Some auxiliary operators in AUT-PI

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For AUT-Π we refer to Zucker [2]. If we omit all those features that the languages of the AUTOMATH family have in common (cf. the description of AUT-QE in D. van Daalen [1]), the basic rules are the following (i) - (vii). Two simplifications are made here. First, we use a symbol τ which may be either type or prop. Secondly, we omit all Π's in expressions of degree 1, which does not make any essential difference. And we use the notation $(x : \alpha) \vdash$ in order to indicate that something is valid in the context extended by x (of type α). As in [1], $[\![x/A]\!]\ Z$ means that in Z we have to replace x by A.

The rules are

(i) $\vdash^1 \tau$

(ii) $$\frac{\vdash^2 \alpha : \tau \qquad (x : \alpha) \vdash^1 P}{\vdash^1 [x : \alpha]\, P}$$

(iii) $$\frac{\vdash^2 \alpha : \tau \qquad (x : \alpha) \vdash^2 Q : P}{\vdash^2 [x : \alpha]Q \ : [x : \alpha]P}$$

(iv) $$\frac{\vdash^3 A : \alpha : \tau \qquad \vdash^2 Q : [x : \alpha]P}{\vdash^2 \{A\}Q \ : \ [\![x/A]\!]\ P}$$

(v) $$\frac{\vdash^2 \alpha : \tau \qquad \vdash^2 Q : [x : \alpha]\tau}{\vdash^2 \Pi Q \ : \ \tau}$$

(vi) $$\frac{\vdash^2 \alpha : \tau \qquad (x : \alpha) \vdash^3 R : Q : \tau}{\vdash^3 [x : \alpha]R : [x : \alpha]Q}$$

(vii) $$\frac{\vdash^3 A : \alpha : \tau \qquad \vdash^3 R : \Pi Q \qquad \vdash^2 Q : [x : \alpha]\tau}{\vdash^3 \{A\}R \ : \{A\}\ Q}$$

We shall now define operators $\theta_1, \theta_2, \ldots, \theta_m$, acting on Q's with

$$\vdash^2 Q : [x_1 : \alpha_1] \ldots [x_m : \alpha_m]\tau$$

The symbols are metalinguistic: $\theta_j Q$ is used in the metalanguage to indicate a certain expression in the language, viz.

$$\theta_1\ Q = [x_1 : \alpha_1] \ldots [x_{m-1} : \alpha_{m-1}]\ \Pi\ \{x_{m-1}\}\ \{x_{m-2}\} \ldots \{x_1\}\ Q$$

$$\theta_2\ Q = [x_1 : \alpha_1] \ldots [x_{m-2} : \alpha_{m-2}]\ \Pi\ [x_{m-1} : \alpha_{m-1}]\ \Pi\ \{x_{m-1}\} \ldots \{x_1\} Q$$

....

$$\theta_{m-1}\ Q = [x_1 : \alpha_1]\ \Pi\ [x_2 : \alpha_2]\ \Pi \ldots \Pi\ [x_{m-1} : \alpha_{m-1}]\ \Pi\ \{x_{m-1}\} \ldots \{x_1\} Q$$

$$\theta_m\ Q = \Pi [x_1 : \alpha_1]\ \Pi\ [x_2 : \alpha_2]\ \Pi \ldots \Pi\ [x_{m-1} : \alpha_{m-1}]\ \Pi\ \{x_{m-1}\} \ldots \{x_1\} Q$$

Note that θ_j is built by starting from the expression just given for $\theta_m Q$ and then omitting the first m-j Π's. If m=1 we just have $\theta_1 Q = \Pi Q$. If m=2 then $\theta_1\ Q =$ $= [x_1 : \alpha_1]\ \Pi\ \{x_1\} Q$ and $\theta_2\ Q = \Pi [x_1 : \alpha_1]\ \Pi\ \{x_1\} Q$. If m=0 none of the θ_j's are defined.

We can now prove the validity of a new rule, viz:

(viii) $$\frac{\vdash^2 \alpha : \tau \qquad \vdash^2 Q : [x_1 : \alpha_1] \ldots [x_m : \alpha_m]\tau}{\vdash^2 \theta_j\ Q : [x_1 : \alpha_1] \ldots [x_{m-j} : \alpha_{m-j}]\tau}$$

for $1 \leq j \leq m$. If j=m-1 it is just the old rule (v).

For shortness we shall write $[x_i]$ and (x_i) instead of $[x_i : \alpha_i]$ and $(x_i : \alpha_i)$.

Let us start from

$$\vdash^2 \alpha : \tau \qquad \vdash^2 Q : [x_1] \ldots [x_m]\tau \tag{1}$$

Applying (iv) we get

$$(x_i) \vdash^2 \{x_1\} Q : [x_1] \ldots [x_m]\tau$$

and m-2 more applications of the same rule leads to

$$(x_1) \ldots (x_{m-1}) \vdash^2 \{x_{m-1}\} \ldots \{x_1\}\ Q : [x_m]\tau$$

Next we apply (v):

$$(x_1) \ldots (x_{m-1}) \vdash^2 \Pi \{x_{m-1}\} \ldots \{x_1\} Q : \tau$$

and by (iii) this gives

$$(x_1)\ldots(x_{m-2}) \models^2 [x_{m-1}]\,\Pi\,\{x_{m-1}\}\ldots\{x_1\}\,Q : [x_{m-1}]\tau \qquad (2)$$

Now m-2 further applications of (iii) gives

$$\models^2 \theta_1 Q : [x_1]\ldots[x_{m-1}]\,\tau$$

On the other hand, if we apply (v) to (2) followed by a single application of (iii) we get

$$(x_1)\ldots(x_{m-3}) \models^2 [x_{m-2}]\,\Pi\,[x_{m-1}]\,\Pi\,\{x_{m-1}\}\ldots\{x_1\}Q : [x_{m-2}]\tau \qquad (3)$$

Now m-3 more applications of (iii) lead to

$$\models^2 \theta_2 Q : [x_1]\ldots[x_{m-2}]\tau.$$

On the other hand, if we apply (v) followed by (iii) to (3) we get

$$(x_1)\ldots(x_{m-4}) \models^2 [x_{m-3}]\,\Pi\,[x_{m-2}]\,\Pi\,[x_{m-1}]\,\Pi\,\{x_{m-1}\}\ldots\{x_1\}Q : [x_{m-3}]\tau.$$

This way we get, indeed

$$\models^2 \theta_j Q : [x_1]\ldots[x_{m-j}]\,\tau \qquad (4)$$

for all j $(1 \leq j \leq m)$.

We shall also show that $\theta_i\theta_j \overset{D}{=} \theta_{i+j}$. More precisely, if $\models^2 Q : [x_1 : \alpha_1]\ldots$ $\ldots[x_m : \alpha_m]\,\tau$, and if $i \geq 1$, $j \geq 1$, $i+j \leq m$, then $\theta_i\theta_j\ Q$ reduces to $\theta_{i+j}\ Q$ by means of repeated β-reduction. First we have (4), i.e.

$$\models^2 [x_1]\ldots[x_{m-j}]\,\Pi\,[x_{m-j+1}]\,\Pi\ldots\Pi[x_{m-1}]\,\Pi\,\{x_{m-1}\}\ldots\{x_1\}Q$$

$$: [x_1]\ldots[x_{m-j}]\tau.$$

Applying θ_i to this we get

$$\models^2 \theta_i\theta_j\ Q : [x_1]\ldots[x_{m-j-i}]\,\tau$$

and

$$\theta_i\,\theta_j\; Q = [y_1]\ldots[y_{m-j-1}]\;\Pi\;[y_{m-j-i+1}]\;\Pi\ldots\Pi\;[\,y_{m-j-1}]\;\Pi\;\{\,y_{m-j-1}\}\ldots$$

$$\ldots\;\{y_1\}\;[x_1]\ldots[x_{m-j}]\;\Pi\;[x_{m-j+1}]\Pi\ldots\Pi[x_{m-1}]\;\Pi\;\{x_{m-1}\}\ldots\{x_1\}Q.$$

The sequence $\{y_{m-j-1}\}\ldots\{y_1\}\;[x_1]\ldots[x_{m-j-1}]$ is annihilated by m applications of β-reduction. After that, we change the names $y_1,\ldots y_{m-j-1}$ into $x_1,\ldots,x_{m-j-1}$, thus arriving at $\theta_{i+j}\;Q$.

Above we extended rule (v) to rule (viii). Similarly we shall extend rule (vii) to the following rule (ix) for $m \geq 1$:

$$\text{(ix)}\qquad \frac{\vdash^3 A : \alpha_1 \qquad \vdash^3 R : \theta_m\;Q \qquad \vdash^2 Q : [x_1 : \alpha_1]\ldots[x_m : \alpha_m]\tau}{\vdash^3 \{A\}\;\;R \quad : \quad \{A\}\,\theta_{m-1}\;Q}$$

If m=1 we have $\theta_m Q = \Pi\,Q$, and $\theta_{m-1}\;Q$ has to be explained as Q itself (θ_o was not defined before).

Rule (ix) is not hard to derive. Noting that $\theta_m\;Q = \Pi\theta_{m-1}\;Q$, and $\vdash^2 \theta_{m-1}\;Q$: : $[x_1 : \alpha_1]\tau$ by (viii), we can apply (vii) with Q replaced by $\theta_{m-1}\;Q$, which leads to $\vdash^3 \{A\}\;R : \{A\}\theta_{m-1}\;Q$.

We note that in all rules formulas of the type $\vdash^3 R : Q$ lead to $\vdash^2 Q : \tau$. Indeed, in (vi) we have $\vdash^2\;\Pi\,[x : \alpha]Q : \tau$ by (v), and in (ix) we have $\vdash^2 \{A\}\theta_{m-1}Q : \tau$ by (iv), according to the typing of $\theta_{m-1}\;Q$ just derived.

Instead of the lower kind of (ix) we may as well get

$$\vdash^3 \{A\}\;R : \;\theta_{m-1}\;\;\{A\}\;Q$$

since $\{A\}\theta_{m-1}\;Q$ reduces to $\theta_{m-1}\;\{A\}\;Q$ by a single beta reduction.

More generally we observe that

$$\{A\}\theta_j\,Q \text{ reduces to } \theta_j\;\;\{A\}\;Q$$

by a single beta reduction if $j < m$.

The symbols θ_j also commute with abstraction : if

$$\vdash^2 Q : [x_1 : \alpha_1]\ldots[x_m : \alpha_m]\tau \text{ then}$$

$$[y : \beta]\;\theta_j\;\;Q \text{ reduces to } \theta_j\;[y : \beta]Q$$

if $j \leq m$.

These observations mean that in composite expressions like $\{\;\}\;\;\theta_i\;\;[\;\;]\;\theta_j\;\{\;\}\;\;\{\;\}\;\theta_k\;\;[\;\;]\;Q$ the θ's may all be shifted to the extreme left.

References:

1. D. van Daalen: A description of AUTOMATH and some aspects of its language theory. Proceedings of the Symposium APLASM. Vol. I, ed. P. Braffort, (Orsay) France (December 1973).
(This paper is reproduced in L.S. van Benthem Jutting, Checking Landau's "Grundlagen" in the AUTOMATH system. Thesis, Technological University Eindhoven, 1977).

2. J.Zucker: Formalization of classical mathematics in AUTOMATH. Colloque international de logique, Clermont-Ferrand 18-25 July 1975. Colloques internationaux du Centre National de la Recherche Scientifique Nr. 249.