

Some auxiliary operators in AUT-PI

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SOME AUXILIARY OPERATORS IN AUT-II.

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For AUT-II we refer to Zucker [2]. If we omit all those features that the languages of the AUTOMATH family have in common (cf. the description of AUT-QE in D. van Daalen [1]), the basic rules are the following (i) - (vii). Two simplifications are made here. First, we use a symbol τ which may be either type or prop. Secondly, we omit all Π 's in expressions of degree 1, which does not make any essential difference. And we use the notation $(x : \alpha) \vdash$ in order to indicate that something is valid in the context extended by x (of type α). As in [1], $\llbracket x/A \rrbracket Z$ means that in Z we have to replace x by A .

The rules are

- (i) $\vdash \tau$
- (ii)
$$\frac{\vdash^2 \alpha : \tau \quad (x : \alpha) \vdash^1 P}{\vdash^1 [x : \alpha] P}$$
- (iii)
$$\frac{\vdash^2 \alpha : \tau \quad (x : \alpha) \vdash^2 Q : P}{\vdash^2 [x : \alpha] Q : [x : \alpha] P}$$
- (iv)
$$\frac{\vdash^3 A : \alpha : \tau \quad \vdash^2 Q : [x : \alpha] P}{\vdash^2 \{A\} Q : \llbracket x/A \rrbracket P}$$
- (v)
$$\frac{\vdash^2 \alpha : \tau \quad \vdash^2 Q : [x : \alpha] \tau}{\vdash^2 \Pi Q : \tau}$$
- (vi)
$$\frac{\vdash^2 \alpha : \tau \quad (x : \alpha) \vdash^3 R : Q : \tau}{\vdash^3 [x : \alpha] R : [x : \alpha] Q}$$
- (vii)
$$\frac{\vdash^3 A : \alpha : \tau \quad \vdash^3 R : \Pi Q \quad \vdash^2 Q : [x : \alpha] \tau}{\vdash^3 \{A\} R : \{A\} Q}$$

We shall now define operators $\theta_1, \theta_2, \dots, \theta_m$, acting on Q 's with

$$\stackrel{2}{\vdash} Q : [x_1 : \alpha_1] \dots [x_m : \alpha_m] \tau$$

The symbols are metalinguistic: $\theta_j Q$ is used in the metalanguage to indicate a certain expression in the language, viz.

$$\theta_1 Q = [x_1 : \alpha_1] \dots [x_{m-1} : \alpha_{m-1}] \Pi \{x_{m-1}\} \{x_{m-2}\} \dots \{x_1\} Q$$

$$\theta_2 Q = [x_1 : \alpha_1] \dots [x_{m-2} : \alpha_{m-2}] \Pi [x_{m-1} : \alpha_{m-1}] \Pi \{x_{m-1}\} \dots \{x_1\} Q$$

.....

$$\theta_{m-1} Q = [x_1 : \alpha_1] \Pi [x_2 : \alpha_2] \Pi \dots \Pi [x_{m-1} : \alpha_{m-1}] \Pi \{x_{m-1}\} \dots \{x_1\} Q$$

$$\theta_m Q = \Pi [x_1 : \alpha_1] \Pi [x_2 : \alpha_2] \Pi \dots \Pi [x_{m-1} : \alpha_{m-1}] \Pi \{x_{m-1}\} \dots \{x_1\} Q$$

Note that θ_j is built by starting from the expression just given for $\theta_m Q$ and then omitting the first $m-j$ Π 's. If $m=1$ we just have $\theta_1 Q = \Pi Q$. If $m=2$ then $\theta_1 Q = [x_1 : \alpha_1] \Pi \{x_1\} Q$ and $\theta_2 Q = \Pi [x_1 : \alpha_1] \Pi \{x_1\} Q$. If $m=0$ none of the θ_j 's are defined.

We can now prove the validity of a new rule, viz:

$$(viii) \quad \frac{\stackrel{2}{\vdash} \alpha : \tau \quad \stackrel{2}{\vdash} Q : [x_1 : \alpha_1] \dots [x_m : \alpha_m] \tau}{\stackrel{2}{\vdash} \theta_j Q : [x_1 : \alpha_1] \dots [x_{m-j} : \alpha_{m-j}] \tau}$$

for $1 \leq j \leq m$. If $j=m-1$ it is just the old rule (v).

For shortness we shall write $[x_i]$ and (x_i) instead of $[x_i : \alpha_i]$ and $(x_i : \alpha_i)$.

Let us start from

$$\stackrel{2}{\vdash} \alpha : \tau \quad \stackrel{2}{\vdash} Q : [x_1] \dots [x_m] \tau \tag{1}$$

Applying (iv) we get

$$(x_1) \stackrel{2}{\vdash} \{x_1\} Q : [x_1] \dots [x_m] \tau$$

and $m-2$ more applications of the same rule leads to

$$(x_1) \dots (x_{m-1}) \stackrel{2}{\vdash} \{x_{m-1}\} \dots \{x_1\} Q : [x_m] \tau$$

Next we apply (v):

$$(x_1) \dots (x_{m-1}) \stackrel{2}{\vdash} \Pi \{x_{m-1}\} \dots \{x_1\} Q : \tau$$

and by (iii) this gives

$$(x_1) \dots (x_{m-2}) \stackrel{\beta}{\vdash} [x_{m-1}] \Pi \{x_{m-1}\} \dots \{x_1\} Q : [x_{m-1}] \tau \quad (2)$$

Now $m-2$ further applications of (iii) gives

$$\stackrel{\beta}{\vdash} \theta_1 Q : [x_1] \dots [x_{m-1}] \tau$$

On the other hand, if we apply (v) to (2) followed by a single application of (iii) we get

$$(x_1) \dots (x_{m-3}) \stackrel{\beta}{\vdash} [x_{m-2}] \Pi [x_{m-1}] \Pi \{x_{m-1}\} \dots \{x_1\} Q : [x_{m-2}] \tau \quad (3)$$

Now $m-3$ more applications of (iii) lead to

$$\stackrel{\beta}{\vdash} \theta_2 Q : [x_1] \dots [x_{m-2}] \tau.$$

On the other hand, if we apply (v) followed by (iii) to (3) we get

$$(x_1) \dots (x_{m-4}) \stackrel{\beta}{\vdash} [x_{m-3}] \Pi [x_{m-2}] \Pi [x_{m-1}] \Pi \{x_{m-1}\} \dots \{x_1\} Q : [x_{m-3}] \tau.$$

This way we get, indeed

$$\stackrel{\beta}{\vdash} \theta_j Q : [x_1] \dots [x_{m-j}] \tau \quad (4)$$

for all j ($1 \leq j \leq m$).

We shall also show that $\theta_i \theta_j \stackrel{D}{=} \theta_{i+j}$. More precisely, if $\stackrel{\beta}{\vdash} Q : [x_1 : \alpha_1] \dots [x_m : \alpha_m] \tau$, and if $i \geq 1$, $j \geq 1$, $i+j \leq m$, then $\theta_i \theta_j Q$ reduces to $\theta_{i+j} Q$ by means of repeated β -reduction. First we have (4), i.e.

$$\begin{aligned} & \stackrel{\beta}{\vdash} [x_1] \dots [x_{m-j}] \Pi [x_{m-j+1}] \Pi \dots \Pi [x_{m-1}] \Pi \{x_{m-1}\} \dots \{x_1\} Q \\ & : [x_1] \dots [x_{m-j}] \tau. \end{aligned}$$

Applying θ_i to this we get

$$\stackrel{\beta}{\vdash} \theta_i \theta_j Q : [x_1] \dots [x_{m-j-i}] \tau$$

and

$$\theta_i \theta_j Q = [y_1] \dots [y_{m-j-1}] \Pi [y_{m-j-i+1}] \Pi \dots \Pi [y_{m-j-1}] \Pi \{y_{m-j-1}\} \dots$$

$$\dots \{y_1\} [x_1] \dots [x_{m-j}] \Pi [x_{m-j+1}] \Pi \dots \Pi [x_{m-1}] \Pi \{x_{m-1}\} \dots \{x_1\} Q.$$

The sequence $\{y_{m-j-1}\} \dots \{y_1\} [x_1] \dots [x_{m-j-1}]$ is annihilated by m applications of β -reduction. After that, we change the names y_1, \dots, y_{m-j-1} into x_1, \dots, x_{m-j-1} , thus arriving at $\theta_{i+j} Q$.

Above we extended rule (v) to rule (viii). Similarly we shall extend rule (vii) to the following rule (ix) for $m \geq 1$:

$$(ix) \quad \frac{\vdash^3 A : \alpha_1 \quad \vdash^3 R : \theta_m Q \quad \vdash^2 Q : [x_1 : \alpha_1] \dots [x_m : \alpha_m] \tau}{\vdash^3 \{A\} R : \{A\} \theta_{m-1} Q}$$

If $m=1$ we have $\theta_m Q = \Pi Q$, and $\theta_{m-1} Q$ has to be explained as Q itself (θ_0 was not defined before).

Rule (ix) is not hard to derive. Noting that $\theta_m Q = \Pi \theta_{m-1} Q$, and $\vdash^2 \theta_{m-1} Q : [x_1 : \alpha_1] \tau$ by (viii), we can apply (vii) with Q replaced by $\theta_{m-1} Q$, which leads to $\vdash^3 \{A\} R : \{A\} \theta_{m-1} Q$.

We note that in all rules formulas of the type $\vdash^3 R : Q$ lead to $\vdash^2 Q : \tau$. Indeed, in (vi) we have $\vdash^2 \Pi [x : \alpha] Q : \tau$ by (v), and in (ix) we have $\vdash^2 \{A\} \theta_{m-1} Q : \tau$ by (iv), according to the typing of $\theta_{m-1} Q$ just derived.

Instead of the lower kind of (ix) we may as well get

$$\vdash^3 \{A\} R : \theta_{m-1} \{A\} Q$$

since $\{A\} \theta_{m-1} Q$ reduces to $\theta_{m-1} \{A\} Q$ by a single beta reduction.

More generally we observe that

$$\{A\} \theta_j Q \text{ reduces to } \theta_j \{A\} Q$$

by a single beta reduction if $j < m$.

The symbols θ_j also commute with abstraction : if

$$\vdash^2 Q : [x_1 : \alpha_1] \dots [x_m : \alpha_m] \tau \text{ then}$$

$$[y : \beta] \theta_j Q \text{ reduces to } \theta_j [y : \beta] Q$$

if $j \leq m$.

These observations mean that in composite expressions like

$\{ \} \theta_i [\] \theta_j \{ \} \{ \} \theta_k [\] Q$ the θ 's may all be shifted to the extreme left.

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(This paper is reproduced in L.S. van Benthem Jutting, Checking Landau's "Grundlagen" in the AUTOMATH system. Thesis, Technological University Eindhoven, 1977).

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