be another. Then by (06)

$$2\alpha-n = |A_x \cap B_y| = |L|+|M|+2 \geq |L|+3 ,$$

which contradicts (5).

So $A_x \cap B_y = L \cup \{x,y\} \subseteq J'$ and (J',R') satisfies (4). □

LEMMA 2 (Reduction). *If* (J,R) *is a c.e. and* J *has a component* L *satisfying*

(6) $$2 \leq |L| \leq 2\alpha-n,$$

then there exists a c.e. (J',R') *such that* $J' \subset J$, $L \nsubseteq J'$, $J-J' \subseteq L$, *and* $k(J') \leq k(J)$. *If, on the other hand,* $L = \{x\}$ *then there exists a c.e.* (J,R^*) *where* $\{A_x,B_x\},\{\hat{A}_x,\hat{B}_x\} \in R^*$.

PROOF. Let $x = \text{first}(L)$ and $y = \text{last}(L)$. If $A_x,\hat{A}_x \notin \cup R$ then we may put $J' = J - \{x\}$ and $R' = R$. Similarly, $B_y,\hat{B}_y$. So suppose that $A_x,B_y \in \cup R$. We may further suppose that $\{A_x,B_y\} \in R$, for if not then we have $B_y \neq C$, $\{A_x,C\},\{B_y,D\} \in R$, say. Consider $R^* = R \cup \{\{A_x,B_y\},\{C,D\}\} - \{\{A_x,C\},\{B_y,D\}\}$. L is a component of $A_x \cap B_y$ and $|L| \leq 2\alpha-n$, so $A_x \cup B_y = S$. So

$$C \cap D = (C \cap A_x \cap D) \cup (C \cap B_y \cap D) \subseteq (C \cap A_x) \cup (B_y \cap D) \subseteq J .$$

Let M be another component of $A_x \cap B_y$. Since $\text{first}(S-L) \notin J$, we have $\text{first}(S-L) \notin C$. Since $C \neq B_y$, it follows that $y \notin C$. So $z_1 = \text{first}(B_y)$ is an endpoint of $C \cap B_y$. If x is an endpoint of $C \cap A_x$, then $C \cap A_x$ has one component properly contained in L and so has another component. In any event:

$$z_2 = \text{last}(A_x) \text{ is an endpoint of } C \cap A_x .$$

So $z_1,z_2 \in C$ are endpoints of M. Since $|M| < 2\alpha-n$, it follows that $M \subseteq C$. In particular, $M \subseteq C \cap A_x \subseteq J$, so

$$A_x \cap B_y = L \cup M \subseteq J$$

and (J,R^*) satisfies (4). So we suppose $\{A_x,B_y\} \in R$.

If $\hat{A}_x \notin \cup R$, then we put

$$J' = J - \{x\} \quad \text{and} \quad R' = R - \{\{A_x, B_y\}\} .$$

Thus we may suppose $\hat{A}_x, \hat{B}_y \in \cup R$. If $\{\hat{A}_x, \hat{B}_y\} \notin R$, then we argue as above, so we suppose $\{\hat{A}_x, \hat{B}_y\} \in R$.

So far we have not used the condition $|L| > 1$, so setting $x = y$ we have already done the second part of the lemma.

If $|L| \geq 2$ then $x \neq y$, so we set

$$J' = J - \{x,y\} \quad \text{and} \quad R' = R - \{\{A_x, B_y\}, \{\hat{A}_x, \hat{B}_y\}\}$$

which completes the proof of the lemma. □

PROOF of theorem. Suppose that (J,R) is a c.e. with least $k(J)$. Let ℓ be the size of the largest components of J and let L be a component of this size.

If $\ell \leq 2\alpha - n$ then by the reduction lemma we may suppose that $\ell = 1$. It follows (05) that $n = 4$. In this case $|J| = 2 = k(J)$ and $|\cup R| = 4$, so $|R| = 2$ which contradicts (4).

If $\ell > 2\alpha - n$ then by the augmentation lemma we may suppose that $\ell \geq \alpha - 2$. If M is a component of $J-L$ then

$$|M| + |L| \leq n-2 .$$

So $|M| \leq n-\alpha \leq 2\alpha-n-1$ and by the reduction lemma we may suppose that $|M| = 1$.

For $x \in L$ we define

$$U_x = \{\{X,Y\} \in R \mid \text{first}_L(X \cap Y) = x\}$$

and

$$V_x = \{\{X,Y\} \in R \mid \text{last}_L(X \cap Y) = x\} .$$

Also, we partition L into three sets in two ways:
Define σ to be 1 if $\ell = \alpha - 2$ and 0 otherwise.
Let P_1 (P_2) be the first (last) $\ell+1+\sigma-\alpha$ points of L.
Let R_1 (R_2) be the last (first) $2\alpha-n-1$ points of L, and Q_1 (Q_2) be the remaining $n-\alpha-\sigma$ points of L.
Evidently $|U_x| \leq 2$ for all $x \in L$. Also since $\{X,Y\} \in R$ implies

$|X \cap Y \cap L| \geq 2\alpha-n-1$ we have that, for $x \in R_1$-first(R_1), $|U_x| = 0$.

Let $z = \text{first}(R_1)$. If $|U_z| > 0$ then suppose $z = \text{first}_L(X \cap Y)$. Then $|X \cap Y - L| = 1$. Let $\{z'\} = X \cap Y - L$. By the reduction lemma, $\{A_{z'},B_{z'}\}$ and $\{\hat{A}_{z'},\hat{B}_{z'}\} \in R$. Clearly, $\text{last}(L) = \text{last}(A_{z'})$ and if $\{X,Y\} \in R$ such that $X \cap Y \ni \text{last}(L)$, then since $X \cap Y \not\ni \text{first}(S-L)$ we must have last(L) as the last point of X or Y. So $\{X,Y\}$ is either $\{A_{z'},B_{z'}\}$ or $\{\hat{A}_{z'},\hat{B}_{z'}\}$. It follows that

$$J' = J - \{z',\text{last}(L)\}$$

and

$$R' = R - \{\{A_{z'},B_{z'}\},\{\hat{A}_{z'},\hat{B}_{z'}\}\}$$

are such that (J',R') satisfies (4) but $k(J') < k(J)$, a contradiction. So for $x \in R_1$, $|U_x| = 0$. Similarly, for $y \in R_2$, $|V_y| = 0$.

Now for $x \in P_1$ there exists an $y \in P_2$ such that $A_x = B_y$. If, in addition, $|U_x| = 2$ then $\{A_x,C\},\{\hat{A}_x,D\} \in R$ for some $C,D \neq A_x$. But neither of these contribute to $|V_y|$ and no other members of $\cup R$ have y as an endpoint. So $|V_y| = 0$. Similary, if $|V_y| = 2$, then $|U_x| = 0$. Otherwise each is ≤ 1. In each case we have

$$|U_x| + |V_y| \leq 2 \ . \tag{8}$$

Now

$$2|R| = \sum_{x\in L} |U_x| + \sum_{y\in L} |V_y| =$$

$$= \sum_{x\in P_1} |U_x| + \sum_{y\in P_2} |V_y| + \sum_{x\in Q_1} |U_x| + \sum_{y\in Q_2} |V_y|$$

from (7). $|P_1| = |P_2|$, so from (8)

$$2|R| \leq 2|P_1| + 4|Q_1| \ ,$$

$$|R| \leq \ell+\sigma+1-\alpha+2(n-\alpha-\sigma) \leq \ell+2n+1-3\alpha \leq \ell \overset{\text{by (1)}}{\leq} |J| \ ,$$

which completes the proof of the theorem. □

SOLUTION by R. TIJDEMAN and J.H. VAN LINT.

We use the following notation. If $C \subset A$, then

$$C^* := \{\{i,j\} \mid i \in C,\ j \in C,\ i \neq j\} .$$

A subset $P = \{\{i_1,i_2\},\dots,\{i_{2n-1},i_{2n}\}\}$ of A^* is called a *pairing* of A, if $i_1,i_2,\dots,i_{2n}$ is a permutation of $1,2,\dots,2n$. $\mathcal{P}$ is the set of all pairings of A.

Given subsets $C_1,C_2,\dots$ of A and a pairing P we define $S_j^{(P)} := P \cap C_j^*$ $(j=1,2,\dots)$.

A set of subsets $\{C_1,C_2,\dots,C_n\}$ is *admissible* if the sets $S_1^{(P)},\dots,S_n^{(P)}$ have a system of distinct representatives for every $P \in \mathcal{P}$.

$N_i := N_i(C_1,C_2,\dots,C_n) :=$ the number of sets C_j which contain i $(j=1,2,\dots,n;\ i=1,2\dots,n)$.

$N := \max\{N_i \mid i=1,2,\dots,2n\}$.

The problem is to determine

$$N^* := \min\{N \mid \{C_1,\dots,C_n\} \text{ is admissible}\}.$$

By HALL's theorem, $\{C_1,C_2,\dots,C_n\}$ is admissible if and only if, for every $P \in \mathcal{P}$ and every $k \le n$, every k-tuple $C_{j_1},C_{j_2},\dots,C_{j_k}$ has the property

$$|S_{j_1}^{(P)} \cup S_{j_2}^{(P)} \cup \dots \cup S_{j_k}^{(P)}| \ge k .$$

LEMMA. *If* $P \in \mathcal{P}$, $C \subset A$ *and* $|C| = n+\ell$, *then* $|C^* \cap P| \ge \ell$.

PROOF. There are $n-\ell$ elements of A which are not in C and these elements occur in at most $n-\ell$ pairs of P. □

Let $28|n$. We divide A into 8 subsets $A_1,A_2,\dots,A_8$ of size $n/4$, where

$$A_i := \{(i-1)\tfrac{n}{4}+1,(i-1)\tfrac{n}{4}+2,\dots,i\tfrac{n}{4}\} .$$

Consider the block design BD($v = 8$, $k = 4$; $b = 14$, $r = 7$, $\lambda = 3$) on $A_1,A_2,\dots,A_8$ displayed in the following table.

Block numbers →	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	A_1	A_1	A_1				A_1	A_1	A_1				A_1	
	A_2	A_2				A_2		A_2		A_2		A_2		A_2
	A_3		A_3		A_3				A_3	A_3	A_3			A_3
	A_4				A_4	A_4	A_4				A_4	A_4	A_4	
				A_5	A_5	A_5		A_5	A_5	A_5			A_5	
			A_6	A_6	A_6		A_6	A_6				A_6		A_6
		A_7		A_7		A_7	A_7		A_7		A_7			A_7
		A_8	A_8	A_8						A_8	A_8	A_8	A_8	

We have numbered the blocks 1 to 14 (blocks vertically).

We now define subsets $C_1, C_2, \ldots, C_n$ of A. First split $1,2,\ldots,n$ into $\frac{1}{14}n$ *groups* of 14 consecutive integers. We number these groups with index t where $t = 1,2,\ldots,\frac{1}{14}n$. Let j be a number in group t, say $j = (t-1)\frac{n}{14} + m$. Then we define:

C_j consists of block number m in the table and furthermore 2t elements (we shall explain in the sequel how these 2t elements are to be chosen) from each set A_i which does not occur in the block. (We shall call the elements of C_j from the block "*standard*" elements, the other 8t elements are called "*additional*" elements of C_j.) A C_j of this kind is called "*a* C_j *of type* m".

Note that the total number of additional elements to be chosen from some A_i equals

$$7 \sum_{t=1}^{\frac{1}{14}n} (2t) = \frac{1}{2}n\left(\frac{1}{14}n+1\right) .$$

Since this number is divisible by n/4 we can make the subsequent choices of the additional elements in such a way that at the end of the construction all the numbers N_i $(i=1,2,\ldots,2n)$ are equal. We have

(1) $$N = \frac{1}{2n}\left\{n^2+8\cdot\frac{1}{2}n\left(\frac{1}{14}n+1\right)\right\} = \frac{9}{14}n + 2$$

for $\{C_1,C_2,\ldots,C_n\}$ constructed above.

Now consider a k-tuple $C_{j_1}, C_{j_2}, \ldots, C_{j_k}$ $(j_1 < j_2 < \ldots < j_k)$.

There are four possibilities to be considered:

(a) Suppose that all 14 types occur among these k sets. Clearly for every $P \in \mathcal{P}$ we then have

$$|S_{j_1}^{(P)} \cup \ldots \cup S_{j_k}^{(P)}| = n .$$

In fact, this is true if at least 12 different types occur (since the block design has $\lambda = 3$).

(b) Suppose there are at most 8 different types occurring among the sets $C_{j_1},\ldots,C_{j_k}$. Then we have:

j_k is a number in group t, where $t \geq 1 + [\frac{k-1}{8}]$,

and hence

$$|C_{j_k}| \geq n+k .$$

Then by lemma 1 we have $|S_{j_1}^{(P)} \cup \ldots \cup S_{j_k}^{(P)}| \geq |S_{j_k}^{(P)}| \geq k$ for every $P \in \mathcal{P}$.

(c) Suppose there are 10 or 11 types occurring among the sets $C_{j_1},\ldots,C_{j_k}$. Since no 2 blocks of the design have 3 elements in common, we see that at most one pair $\{A_i,A_j\}$ is not contained in the C's. Certainly, every A_i is in some C_{j_ν}. So w.l.o.g. we may assume that types 1, 2, 8 (and maybe one other type) do not occur, i.e. $\{A_1,A_2\}$ is missing. For any $P \in \mathcal{P}$ there are at least 3n/4 pairs $\{i,j\}$ of P occurring as a pair of standard elements in some C_{j_ν}.

We denote by α_i the number of sets C_{j_ν} among $C_{j_1},\ldots,C_{j_k}$ which are of type i ($i=1,\ldots,14$; $i\neq 1,2,8$). Consider a type which does occur and which contains all of A_1 or all of A_2, say (again w.l.o.g.) type 9. The set C_{j_ν} of type 9 with largest index contains at least $2\alpha_9$ pairs $\{i,j\}$ where $i \in A_1$ (standard) and $j \in A_2$ (additional). Hence

$$|S_{j_1}^{(P)} \cup \ldots \cup S_{j_k}^{(P)}| \geq \frac{3}{4}n + 2\alpha_9 \geq \frac{10}{14}n + \alpha_9 \geq$$

$$\geq (\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}+\alpha_{14}) + \alpha_9 = k .$$

(d) It remains to consider the possibility that there are 9 types which occur among the sets $C_{j_1},\ldots,C_{j_k}$. It is now possible that 2 pairs $\{A_i,A_j\}$ do not occur in any set. (If this is not so, we are finished by (c) above.)

Suppose 2 pairs do not occur. One can easily check that there are two typical cases:

(i) types 1, 2, 3, 8, 9 do not occur;

(ii) types 1, 2, 5, 8, 9 do not occur.

In case (i) 3n/4 pairs of any P occur as standard pairs in $S_{j_1}^{(P)} \cup \ldots \cup S_{j_k}^{(P)}$; in case (ii) n/2 pairs of any P occur as standard pairs in $S_{j_1}^{(P)} \cup \ldots \cup S_{j_k}^{(P)}$. In case (i) sets of type 11 and 14 improve the lower bound for $|S_{j_1}^{(P)} \cup \ldots \cup S_{j_k}^{(P)}|$; in case (ii) sets of type 6, 7, 9 and 10 improve the lower bound for $|S_{j_1}^{(P)} \cup \ldots \cup S_{j_k}^{(P)}|$.

Exactly as in (c) above we take such a set with largest index and see how many pairs of the excluded type can be counted using "additional" elements. We are finished if:

Case (i) $\frac{3}{4}n + 2\alpha_{11} \geq k$ (this is so by (c)).

Case (ii) $\frac{1}{2}n + 4\alpha_i \geq k$, for i = 6 or 7 or 9 or 10, i.e. if $\alpha_i \geq n/42$ for one of these values. If this were not so we would have

$$\frac{1}{2}n+3\alpha_6 \geq \frac{3}{42}n+\alpha_3+\alpha_{12}+\alpha_{13}+\alpha_{14} ,$$

a contradiction since $\alpha_i \leq n/14$ for all i.

By (a),(b),(c),(d) the conditions of HALL's theorem are satisfied. From (1) it follows that

(2) $$N^* \leq \frac{9}{14}n + 2 \qquad \text{if } 28|n .$$

Hence for arbitrary n we have

$$N^* \le \frac{9}{14}n + 20.$$

REMARK. If this proof can be extended to block design BD(2k,k; 4k-2,2k-1,k-1) then the constant factor of n would be $\frac{5k-2}{4(2k-1)}$. Hence one cannot reach the constant $\frac{1}{2}$ without further refinements.

SOLUTION by P. ERDÖS and J. SPENCER. (The following discussion of the problem for large values of n was written by N.G. DE BRUIJN and mainly based on contributions by P. ERDÖS and J. SPENCER.)

Let us call $\{C_1,\ldots,C_\ell\}$ "*good*" if it has the property mentioned in the first five lines of the problem, and "*bad*" otherwise. For a good set we define

$$\phi(\{C_1,\ldots,C_\ell\}) := \max_{1\le j\le 2n} \psi(j) ,$$

where

$$\psi(j) := |\{i \mid 1 \le i \le \ell,\ j \in C_i\}| .$$

For every positive integer n we define $\lambda(n)$ to be the minimum of $\phi(\{C_1,\ldots,C_\ell\})$, where the minimum is taken over all ℓ and all good sets. With this notation, the question of the problem is to show that $\lambda(n) \le [\frac{3n+2}{4}]$ $(n > 3)$. Here we prove

(i) $\qquad \lambda(n) = O(n^{\frac{1}{2}}(\log n)^{\frac{1}{2}}) \qquad (n\to\infty)$;

(ii) $\qquad \liminf_{n\to\infty} \lambda(n)n^{-\frac{1}{2}} \ge 2^{\frac{1}{2}} e^{-\frac{1}{2}}$.

P. ERDÖS showed by a probability argument that $\lambda(n) = o(n)$. J. SPENCER used a different probability argument, leading to $\lambda(n) = O(n^{\frac{1}{2}} \log n)$. The proof of (i) presented here is only a slight modification of SPENCER's proof. It also shows that we need not take ℓ/n much larger than 1.

PROOF of (i). Let α be any constant > 1. We shall show that there exist numbers $\beta > 0$, $\gamma > 0$, $N_0 > 0$ such that for all $n > N_0$ there exists a good set $\{C_1,\ldots,C_\ell\}$ with $\ell < \alpha n$, $\phi(\{C_1,\ldots,C_\ell\}) < \beta(n \log n)^{\frac{1}{2}}$, and such that $|C_i| < \gamma(n \log n)^{\frac{1}{2}}$ $(i=1,\ldots,2n)$.

We fix ℓ by taking $\ell = [\alpha n]$, and we fix a number θ (independent of n) with $\theta > 3(\alpha-1)^{-1}$. We next select $C_1,\ldots,C_\ell$ at random, in the sense that the $2n\ell$ events $j \in C_i$ $(1 \le j \le 2n,\ 1 \le i \le \ell)$ are independent, each with probability λ, where $\lambda := (\theta n^{-1}\log n)^{\frac{1}{2}}$, which is less than 1 if n is large enough. We shall give upper estimates for the probability that $\{C_1,\ldots,C_\ell\}$ is good and that it satisfies the inequalities involving β and γ.

If $\{C_1,\ldots,C_\ell\}$ is bad then there is at least one partitioning into pairs $\{x_1,y_1\},\ldots,\{x_n,y_n\}$ for which there are no suitable $i_1,\ldots,i_n$. By the KÖNIG-HALL theorem, there is a number t $(1 \le t \le n)$ and there are t indices $i_1,\ldots,i_t$ such that there are more than $\ell-t$ C_i's that contain none of the pairs $\{x_{i_1},y_{i_1}\},\ldots,\{x_{i_t},y_{i_t}\}$. We can now forget about the partition we started from, and say that if $\{C_1,\ldots,C_\ell\}$ is bad, then there exists a number t $(1 \le t \le n)$, a set of disjoint pairs $\{x_{i_1},y_{i_1}\},\ldots,\{x_{i_t},y_{i_t}\}$, and a set $K \subset \{1,\ldots,\ell\}$ with $|K| = \ell-t$, such that none of the pairs is contained in any C_i with $i \in K$. The probability P_n that this happens can be estimated above by

$$P_n \le \sum_{t=1}^{n} (2n)^{2t}\binom{\ell}{t}(1-\lambda^2)^{(\ell-t)t} .$$

We have $\ell-t \ge (\alpha-1)n-1$, whence

$$P_n \le \sum_{t=1}^{n} (4n^2\ell(1-\theta n^{-1}\log n)^{(\alpha-1)n-1})^t .$$

Since $n^2\ell(1-\theta n^{-1}\log n)^{(\alpha-1)n} = O(n^{3-\theta(\alpha-1)})$, we have $\lim_{n\to\infty} P_n = 0$.

From the theory of the binomial distribution, in particular from the inequality

$$\sum_{j=\mu m} \lambda^j(1-\lambda)^{m-j}\binom{m}{j} \le e^{-\mu m\rho}(\lambda e^{\rho}+1-\lambda)^m \le e^{-\mu m\rho+\lambda m(e^{\rho}-1)}$$

(with $\lambda < \mu < 1$, $\rho > 0$ such that $(e^{\rho}-1)/\rho < \mu/\lambda$), it easily follows that if ρ is a constant $> \theta$, then with a probability Q_n (with $\lim_{n\to\infty} Q_n = 1$) we have that everyone of the elements $1,\ldots,2n$ belongs to less than $(\rho n^{-1}\log n)^{\frac{1}{2}}\cdot\ell$ of the C_i's, and that each C_i has less than $(\rho n^{-1}\log n)^{\frac{1}{2}}\cdot 2n$ elements. Now taking $\beta = \alpha\rho^{\frac{1}{2}}$, $\gamma = \frac{1}{2}\rho^{\frac{1}{2}}$, we infer that the set of properties mentioned in the beginning of this proof has positive probability as soon as n is large enough.

PROOF of (ii). Let $\{C_1,\ldots,C_\ell\}$ be good, and denote for $j=1,\ldots 2n$, by $S(j)$ the set of i's with $j \in C_i$. For every partition into pairs $\{x_1,y_1\},\ldots,\{x_n,y_n\}$ there is a set of choices $p(1)\in S(1),\ldots,p(2n)\in S(2n)$ such that $p(x_1) = p(y_1),\ldots,p(x_n) = p(y_n)$ and $p(x_u) \neq p(x_v)$ if $u \neq v$. Such a set of choices determines the partition uniquely. It follows that the number of partitions is at most $\prod_{j=1}^{2n} |S(j)|$. Since $|S(j)| \le \phi(\{C_1,\ldots,C_\ell\})$, it follows that

$$\frac{(2n)!}{2^n\, n!} \le (\phi(\{C_1,\ldots,C_\ell\}))^{2n} ,$$

whence $\lambda(n) \ge ((2n)!/(2^n\, n!))^{1/2n}$, and this is asymptotically equivalent to $(2e^{-1}n)^{\frac{1}{2}}$. □

SOLUTION by P. ERDÖS. (The following discussion of the problem for $\ell = n$ and large values of n was written by J.H. VAN LINT and based on contributions by P. ERDÖS.)

Let $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ be given (small). Take n so large that

$$2(\tfrac{3}{4})^{\varepsilon_1 n} n^3 < \varepsilon_2 .$$

There are

$$M_n := \binom{2n}{[n(1+\varepsilon_1)]}$$

ways of choosing a subset C of A for which

(1) $$|C| = [n(1+\varepsilon_1)] .$$

We shall estimate the number of ways of choosing $\{C_1,C_2,\ldots,C_n\}$ such that all C_i satisfy (1) and $\{C_1,C_2,\ldots,C_n\}$ is bad. If $C_1,C_2,\ldots,C_{n-t}$ do not contain any of the disjoint pairs $\{x_1,y_1\},\ldots,\{x_t,y_t\}$ then clearly $t \le (1-\varepsilon_1)n$. For a given pair $\{x,y\}$ there are $< \frac{3}{4}M_n$ sets C satisfying (1) and not containing $\{x,y\}$. For a given t-tuple

$\{x_1,y_1\},\ldots,\{x_t,y_t\}$ of disjoint pairs there are less than

$$\binom{(\frac{3}{4})^t M_n}{n-t}$$

ways of choosing n-t subsets $C_1,C_2,\ldots,C_{n-t}$ all satisfying (1) such that none of them contains any of the given pairs. The contribution due to this t-tuple when we count the bad sets, is therefore at most

$$\binom{(\frac{3}{4})^t M_n}{n-t}\binom{M_n}{t} \le C(\tfrac{3}{4})^{t(n-t)} n^t \binom{M_n}{n}, \tag{2}$$

where C is a constant. The t-tuple can be chosen in at most $(2n^2)^t$ ways. Summing over all t not greater than $[(1-\varepsilon_1)n]+1$ we find at most

$$C \sum_{t=1}^{[n(1-\varepsilon_1)]+1} 2^t n^{3t} (\tfrac{3}{4})^{t(n-t)} \binom{M_n}{n} < C'\varepsilon_2 \binom{M_n}{n} \tag{3}$$

bad n-tuples satisfying (1).

We have thus shown that if the sets $C_1,C_2,\ldots,C_n$ are chosen at random, subject to (1), then the probability that $\{C_1,C_2,\ldots,C_n\}$ is good tends to 1 for $n\to\infty$. In the same way as in the previous solution one can show that for these good choices the probability that $\lambda(n) < \frac{1}{2}n(1+2\varepsilon_1)$ tends to 1 for $n\to\infty$.

This proves that if $\ell = n$ then

$$n^{-1}\lambda(n) \to \tfrac{1}{2} \qquad (n\to\infty). \tag{4}$$

351. Prove that for all $\lambda \in \mathbb{R}$

$$\sum_{r=0}^{m} \sum_{k=0}^{r} (-1)^k \binom{m+1}{k} (r-k+\lambda)^m = m!$$

and that

$$\sum_{k=0}^{r} (-1)^k \binom{m+1}{k} (r-k+\lambda)^m \ge 0 \text{ for } 0 \le \lambda \le 1.$$

(H.G. TER MORSCHE)

Solutions by F.J.M. BARNING, J. BOERSMA, A.A. JAGERS, J.H. VAN LINT, H.G. TER MORSCHE, F.W. STEUTEL and P.C.G. DE VRIES.

SOLUTION by F.W. STEUTEL.

If $X_1, X_2, \ldots, X_{n+1}$ are independent random variables, uniformly distributed on $(0,1)$, then the probability density function $u_{m+1}(x)$ of $X_1+X_2+\ldots+X_{m+1}$ is given by (cf. W. FELLER, *An introduction to probability theory and its applications*, Vol. II, Wiley, New York, 1966, 2nd ed., p.27)

$$(1) \qquad u_{m+1}(x) = \frac{1}{m!} \sum_{0\le k\le x} (-1)^k \binom{m+1}{k} (x-k)^m ,$$

which, of course, is non-negative. Putting $x = \lambda+r$ in (1) with $0 \le \lambda \le 1$ and r a non-negative integer, we obtain the second assertion:

$$u_{m+1}(\lambda+r) = \frac{1}{m!} \sum_{k=0}^{r} (-1)^k \binom{m+1}{k} (\lambda+r-k)^m \ge 0 .$$

As $\int_0^{m+\lambda} u_m(x)\,dx = 1$ for all $\lambda \ge 0$, we have

$$(2) \qquad (m-1)! = \int_0^{m+\lambda} \sum_{0\le k\le x} (-1)^k \binom{m}{k} (x-k)^{m-1}\,dx =$$

$$= \sum_{k=0}^{m} (-1)^k \binom{m}{k} \int_k^{m+\lambda} (x-k)^{m-1}\,dx =$$

$$= \frac{1}{m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m+\lambda-k)^m .$$

On the other hand we have, using $\binom{m+1}{k} = \binom{m}{k} + \binom{m}{k-1}$,

$$\sum_{r=0}^{m} \sum_{k=0}^{r} (-1)^k \binom{m+1}{k} (r+k-\lambda) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m+\lambda-k)^m ,$$

which by (2) proves the first assertion for $\lambda \ge 0$, and hence, because of the analyticity of the expression involved, for all complex λ.

SOLUTION by J.H. VAN LINT.

Let

$$F_m(x;\lambda) := \sum_{n=0}^{\infty} (n+\lambda)^m x^n .$$

Then we have

(1) $$\lambda F_m(x;\lambda) + xF_m'(x;\lambda) = F_{n+1}(x;\lambda)$$

and since $F_0(x) = (1-x)^{-1}$ we find that

(2) $$F_m(x;\lambda) = P_m(x;\lambda)(1-x)^{-m-1} ,$$

where P_m is a polynomial of degree m. By substitution in (1) we find

$$P_{m+1}(1;\lambda) = (m+1)P_m(1;\lambda) ,$$

i.e.

(3) $$P_m(1;\lambda) = m! .$$

From (2) it follows that

$$P_m(x;\lambda) = (1-x)^{m+1} \sum_{n=0}^{\infty} (n+\lambda)^m x^n =$$

$$= \sum_{r=0}^{m} x^r \sum_{k=0}^{r} (-1)^k \binom{m+1}{k} (r-k+\lambda)^m ,$$

i.e. the sum to be determined is $P_m(1;\lambda) = m!$.

From (1) and (2) we have

$$P_{m+1}(x;\lambda) = \{\lambda(1-x)+(m+1)x\}P_m(x;\lambda) + x(1-x)P_m'(x;\lambda) .$$

If

$$P_m(x;\lambda) = \sum_{k=0}^{m} a_k x^k \quad \text{and} \quad P_{m+1}(x;\lambda) = \sum_{k=0}^{m+1} b_k x^k ,$$

then

(4) $$b_k = (m+2-k-\lambda)a_{k-1} + (\lambda+k)a_k .$$

Now, if $\lambda \in [0,1]$, then from $P_0(x;\lambda) = 1$ it follows by induction that all coefficients of $P_m(x;\lambda)$ are positive.