The difference-differential equation F'(x)=e\sp {ax+\beta}F(x-1), I and II

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MATHEMATICS

THE DIFFERENCE-DIFFERENTIAL EQUATION $F'(x) = e^{\alpha x+\beta} F(x-1)$ I + II

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1. *Introduction*

1.1. In a paper dedicated to a special partition problem KURT MAHLER [5] considered the functional equation $f'(y) = f(qy)$ for $y > 0$, where q is a constant $(0 < q < 1)$. On substituting $y = q^{-x}$, $f(q^{-x}) = F(x)$ this equation is transformed into the difference-differential equation

$$F'(x) = (\log\ 1/q) \cdot q^{-x} F(x-1).$$

We shall study the slightly more general equation

$$F'(x) = e^{\alpha x+\beta} F(x-1), \tag{1.1}$$

where α and β are constants; α is positive, but β may be complex. If β is real, (1.1) is easily transformed into MAHLER's equation, simply by a shift $x \to x + a$.

We shall study (1.1) for real values of x, and in particular the behaviour of the solutions if $x \to \infty$. It is, of course, possible to investigate solutions that are analytical in regions of the complex x-plane, but this is actually easier than the theory for x real, in the same sense as the theory of the LAURENT series is simpler than the one of FOURIER series. And, a theory for the solutions in the complex plane can be easily obtained from the results of this paper.

1.2. If B is a real number, then a real or complex function $F(x)$, defined for $x \geqslant B - 1$, is said to be "a solution of (1.1) for $x \geqslant B$", if it is continuous for $x \geqslant B$, continuous by parts for $B - 1 \leqslant x \leqslant B$, and if it satisfies (1.1) for all $x \geqslant B$ except those for which $x - 1$ is a point of discontinuity. It has to be understood that $F(x)$ is continuous to the right at $x = B$.

Evidently any function that is continuous by parts in $B - 1 \leqslant x \leqslant B$, can be continued uniquely to a solution for $x \geqslant B$. However, $F(x)$ is not uniquely determined by its values in $B - 1 \leqslant x < B$; the value of $f(B)$ is not irrelevant.

If $F(x)$ is a solution for $x \geqslant B$, then it has n successive continuous derivatives for $x \geqslant B + n + 1$. A similar situation would arise on taking a wider definition of "solution". The most general and the most natural thing to do would be to define a solution as a distribution on $-\infty < x < \infty$ in the sense of L. SCHWARTZ. But even these would be solutions in the sense of the above definition from a certain B onward. And, as translations $x \to x + a$ simply mean taking new values of β, and are trivially disposed of in questions concerning the asymptotic behaviour, our naive definition will be suitable for most purposes.

1.3. In the case that β is real, MAHLER [5] proved, that any solution of (1.1) satisfies $F(x) = e^{H(x)} \cdot O(1)$, and if $F(x) > 0$ in some interval $b \leqslant x \leqslant b + 1$, even that $F(x) = e^{H(x)} \cdot e^{O(1)}$. Here $H(x)$ is defined by

$$(1.2) \qquad \begin{cases} H(x) = \frac{1}{2}\alpha(x - \alpha^{-1}\log x)^2 + (1 + \beta + \frac{1}{2}\alpha - \log\alpha)\,x + \\ + (-1 + \alpha^{-1}\log\alpha - \alpha^{-1}\beta)\log x - \frac{1}{2}\log\alpha + \frac{1}{2}\gamma^2\alpha^{-1} \\ - \frac{1}{2}\alpha^{-2}x^{-1}\log^2 x + \alpha^{-2}(\alpha + \beta - \log\alpha)\,x^{-1}\log x. \end{cases}$$

The last four terms are irrelevant in this connection; the last two terms play a role in (1.3), whereas the constant terms (for the definition of γ see sec. **1.9**) only serve normalization purposes.

1.4. Recently the author [2] proved, as a special case of more general equations with asymptotically constant coefficients of a singular type, that to any solution $F(x)$ of (1.1) (with β real) there corresponds a function $\psi(y)$ of period 1, with derivatives of all orders, such that

$$(1.3) \qquad F(x) = e^{H(x)}\,\{\psi(x - \alpha^{-1}\log x + \alpha^{-2}\,x^{-1}\log x) + O(x^{-1})\}.$$

A similar asymptotic behaviour, with a specified function ψ, was obtained (see [1]) in MAHLER's partition problem, that corresponds to the equation $f(x) - f(x-1) = f(qx)$, with simple initial conditions.

1.5. In the present paper the following results are obtained.

1.51. (1.3) is still true if β is complex (sec. 8).

1.52. An infinite set of linearly independent solutions $F_n(x)$ ($n = 0, \pm 1, \pm 2, \ldots$) are constructed (sec. 4); any other solution can be expressed in terms of these by finite or infinite linear combination. These $F_n(x)$ correspond, in the sense of (1.3), to $\psi_n(y) = C_n\,e^{2\pi n i y}$, where the C_n are constants ($C_n = \exp(-2\pi^2 n^2\alpha^{-1} + 2\pi n i\gamma\alpha^{-1})$). Further, all $F_n(x)$ are, by $F_n(x) = F_0(x + 2\pi n i/\alpha)$, obtained from one and the same integral transcendental function $F_0(x)$.

1.53. If a solution $F(x)$ is given by its values in $B - 1 \leqslant x \leqslant B$, then $\psi(x)$ (cf. (1.3)) can be expressed explicitly in terms of these values (see sec. **8.8**).

1.54. In [2] a uniqueness theorem was proved for a class of difference-differential equations. It states that the function ψ, describing the asymptotically periodic behaviour of a solution, vanishes identically only if the solution vanishes identically. The conditions given in [2] for that theorem are not satisfied in our present case, nevertheless the result holds (see sec. **8.4**).

1.55. The result (1.3) is best possible in a certain sense (see sec. **8.5**).

1.6. In the present paper we mainly use methods that are frequently used in the theory of linear differential equations, e.g. GREEN function, adjoint equation, biorthogonal system, LAPLACE transform and saddle-point analysis. Remarkably these methods work wonderfully for our equation, and almost everything can be calculated explicitly.

As far as routine techniques are involved, many details will be omitted.

1.7. The paper can be read independently of the papers quoted. Especially, the result (1.3) will be proved independently for systematical reasons. It was proved in [2] for real β only; the reality of the equations was essential in that paper. In the present paper, however, it does not matter anywhere whether β is real or complex.

1.8. Many of the methods developed in the sequel can be used for similar simple equations, e.g. $F'(x) = e^{\alpha x+\beta} F(x-1)$ if α is not positive, $F'(x) = xF(x-1)$, $xF'(x) + F(x) = F(x-1)$, $xF'(x) + F(x-1) = 0$. The latter two play a role in number theoretical problems (see, also for further references, [3] and [4]).

1.9. *Notations.* We shall often use $\gamma = \beta + \frac{1}{2}\alpha - \log\alpha$ as a parameter instead of β, and $u = x + \alpha^{-1}\gamma$ as a variable instead of x. If x runs through the real axis, then u runs through a parallel line in the complex plane (as β may be complex).

2. *The adjoint equation and the inner product* $\{F, G\}$

2.1. Consider a linear difference-differential operator of the form

$$\Lambda f(x) = w(x)\, f'(x) + p(x)\, f(x) - q(x)\, f(x-1). \tag{2.1}$$

If we would have a pure differential operator, we would define the adjoint operator Λ^* such that $\int_a^b (g\Lambda f - f\Lambda^* g)\, dx$ only depends on the values of f and g and their derivatives in a and b. As we actually have a difference-differential operator, we also have to take into consideration the values of f and g and their derivatives over intervals around a and b, intervals whose length equals the "span" of the operator (here the span equals 1). In fact, writing

$$\Lambda^* g(x) = -\{w(x)\, g(x)\}' + p(x)\, g(x) - q(x+1)\, g(x+1), \tag{2.2}$$

we have formally

$$\int_a^b \{g(x)\, \Lambda\, f(x) - f(x)\, \Lambda^*\, g(x)\}\, dx = \varphi(b) - \varphi(a), \tag{2.3}$$

where

$$\varphi(x) = w(x)\, f(x)\, g(x) + \int_{x-1}^{x} q(t+1)\, f(t)\, g(t+1)\, dt. \tag{2.4}$$

If f and g satisfy $\Lambda f = 0$, $\Lambda^* g = 0$, respectively, then $\varphi(x)$ is, by (2.3), an invariant of f and g in the sense that it does not depend on x. Its value can be called the inner product $\{f, g\}$, as it is bilinear over the linear solution spaces of $\Lambda f = 0$ and $\Lambda^* g = 0$.

2.2. We now state the things more exactly for our equation (1.1). The adjoint equation is

$$(2.5) \qquad -G'(x) - e^{\alpha x+\alpha+\beta}\, G(x+1) = 0.$$

$G(x)$ will be called a solution of (2.5) for $x \leqslant C$ if $G(x)$ is defined for $x \leqslant C + 1$, continuous for $x \leqslant C$ and continuous by parts for $x \leqslant C + 1$, and if it satisfies (2.5) for all $x \leqslant C$ except those for which $x + 1$ is a point of discontinuity. In other words, $G(x)$ is a solution of (2.5) for $x \leqslant C$, if $G(-x)$ is a solution (in the sense of sec. 1.2) for $x \geqslant -C$ of the equation $f'(x) = e^{-\alpha x+\alpha+\beta}\, f(x-1)$.

Now let B and C be real numbers, and assume $B < C$. Let $F(x)$ be a solution of (1.1) for $x \geqslant B$, and let $G(x)$ be a solution of (2.5) for $x \leqslant C$. Then the expression

$$(2.6) \qquad \{F(x), G(x)\} = \varphi(x) = F(x)\, G(x) + \int_{x-1}^{x} e^{\alpha t+\alpha+\beta}\, F(t)\, G(t+1)\, dt$$

has a meaning for $B \leqslant x \leqslant C$, and it is independent of x in that interval.

2.3. If both F and G are entire functions which satisfy (1.1) and (2.5), respectively, then $\varphi(x)$ is a constant for all complex values of x.

3. *The Green function $K(y, x)$*

3.1. Let η be a real number. Then denote by $K(\eta, x)$ the solution of (1.1) for $x \geqslant \eta$, that is defined by the initial conditions

$$(3.1) \qquad K(\eta, x) = 0 \quad (\eta - 1 \leqslant x < \eta), \qquad K(\eta, \eta) = 1.$$

Moreover, we define $K(\eta, x) = 0$ for $x < \eta - 1$. So we have, in a terminology that we did not adopt, $\Lambda K(\eta, x) = \delta(x - \eta)$ $(-\infty < x < \infty)$, where δ indicates the Dirac δ-function.

3.2. Accordingly, if ξ is a real number, we define $K^*(y, \xi)$ as a solution of (2.5) for $y \leqslant \xi$, defined by

$$(3.2) \qquad K^*(y, \xi) = 0 \quad (y > \xi), \qquad K^*(\xi, \xi) = 1.$$

Again, $\Lambda^* K^*(y, \xi) = \delta(y - \xi)$.

3.3. It is not difficult to show that $K(\eta, \xi)$ and $K^*(\eta, \xi)$ are identical. This is trivial if $\xi \leqslant \eta$. Now assume $\xi > \eta$. Write $K(\eta, x) = f(x)$, $K^*(y, \xi) = g(y)$. Applying (2.3), with $a = -\infty$, $b = +\infty$, we obtain $\varphi(-\infty) = \varphi(+\infty) = 0$, and hence

$$\int_{-\infty}^{\infty} g(x)\, \Lambda f(x)\, dx - \int_{-\infty}^{\infty} f(y)\, \Lambda^* g(y)\, dy = 0.$$

As $\Lambda f(x) = \delta(x - \eta)$, $\Lambda^* g(y) = \delta(y - \xi)$, we infer that $g(\eta) = f(\xi)$, and so $K(\eta, \xi) = K^*(\eta, \xi)$. It is of course easy to turn this unorthodox argument into a rigorous proof.

3.4. The Green function $K(y, x)$ enables us to express arbitrary solutions of (2.1) and (2.5) in terms of their initial values.

Let $F(x)$ be a solution of (1.1) for $x \geqslant B$. Let ξ be any number $\geqslant B$. Then $K^*(y, \xi)$ is a solution of (2.5) for $y \leqslant \xi$. Therefore, by sec. **2.2**, the inner product

$$(3.3) \qquad \{F(t), K(t, \xi)\} = F(t)\, K(t, \xi) + \int_{t-1}^{t} e^{\alpha s+\alpha+\beta}\, F(s)\, K(s+1, \xi)\, ds$$

is defined and constant in the interval $B \leqslant t \leqslant \xi$. If $t = \xi$, the integral in (3.3) vanishes, and we infer $(B \leqslant t \leqslant \xi)$

$$(3.4) \qquad F(\xi) = \{F(t), K(t, \xi)\} = F(B)\, K(B, \xi) + \int_{B-1}^{B} e^{\alpha s+\alpha+\beta}\, F(s)\, K(s+1, \xi)\, ds.$$

Analogously, if $G(x)$ is a solution of (2.5) for $x \leqslant C$, we have $(\eta \leqslant t \leqslant C)$

$$(3.5) \qquad G(\eta) = \{K(\eta, t), G(t)\} = K(\eta, C)\, G(C) + \int_{C-1}^{C} e^{\alpha s+\alpha+\beta}\, K(\eta, s)\, G(s+1)\, ds.$$

Finally, by (3.4) as well as by (3.5), we infer

$$(3.6) \qquad \{K(\eta, t),\ K(t, \xi)\} = K(\eta, \xi) \quad (\eta \leqslant t \leqslant \xi).$$

3.5. It is not difficult to obtain an explicit formula for $K(y, x)$. Write $f_0(x) = 0$ $(x < y)$, $f_0(x) = 1$ $(x \geqslant y)$, and $f_n(x) = \int_{-\infty}^{x} e^{\alpha t+\beta} f_{n-1}(t-1)\, dt$ $(n = 1, 2, 3, \ldots)$. Then we have $K(y, x) = f_0(x) + f_1(x) + f_2(x) + \ldots$. Carrying out the integrations, we find, for all x and y

$$(3.7) \qquad K(y, x) = \sum_{0 \leqslant n \leqslant x-y} \frac{e^{\frac{1}{2}n^2\alpha + n(\alpha y+\gamma)}}{n!} \left(e^{\alpha(x-y-n)} - 1\right)^n,$$

(where $\gamma = \frac{1}{2}\alpha + \beta - \log \alpha$). The term $n = 0$ has to be interpreted as 1 if $x = y$. If $x < y$, the sum is vacuous, and its value has to be interpreted as 0.

3.6. If we combine (3.4) and (3.7), we have expressed an arbitrary solution $F(x)$ of (1.1) explicitly in terms of its initial values. This result could be used in order to obtain the asymptotic expansion of $F(x)$. We shall not take this line; in the first place, for asymptotic analysis the finite sum (3.7) is less appropriate than the solutions in the form of integrals, developed in the next section, and secondly, it seems that the uniqueness theorem (see sec. **1.54**) can not be obtained that way.

4. *Biorthogonal system of special solutions*

4.1. If we try to solve (1.1) by a LAPLACE integral $F(x) = \int e^{xt}\, \varphi(t)\, dt$, we easily find that, for $n = 0, \pm 1, \pm 2, \ldots$, the functions

$$(4.1) \qquad F_n(x) = \int_{-\infty}^{\infty} \exp\left(\alpha u z - \tfrac{1}{2}\alpha z^2 + 2\pi n i z\right) \frac{dz}{\Gamma(1+z)}$$

are solutions of (1.1). Again

$$(4.2) \qquad \gamma = \tfrac{1}{2}\alpha + \beta - \log \alpha, \qquad u = x + \alpha^{-1}\gamma.$$

Obviously

$$(4.3) \qquad F_n(x) = F_0(x + 2\pi n i \alpha^{-1}).$$

$F_0(x)$ is an entire function, and so are all $F_n(x)$. They all satisfy (1.1) for all real and complex values of x.

4.2. Using HANKEL's integral for $\Gamma^{-1}(1+z)$ and changing the order of integration, we obtain another integral representation for $F_0(x)$:

$$F_0(x) = -i(2\pi\alpha)^{-\frac{1}{2}} \int_V \exp\left\{\frac{1}{2\alpha}(\alpha u - w)^2 + e^w\right\} dw. \tag{4.4}$$

The path V consists of the line segments $(-\pi i + \infty, -\pi i)$, $(-\pi i, \pi i)$, $(\pi i, \pi i + \infty)$. According to (4.3), the function $F_n(x)$ can be found by integrating the same integrand along the path $V - 2\pi ni$.

4.3. Dealing similarly with the adjoint equation (2.5), we obtain as solutions the entire functions

$$G_n(x) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \exp(-\alpha uz + \tfrac{1}{2}\alpha z^2 - 2\pi niz).\ \Gamma(z)\,dz, \tag{4.5}$$

where again

$$G_n(x) = G_0(x + 2\pi ni\alpha^{-1}). \tag{4.6}$$

Now using EULER's integral for $\Gamma(z)$ and changing the order of integration, we find

$$G_n(x) = (2\pi\alpha)^{-\frac{1}{2}} \int_{-\infty-2\pi ni}^{\infty-2\pi ni} \exp\left\{-\frac{1}{2\alpha}(\alpha u - w)^2 - e^w\right\} dw. \tag{4.7}$$

It is remarkable that the integrands of (4.4) and (4.7) are reciprocal. The same happens, roughly, with (4.1) and (4.5).

4.4. The sets $\{F_n\}$ and $\{G_n\}$ form a *biorthogonal system*, i.e.

$$\{F_n, G_m\} = \delta_{nm}, \tag{4.8}$$

where $\delta_{nm} = 1$ if $n = m$, $\delta_{nm} = 0$ if $n \neq m$. This will follow from the asymptotic formulas for F_n and G_n (see sec. **5.4**). Another proof could be deduced from integral formulas for the product $F_0(x)\,G_0(y)$, for instance from (6.6).

4.5. We give some attention to other special solutions that, however, will not be used in the sequel.

4.51. MAHLER [1]) indicated the following special solutions of (1.1):

$$F(x,\theta) = \sum_{-\infty}^{\infty} e^{\alpha u(n+\theta)}\, e^{-\frac{1}{2}\alpha(n+\theta)^2}\, \Gamma^{-1}(n+\theta+1). \tag{4.9}$$

These functions can be expanded in terms of our $F_n(x)$, for application of POISSON's formula leads to (cf. (4.1))

$$F(x,\theta) = \sum_{-\infty}^{\infty} e^{-2\pi ni\theta} F_n(x). \tag{4.10}$$

The series converges rapidly: for x fixed, we have

$$F_n(x) = O\{\exp(-2\pi^2n^2\alpha^{-1} + C\,|n|)\} \qquad \text{(see (5.15))}.$$

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[1]) Private communication (1946); the case $\theta = 0$ occurs in MAHLER's paper [5].

4.52. There is a solution of the adjoint equation that is connected in a similar way to (4.7):

$$(4.11)\qquad G(x,\theta)=\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\sum_{-\infty}^{\infty}\exp\left\{-\frac{1}{2\alpha}(\alpha n-a\theta-\gamma)^2-e^{\alpha(x+n-\theta)}\right\}.$$

$$(4.12)\qquad G(x,\theta)=\sum_{-\infty}^{\infty}\exp\{2\pi ni(\gamma\alpha^{-1}+\theta)-2\pi n^2\alpha^{-1}\}\,G_n(x).$$

5. *Asymptotic formulas for* $F_0(x)$ *and* $G_0(x)$

We want information concerning the behaviour of $F_n(x)$ and $G_n(x)$ for x real, if at least one of $|n|$, x is a large positive number. So by (4.3) and (4.6) it is sufficient to know the behaviour of $F_0(x)$ and $G_0(x)$ in every half-plane Re $x \geqslant a$, where $|x|$ is large.

In this section we shall apply saddle-point analysis to the integrals (4.4) and (4.7). It turns out that suitable formulas can be obtained in wider regions than strictly necessary, viz. $|\arg x| < \pi - \delta$ for $F_0(x)$ and $|\arg x| < \frac{3}{4}\pi - \delta$ for $G_0(x)$. In the region $|\arg(-x)| < \frac{1}{4}\pi - \delta$ there will hold an essentially different formula for $G_0(x)$, but its investigation is not needed for our present purposes.

5.1. We shall estimate $F_0(x)$ if $|x| \to \infty$ and

$$(5.1)\qquad |\arg x| < \pi - \delta,$$

where δ is an arbitrarily small positive number.

The saddle points ζ of (4.4) satisfy (as before, $\alpha x - \gamma = \alpha u$)

$$(5.2)\qquad \zeta - \alpha u + \alpha e^{\zeta} = 0.$$

Therefore, if x is in (5.1) and $|x|$ is large, there is just one saddle point in the region $|\mathrm{Im}\,\zeta| < \pi$, $Re\,\zeta > 0$. Henceforth ζ denotes this special saddle point.

ζ is easily localized:

$$(5.3)\qquad \zeta = \log u - (\alpha u)^{-1}\log u + O(|u|^{-2}\log^2|u|),$$

where the principal value of the logarithm is used.

In (4.4) we now substitute $w = \zeta + \omega$:

$$(5.4)\qquad F_0(x) = -i(2\pi\alpha)^{-\frac{1}{2}}\exp\left(\frac{\lambda^2+2\lambda}{2\alpha}\right)\int_{V-\zeta}\exp\left\{\frac{\omega^2}{2\alpha}+\frac{\lambda}{\alpha}(e^{\omega}-\omega-1)\right\}d\omega,$$

where

$$(5.5)\qquad \lambda = \alpha e^{\zeta} = \alpha u - \zeta.$$

It follows from (5.3) and (5.5) that, for large $|x|$, (5.1) implies that $|\lambda|$ is large, and $|\arg\lambda| < \pi - \delta'$ (for some $\delta' > 0$).

For the integral (5.4) the saddle point is $\omega = 0$. The difficulty is, of course, to choose a suitable integration path. To this end we first assume

$$(5.6)\qquad 0 \leqslant \arg\lambda < \pi - \delta';$$

the other half of the sector can be dealt with analogously.

If (5.6) holds, the path of integration can be replaced by a curve on which $e^{\omega} - \omega - 1$ has the constant argument $\frac{1}{2}(\pi + \delta')$. This curve starts at $(-2\pi + \frac{1}{2}\pi + \frac{1}{2}\delta')i + \infty$ and it ends at $\frac{1}{2}(\pi + \delta')i + \infty$. On the curve we have $\omega^2 = O\{|e^{\omega} - \omega - 1|\}$. The existence of this curve is easily seen from the conformal mapping of the strip $|\mathrm{Im}\,\omega| < 2\pi$ by the function

$$\xi = \{2(e^{\omega} - \omega - 1)\}^{\frac{1}{2}}.$$

The image in the ξ-plane consists of the entire ξ-plane, cut along two hyperbolic arcs described by $(\mathrm{Im}\,\xi)\cdot(\mathrm{Re}\,\xi) = \pm 2\pi$, $\mathrm{Re}\,\xi \leqslant -|\mathrm{Im}\,\xi|$. Now the image of the line

$$\xi = t \exp\{\tfrac{1}{4}(\pi + \delta')i\} \qquad (-\infty < t < \infty)$$

has the required properties.

The asymptotic expansion for large λ in the sector (5.6) can now be obtained in the usual way. In the neighbourhood of $\omega = 0$ we have a power series

$$(5.7) \qquad e^{\omega^2/2\alpha} \frac{d\omega}{d\xi} = a_0 + a_1\xi + a_2\xi^2 + \ldots; \qquad a_0 = 1.$$

The final result is that for $M = 1, 2, 3, \ldots$ we have, if $0 \leqslant \arg \lambda < \pi - \delta'$, $|\lambda| \to \infty$,

$$(5.8) \quad F_0(x) = \exp\left(\frac{\lambda^2+2\lambda}{2\alpha}\right)\left\{\sum_{k=0}^{M-1} (-1)^k a_{2k} \frac{(2k)!}{k!} (\tfrac{1}{2}\alpha)^k \lambda^{-k-\frac{1}{2}} + O(|\lambda|^{-M-\frac{1}{2}})\right\}.$$

As analogous considerations lead to the same formula for λ in the sector $-\pi + \delta' < \arg \lambda \leqslant 0$, the asymptotic expansion (5.8) holds uniformly for all large x in the sector (5.1). The constant implied in the O-symbol of course depends on M.

5.2. For the integral (4.7) ($n = 0$) we have exactly the same saddle point. So we shall use ζ and λ with the same meaning as in sec. **5.1.** We find (cf. (5.4))

$$(5.9) \quad G_0(x) = (2\pi\alpha)^{-\frac{1}{2}} \exp\left(-\frac{\lambda^2+2\lambda}{2\alpha}\right) \int_{-\infty-\zeta}^{\infty-\zeta} \exp\left\{-\frac{\omega^2}{2\alpha} - \frac{\lambda}{\alpha}(e^{\omega} - \omega - 1)\right\} d\omega.$$

Comparing this with the previous case, an extra difficulty arises. For, if ω is in the left half-plane, and $|\omega|$ large, the term with $e^{\omega} - \omega - 1$ is no longer the dominating term in the exponent. Therefore, we cannot take an integration path that asymptotically makes an angle of more than $\frac{1}{4}\pi$ with the negative real axis. The effect is, that $\omega = 0$ is the dominating saddle point only if $|\arg \lambda| < \frac{3}{4}\pi - \delta'$; otherwise the saddle point in the neighbourhood of $\omega = \lambda$ will furnish the main contribution to the asymptotic behaviour.

For our present purposes the behaviour of $G_0(x)$ for x far in the left half-plane is irrelevant; we therefore assume $|\arg \lambda| < \frac{3}{4}\pi - \delta'$, and we first consider $0 \leqslant \arg \lambda < \frac{3}{4}\pi - \delta'$. As the integration path we can now take a curve through the point $\omega = 0$, on which $e^{\omega} - \omega - 1$ has the constant argument $-\frac{1}{4}\pi$. The part of this curve to the right of the point $\omega = 0$ is easily found by the process used in sec. **5.1**; for the part to the left of $\omega = 0$ it is better to use the image of the left ω-half-plane under the mapping $\eta = e^{\omega} - \omega - 1$.

Again following usual techniques, we finally obtain

$$(5.10)\quad G_0(x) = \exp\left(-\frac{\lambda^2+2\lambda}{2\alpha}\right)\left\{\sum_{k=0}^{M-1}(-1)^k\, b_{2k}\frac{(2k)!}{k!}(\tfrac{1}{2}\alpha)^k \lambda^{-k-\frac{1}{2}} + O\left(|\lambda|^{-M-\frac{1}{2}}\right)\right\},$$

uniformly for $|\lambda| \to \infty$, $|\arg \lambda| < \frac{3}{4}\pi - \delta'$. The b_k's are defined as the coefficients of the power series for $\exp(-\omega^2/2\alpha)\cdot d\omega/d\xi$ (cf. (5.7)), so that $b_0 = 1$. Finally, λ is defined as in sec. **5.1**, so that (5.10) also refers to $|x| \to \infty$, $|\arg x| < \frac{3}{4}\pi - \delta$.

5.3. We now give some immediate applications of the results of secs. **5.1** and **5.2**. Most of them will be used in the next sections.

Formulas labelled "$|x| \to \infty$" hold uniformly in any fixed half-plane $\operatorname{Re} x \geqslant C$; "$x \to \infty$" refers to real values of x.

It follows from (5.5) that $\lambda\,(= \alpha u - \zeta)$ satisfies $\lambda = \alpha \exp\{\alpha u - \lambda\}$. Hence (cf. (5.3))

$$(5.11)\qquad \lambda = \alpha u - \log u + P(u^{-1}, u^{-1}\log u) \qquad (|x| \to \infty),$$

where P denotes a power series in two variables, convergent if both variables are sufficiently small in absolute value. And, $\log u$ denotes the principal value. We give the first few terms explicitly $(|x| \to \infty)$:

$$(5.12)\qquad \lambda = \alpha u - \log u + \frac{\log u}{\alpha u} + \frac{\log^2 u}{2\alpha^2 u^2} - \frac{\log u}{\alpha^2 u^2} + O\left(\frac{\log^3 u}{u^3}\right).$$

It follows that $(|x| \to \infty)$

$$(5.13)\quad \lambda^2 = \alpha^2 u^2 - 2\alpha u \log u + 2\log u + \log^2 u - \frac{\log^2 u}{\alpha u} - \frac{2\log u}{\alpha u} + O\left(\frac{\log^3 u}{u^2}\right)$$

$$(5.14)\qquad \log \lambda = \log u + \log \alpha - \frac{\log u}{\alpha u} + O\left(\frac{\log^2 u}{u^2}\right).$$

Now (5.12), (5.13), (5.14), (5.8) show that $(|x| \to \infty)$

$$(5.15)\quad \left\{\begin{aligned} \log F_0(x) &= \tfrac{1}{2}\alpha u^2 - u\log u + u + \frac{\log^2 u}{2\alpha} - \tfrac{1}{2}\log u - \\ &\quad - \tfrac{1}{2}\log \alpha - \frac{\log^2 u}{2\alpha^2 u} + \frac{\log u}{2\alpha u} + O\left(\frac{\log^3 u}{u^2}\right), \end{aligned}\right.$$

and expressed in terms of x we obtain, with the notation of (1.2),

$$(5.16)\qquad F_0(x) = e^{H(x)}\,\{1 + O(x^{-1})\}. \qquad (|x| \to \infty).$$

Formulas for $G_0(x)$ are easily obtained from formulas for $F_0(x)$ on using (cf. (5.8), (5.10)),

$$(5.17)\quad \left\{\begin{aligned} F_0(x)\,G_0(x) &= \lambda^{-1} + O(\lambda^{-2}) = \alpha^{-1}u^{-1}\left(1 + \frac{\log u}{\alpha u} + O\left(\frac{1}{u}\right)\right) \\ &= \alpha^{-1}x^{-1}\left(1 + \frac{\log x}{\alpha x} + O\left(\frac{1}{x}\right)\right)\ (|x| \to \infty). \end{aligned}\right.$$

We shall need formulas for $F_0(x+\varrho)/F_0(x)$, if $|x| \to \infty$, and ϱ is fixed, or small with respect to x. For our purposes it is sufficient to take $\varrho = O(x^{\frac{1}{4}})$. It easily follows from (5.15) that

$$(5.18)\quad \begin{cases} \log \dfrac{F_0(x+\varrho)}{F_0(x)} = \varrho \left\{ \alpha x - \log x + \gamma + \dfrac{\log x}{\alpha x} - \left(\dfrac{1}{2} + \dfrac{\gamma}{\alpha}\right) \dfrac{1}{x} \right\} + \\ \qquad + \dfrac{1}{2} \varrho^2 \left(\alpha - \dfrac{1}{x}\right) + o\left(\dfrac{1}{x}\right). \qquad (|x| \to \infty,\ \varrho = O(x^{\frac{1}{4}})). \end{cases}$$

We make two applications of (5.18). First, taking $\varrho = 2\pi n i/\alpha$, we obtain

$$(5.19)\quad \begin{cases} F_n(x) = F_0(x) \exp \left[-\dfrac{2\pi^2 n^2}{\alpha} + \dfrac{2\pi n i}{\alpha} \left(\alpha x - \log x + \gamma + \dfrac{\log x}{\alpha x} \right) + \right. \\ \qquad \left. + O\left(\dfrac{1}{x}\right) \right]. \qquad (n \text{ fixed},\ x \to \infty). \end{cases}$$

On the other hand, if $|n| \to \infty$ we have, uniformly in any fixed real interval $a \leqslant x \leqslant b$

$$(5.20)\quad \begin{cases} F_n(x) = F_n(0) \exp \left[\left\{ 2\pi n i - \log \dfrac{2\pi |n|}{\alpha} - \dfrac{1}{2}\pi i \operatorname{sgn} n + \gamma \right\} x + \right. \\ \qquad \left. + \dfrac{1}{2} \alpha x^2 + O\left(|n|^{-1} \log |n|\right) \right]. \end{cases}$$

5.4. The biorthogonality of the sets $\{F_n\}$ and $\{G_n\}$ (see sec. **4.4**) follows immediately from the asymptotic formulas. For, if n and m are fixed, we have, by (5.17), (5.19),

$$F_n(x)\, G_m(x) = o(1) \qquad (x \to \infty),$$

$$e^{\alpha x + \alpha + \beta}\, F_n(x)\, G_m(x+1) = \exp\{2\pi^2(m^2 - n^2)\,\alpha^{-1} + \\ + 2\pi i(n-m)\,(\alpha x + \gamma - \log x)\,\alpha^{-1} + o(1)\} \qquad (x \to \infty).$$

It follows that the expression (2.6) for the inner product, with $F = F_n$, $G = G_m$, is $\delta_{nm} e^{o(1)}$. As that expression actually does not depend on x, we infer that $\{F_n, G_m\} = \delta_{nm}$.

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MATHEMATICS

THE DIFFERENCE-DIFFERENTIAL EQUATION
$F'(x) = e^{\alpha x+\beta}\, F(x-1)$
II

BY

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6. *Series development of* $K(y, x)$

6.1. We shall show in this section that the Green function (sec. 3) can be expressed in terms of the functions of the biorthogonal system:

$$K(y, x) = \sum_{-\infty}^{\infty} F_n(x)\, G_n(y) \qquad (x>y \text{ or } y-1<x<y). \tag{6.1}$$

The convergence is absolute if $x>y$, as the terms are $O(|n|^{y-x-1})$. For $y-1<x<y$, where K vanishes, the series is convergent, but not absolutely. The behaviour of $F_n(x)\, G_n(y)$ for x, y fixed, and $n \to \infty$, follows from (5.17) and (5.18):

$$\left\{\begin{aligned} F_n(x)\, G_n(y) &= -\, i \exp\{\gamma(x-y) + \tfrac{1}{2}\alpha(x-y)^2\}\cdot \\ &\cdot \exp\{2\pi n i(x-y) - \tfrac{1}{2}\pi i(x-y)\operatorname{sgn} n\}\cdot\left(\frac{2\pi|n|}{\alpha}\right)^{y-x-1}\cdot\left\{1+O\left(\frac{\log|n|}{|n|}\right)\right\}.\end{aligned}\right. \tag{6.2}$$

6.2. We first express $F_0(x)\, G_0(y)$ in the form of a single integral ((6.6) below). For the time being, x and y may be complex, but $x-y=t$ is assumed to be real.

In (4.4) we may shift the path of integration in the horizontal direction over an arbitrary distance. Therefore, if w and η are real, we have

$$F_0(x) = -i(2\pi\alpha)^{-\frac{1}{2}} \int_{V-\eta} \exp\{\tfrac{1}{2}\alpha^{-1}(\alpha x+\gamma-z-w)^2 + e^{z+w}\}\, dz.$$

From (4.7) we now obtain

$$F_0(x)\, G_0(y) = (2\pi i\alpha)^{-1} \int_{-\infty}^{\infty} dw \int_{V-\eta} \Phi(w, z)\, dz \qquad (\eta \text{ real}), \tag{6.3}$$

where

$$\Phi(w, z) = \exp\left\{\frac{(z-\alpha t)^2}{2\alpha} - \frac{(z-\alpha t)(\alpha y+\gamma-w)}{\alpha} + e^w(e^z-1)\right\},$$

and $x = y+t$.

Although the integrals for F_0 and G_0 that we used were absolutely convergent, the same cannot always be said of the repeated integral (6.3), as shifting the path over variable distances may spoil the absolute convergence. However, if we assume

$$t<0,\quad 0<\eta<-\alpha t, \tag{6.4}$$

the absolute convergence can be established. Furthermore, we can change the order of integration in that case, and then the inner integral can be evaluated by means of Euler's formula for the Γ-function. It turns out that under (6.4) we have, for all real or complex values of y,

$$(6.5)\quad \left\{ \begin{aligned} F_0(y+t)\,G_0(y) = \frac{1}{2\pi i\alpha}\int_{V-\eta} \exp\left\{\frac{(\alpha y+\gamma)(\alpha t-z)}{\alpha}+\frac{(z-\alpha t)^2}{2\alpha}\right\}\cdot \\ \cdot(1-e^z)^{\frac{\alpha t-z}{\alpha}}\,\Gamma\left(\frac{z-\alpha t}{\alpha}\right)dz. \end{aligned}\right.$$

In this integral we are relatively free to change the path of integration, as there is no longer a term with e^z in the exponent. The path encircles the origin, that is a branch-point of the integrand. We moreover assume $-1<t<0$, and then the integral along a path leading to the origin converges. Therefore, we can, in the usual way, transform the path into a path leading from $+\infty$ to 0 and back from 0 to ∞, taking different branches of the multivalued integrand both times. We thus obtain (for convenience substituting $z=\alpha s$)

$$(6.6)\qquad F_0(y+t)\,G_0(y) = \int_0^\infty \exp\{(\alpha y+\gamma)(t-s)+\tfrac{1}{2}\alpha(s-t)^2\}\cdot\frac{(e^{\alpha s}-1)^{t-s}\,ds}{\Gamma(1-s+t)},$$

and this has been proved under the condition $-1<t<0$, and for all real or complex values of y.

As both sides of (6.6) are analytical functions of t in the half plane $\operatorname{Re} t>-1$, we observe that (6.6) remains valid for $-1<t<\infty$.

6.3. Assume t real, and $t>-1$. Using

$$F_n(x)=F_0(x+2\pi n i\alpha^{-1}),\quad G_n(y)=G_0(y+2\pi n i\alpha^{-1}),$$

we obtain from (6.6), after the substitution $s=t-v$:

$$(6.7)\qquad F_n(y+t)\,G_n(y)=\int_{-\infty}^{\infty} e^{2\pi n i v}\,\varphi(v)\,dv,$$

where

$$(6.8)\quad \left\{ \begin{aligned} \varphi(v) &= \exp\{(\alpha y+\gamma)\,v+\tfrac{1}{2}\alpha v^2\}\cdot(e^{\alpha(t-v)}-1)^v\,\Gamma^{-1}(1+v) && (v<t), \\ \varphi(v) &= 0 && (v\geqslant t). \end{aligned}\right.$$

The function $\varphi(v)$ has a continuous derivative everywhere except possibly at the point $v=t$. If $t>0$, $\varphi(v)$ is continuous at $v=t$; if $-1<t<0$ it tends to infinity when $t\to v$ from the left, but the integral is still absolutely convergent. If $t=0$, there is a jump at $t=v$. Finally, the integral of $\varphi(v)$ converges absolutely at $v=\pm\infty$. Under these circumstances we may apply Poisson's formula if $t>-1$, $t\neq 0$, for then there is no singularity at any point $v=0,\pm1,\pm2,\ldots$. We obtain from (6.7) by Poisson's formula

$$\sum_{-\infty}^{\infty}\varphi(n)=\sum_{-\infty}^{\infty}F_n(y+t)\,G_n(y)\qquad(t>-1,\ t\neq 0).$$

By (3.7), the series on the left equals $K(y, y+t)$, and so we have proved

(6.1). It is not necessary to interpret the series on the right as $\lim_{N\to\infty} \sum_{-N}^{N}$, for it can be seen from (6.2) that the series converges in the ordinary sense. It converges in the restricted sense if $t=0$, and then its sum is not equal to $K(y, y)$, but it is $\frac{1}{2}\{K(y, y+0)+K(y, y-0)\}=\frac{1}{2}$.

6.4. If a and b satisfy $0<a<b$, then the series (6.1) converges absolutely and uniformly for $x-b \leqslant y \leqslant x-a$ (x fixed). This easily follows from (6.2).

6.5. If a and b satisfy $-1<a<b$, then the series (6.1) converges in the mean for $x-b \leqslant y \leqslant x-a$. More precisely, if c and d are given ($c<d$), then

$$\lim_{N\to\infty} \int_a^b \Big| \sum_N^{\infty} F_n(x)\, G_n(x-t) \Big| \, dt = 0, \tag{6.9}$$

uniformly for $c \leqslant x \leqslant d$, and the same holds for $\sum_{-\infty}^{-N}$.

Formula (6.9) can be deduced from (6.2) and from the fact that

$$\lim_{N\to\infty} \int_a^b \Big| \sum_N^{\infty} e^{2\pi n i t}\, n^{-t-1} \Big| \, dt = 0. \tag{6.10}$$

The latter result can be proved as follows. We lose nothing by assuming $b<1$. Then $\sum_1^n e^{2\pi\nu i t} = O\,\{\min\,(n, |t|^{-1})\}$ $(a \leqslant t \leqslant b)$. Now partial summation of the sum occurring in (6.10) shows that the integral is less than a constant times

$$\int_a^b N^{-t-1} \min\,(N, |t|^{-1})\, dt + \int_a^b \sum_N^{\infty} n^{-t-2} \min\,(n, |t|^{-1})\, dt. \tag{6.11}$$

By changing the order of summation and integration, we easily verify that the expression (6.11) is $O(N^{-1-a})=o(1)$.

7. *Series expansion of arbitrary solutions*

7.1. Let $F(x)$ be a solution of (1.1) for $x \geqslant B$ (see sec. **1.2**). Then, by (3.4) and (6.1) we have (cf. sec. **6.4**)

$$F(x) = \sum_{-\infty}^{\infty} F_n(x)\, \{F, G_n\} \qquad (x>B+1), \tag{7.1}$$

and the series converges absolutely and uniformly with respect to x in any finite closed x-interval entirely to the right of $B+1$. Furthermore, it follows from sec. **6.5** that (7.1) is still true for $x>B$. The convergence is still uniform with respect to x in any finite and closed interval entirely to the right of B, but need no longer be absolute.

7.2. Needless to say, analogous things hold for the adjoint equation: if $G(y)$ is a solution of the adjoint equation for $y \leqslant C$, then

$$G(y) = \sum_{-\infty}^{\infty} G_n(y)\, \{F_n, G\}$$

for $y<C$. The convergence is absolute if $y<C-1$.

7.3. From sec. **7.1** and sec. **7.2** we infer that the biorthogonal system is complete in the sense that it cannot be extended. It is even complete in the following sense: If $f(x)$ is any absolutely integrable function over

$[a-1, a]$, then to any $\varepsilon > 0$ we can find a linear combination $\phi(x) = \sum_{-N}^{N} c_n F_n(x)$, such that

$$\int_{a-1}^{a} |f(x)-\phi(x)|\,dx + |f(a)-\phi(a)| < \varepsilon. \tag{7.2}$$

This can be shown on approximating $f(x)$ by a function $f_1(x)$ having two continuous derivatives, such that

$$\int_{a-1}^{a} |f_1(x)-f(x)|\,dx + |f_1(a)-f(a)| < \tfrac{1}{2}\varepsilon,$$

and such that

$$f_1'(a) = e^{\alpha a+\beta} f_1(a-1) \quad , \quad f_1''(a) = e^{\alpha a+\beta} f_1'(a-1) + \alpha e^{\alpha a+\beta} f_1(a-1).$$

This function f_1 can be continued over the interval $a-3 \leqslant x \leqslant \infty$, in such a way that it is a solution of (1.1) for $x \geqslant a-2$. Now applying (7.1), with $B=a-2$, we obtain a linear combination $\varphi(x)$ such that

$$|f_1(x)-\varphi(x)| < \tfrac{1}{4}\varepsilon \text{ in } a-1 \leqslant x \leqslant a.$$

This leads to (7.2).

8. *Asymptotic behaviour of arbitrary solutions*

8.1. Let $F(x)$ be an arbitrary solution of (1.1) for $x \geqslant B$, and write $\{F, G_n\} = c_n$. Hence $F(x) = \sum_{-\infty}^{\infty} c_n F_n(x)$ $(x > B)$. It follows from (5.17), (5.20), (5.15) and (4.3) that

$$c_n = O\left\{\exp\left(\frac{2\pi^2 n^2}{\alpha} - \frac{\pi^2|n|}{\alpha} - \frac{\log^2|n|}{2\alpha} - \frac{2\pi n\gamma i}{\alpha} + C\log|n|\right)\right\}, \tag{8.1}$$

where C is a constant. Denote by ψ the function

$$\psi(t) = \sum_{-\infty}^{\infty} c_n \exp\left\{-\frac{2\pi^2 n^2}{\alpha} + \frac{2\pi n i\gamma}{\alpha} + 2\pi n i t\right\}. \tag{8.2}$$

Hence ψ is a periodic function of t, with period 1, and it is analytic in a symmetrical horizontal strip of width π/α. We shall prove (1.3), or, what is the same thing,

$$F(x) = F_0(x)\{\psi(x - \alpha^{-1}\log x + \alpha^{-2}x^{-1}\log x) + O(x^{-1})\}, \tag{8.3}$$

for x real, $x \to \infty$. We cannot claim its validity in the strip just mentioned, for $F(x)$ is given only if x is real, and $F(x)$ is not assumed to be analytical.

It is obvious from (5.19) that (8.3) holds if $F(x)$ is any finite linear combination of the F_n's. The difficulty lies in the general case, where we have to obtain certain estimates that are uniform in both n and x.

8.2. In the following, C denotes a suitable positive number, independent of n and x, not necessarily the same at each occurrence. We shall assume $|n| \geqslant 2$, $x > C$. From (8.1) and (8.2) we obtain, for t real,

$$\left| c_n \exp\left\{-\frac{2\pi^2 n^2}{\alpha} + 2\pi n i t + \frac{2\pi n i\gamma}{\alpha}\right\}\right| < C\,e^{-C|n|}. \tag{8.4}$$

We next estimate $c_n F_n(x)/F_0(x)$. Write $2\pi n i/\alpha = \Delta$. Noting that

$$\operatorname{Re}\{\log^2(x+\Delta)-\log^2 x-\log^2\Delta\}<C,$$

we infer from (5.15) and (8.1) that

$$\log|c_n F_n(x)/F_0(x)| < \operatorname{Re}\{-(x+\Delta)\log(x+\Delta)+ \\ +x\log x+\Delta\log\Delta\}+C\log|n|.$$

The first term on the right equals

$$-|\Delta|\cdot\left|\arg\left(1+\frac{x}{\Delta}\right)\right|-x\log\left|1+\frac{\Delta}{x}\right|.$$

Hence we obtain as upper bounds

$$\log|c_n F_n(x)/F_0(x)|<\begin{cases}-C\,|n| & (|n|<x)\\ -C\,x & (x\leqslant |n|<x^2)\\ -C\,x\log n & (|n|\geqslant x^2).\end{cases} \tag{8.5}$$

Finally, assuming $|n|<x^{\frac{1}{4}}$, we easily calculate from (5.18)

$$\frac{F_n(x)}{F_0(x)}\exp\left(\frac{2\pi^2 n^2}{\alpha}-\frac{2\pi n i\gamma}{\alpha}\right)-\exp\left\{2\pi n i\left(x-\frac{\log x}{\alpha}+\frac{\log x}{\alpha^2 x}\right)\right\}=O(n^2x^{-1}),$$

and it follows, by (8.4), that

$$\left|c_n\left\{\frac{F_n(x)}{F_0(x)}-\exp\left\{-\frac{2\pi^2n^2}{\alpha}+\frac{2\pi n i\gamma}{\alpha}+2\pi n i\left(x-\frac{\log x}{\alpha}+\frac{\log x}{\alpha^2 x}\right)\right\}\right\}\right|<\frac{e^{-C|n|}}{x}. \tag{8.6}$$
$$(|n|<x^{\frac{1}{4}}).$$

On summing the expression between vertical bars on the left over all values of n, we obtain

$$\left|\frac{F(x)}{F_0(x)}-\psi(x-\alpha^{-1}\log x+\alpha^{-2}x^{-1}\log x)\right|<\sum_{|n|\leqslant x^{\frac{1}{4}}}x^{-1}e^{-C|n|}+ \\ +\sum_{|n|>x^{\frac{1}{4}}}Ce^{-C|n|}+\sum_{x^{\frac{1}{4}}<|n|<x}e^{-C|n|}+\sum_{x\leqslant|n|\leqslant x^2}e^{-Cx}+\sum_{|n|\geqslant x^2}n^{-Cx}.$$

As all these sums are $O(x^{-1})$, we have proved (8.3).

8.3.. The analysis of sec. **8.2** also shows the following refinement of (8.3): If B is given, then numbers $B_1>B$ and $C>0$ can be found, such that for every function F that is a solution for $x\geqslant B$, we have

$$\left|\frac{F(x)}{F_0(x)}-\psi(x-\alpha^{-1}\log x+\alpha^{-2}x^{-1}\log x)\right|<Cx^{-1}\max_{B-1\leqslant t\leqslant B}|F(t)|\quad(x\geqslant B_1).$$

8.4. To every solution $F(x)$ (whatever B may be) there corresponds, according to (8.3), a periodic function ψ. The correspondence is linear, so that we have a linear mapping of the linear space of all solutions into the linear space of periodic functions (with period 1). We denote it by $\psi=\mathbf{T}f$. This mapping is one-to-one (if solutions which differ only in a finite number of points in an interval $(B-1,\ B)$ are identified). For, $\mathbf{T}f=0$ implies that all c's are zero (cf. (8.2)), and so $F=\sum c_n F_n=0$.

8.5. Formula (8.3) is best possible in the sense that there is no similar formula with a smaller O-term. Suppose that there is a real continuous function $\eta(x)$ ($\eta(x) \to \infty$ if $x \to \infty$), such that every solution of (1.1) has the form

$$F(x) = F_0(x)\ [\psi\{\eta(x)\} + o(x^{-1})], \tag{8.7}$$

where ψ is periodic mod 1, and of course depending on F. A contradiction is obtained from the fact that in the behaviour of $F_n(x)/F_0(x)$ (for $x \to \infty$) there is a term $2\pi^2 n^2/\alpha^2 x$ that is not linear in n. For, the following refinement of (5.19) follows from (5.18) ($|x| \to \infty$, n fixed):

$$F_n(x) = F_0(x) \exp\left\{-\frac{2\pi^2 n^2}{\alpha} + \frac{2\pi n i}{\alpha}\left[\alpha x - \log x + \gamma + \right.\right.$$
$$\left.\left. + \frac{\log x}{\alpha x} - \left(\frac{1}{2} + \frac{\gamma}{\alpha}\right)\frac{1}{x}\right] + \frac{2\pi^2 n^2}{\alpha^2 x} + o\left(\frac{1}{x}\right)\right\}.$$

We infer that

$$\{F_1^2(x)\,F_{-2}(x)\,F_0^{-3}(x)\} = \exp\,(-12\,\pi^2\alpha^{-1} + 12\,\pi^2\alpha^{-2}x^{-1} + o(x^{-1})),$$

but (8.7) would show that the same expression is $\psi_1\{\eta(x)\} + o(x^{-1})$, where ψ_1 is periodic with period 1. Therefore, $\psi_1(\eta(x)) = C_1 + C_2 x^{-1} + o(x^{-1})$, $C_2 \neq 0$. Making x to run through a sequence of values $x_1, x_2, \ldots$ with the property that the numbers $\eta(x_i)$ are integers, we obtain a contradiction, as $C_2 \neq 0$.

8.6. Explicit expressions for the operator $\mathbf{T}$ are easily obtained. We have, by (7.1) and (8.2),

$$\psi(t) = \mathbf{T}F(x) = \sum_{-\infty}^{\infty} \{F, G_n\}\ \exp\,\{-2\pi n^2\alpha^{-1} + 2\pi n i\,(t + \gamma\alpha^{-1})\}. \tag{8.8}$$

Especially we obtain for $\Psi\,(t;\,y) = \mathbf{T}\,K(y,\,x)$ a series that, in the notation of (4.12), equals $\Psi(t;\,y) = G(y,\,t)$. This function provides the kernel of $\mathbf{T}$ in the following sense

$$\mathbf{T}\,F(x) = \{F(y),\,G(y,\,t)\} = F(B)\,G(B,\,t) + \int_{B-1}^{B} e^{\alpha s + \alpha + \beta}\,F(s)\,G(s+1,\,t)\,ds.$$

8.7. The special solutions $G_n(x)$ of the adjoint equation show the same type of asymptotical periodicity as $F_n(x)$, in the sense that $G_n(x)/G_0(x)$ tends to a periodic function of $x - \alpha^{-1}\log x + \alpha^{-2}x^{-1}\log x$ as $x \to \infty$. However, the analogue of (8.3) holds only under severe restrictions. In the first place, not every solution of the adjoint equation for $x \leqslant C$ can be continued up to $x = +\infty$. And, if a series $\sum d_n\,G_n(x)$ happens to converge for all x, which occurs if and only if

$$d_n = O\left\{\exp\left(-\frac{2\pi^2 n^2}{\alpha} + \frac{\pi^2|n|}{\alpha} + \frac{\log^2|n|}{2\alpha} - p\log|n|\right)\right\}$$

holds for every real p, then the series which corresponds to (8.2), viz.

$$\sum_{-\infty}^{\infty} d_n \exp\left\{+\frac{2\pi^2 n^2}{\alpha} - \frac{2\pi n i\gamma}{\alpha} - 2\pi n i t\right\}$$

need not converge. If it converges, then $\sum d_n\,G_n(x)$ is analytical in the infinite strip $|\mathrm{Im}\,x| < \frac{1}{2}\pi/\alpha$.

(For References see at the end of part I of this paper).