On the zeros of composition-polynomials

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1. Introduction.

Recently [1]) we proved some inequalities, expressing that the zeros of the derivative of a polynomial lie, in the mean, closer to a given line or a given point in the complex plane than the zeros of the polynomial itself. One may expect that similar inequalities are valid if, instead of the derivative, a polynomial derived in a more general way from the given one is considered. Here we shall prove inequalities of this type for polynomials obtained by composition from two given polynomials.

This composition is defined in the following way [2]): if

$$A(z) \equiv \binom{n}{0} a_0 + \binom{n}{1} a_1 z + \binom{n}{2} a_2 z^2 + \ldots + \binom{n}{n} a_n z^n$$

and

$$B(z) \equiv \binom{n}{0} b_0 + \binom{n}{1} b_1 z + \binom{n}{2} b_2 z^2 + \ldots + \binom{n}{n} b_n z^n$$

are the two given polynomials, then the composition-polynomial is

$$AB(z) \equiv \binom{n}{0} a_0 b_0 + \binom{n}{1} a_1 b_1 z + \binom{n}{2} a_2 b_2 z^2 + \ldots + \binom{n}{n} a_n b_n z^n.$$

If $B(z) \equiv nz(1+z)^{n-1}$ we have $AB(z) \equiv zA'(z)$. For this special choice of $B(z)$ most of the theorems proved in this paper give rise to results already proved in I and II.

Throughout this paper the zeros of $A(z)$, $B(z)$ and $AB(z)$ will be denoted by $\alpha_1, \ldots, \alpha_n$; $\beta_1, \ldots, \beta_n$ and $\gamma_1, \ldots, \gamma_n$, respectively. Furthermore we put

$$\{A, B\} = \binom{n}{0} a_0 b_n - \binom{n}{1} a_1 b_{n-1} + \ldots + (-1)^n \binom{n}{n} a_n b_0.$$

We shall often use the following well-known theorem of J. H. GRACE

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1) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 1037—1044 (1946) = Indagationes Mathematicae **8**, 635—643; Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 458—464 (1947) = Indagationes Mathematicae **9**, 264—270. These papers are referred to as I and II.

2) This way of composition was introduced by G. SZEGÖ, Math. Zeitschrift, **13**, 28—55 (1922).

("Grace's Apolarity Theorem") [3]), which is rather important in problems concerning the geometric properties of the roots of polynomials:

Theorem 1: *If the two polynomials $A(z)$ and $B(z)$ satisfy $\{A,B\}=0$, then any circle-domain* [4]) *containing all zeros of $A(z)$ contains at least one zero of $B(z)$.*

From this theorem we deduce our fundamental

Lemma: *Let $A(z)$ and $B(z)$ be two polynomials and let D_1 and D_2 be the two closed domains determined by a circle (or a straight line) C. The polynomial $A^*(z)$ is derived from $A(z)$ by replacing the zeros of $A(z)$ which lie in D_1 by their images with respect to C, and is normed such that $|A^*(z)|=|A(z)|$ on C* [5]); *$B^{**}(z)$ is obtained from $B(z)$ in a similar way, but now the zeros in D_2 are replaced by their images with respect to C. Then we have*

$$|\{A,B\}| \leqslant |\{A^*,B^{**}\}|. \quad \ldots \ldots \ldots \quad (1)$$

Proof: We may suppose that $A(z)$ and $B(z)$ are $\neq 0$ on C; the general case then follows by an argument of continuity.

Since no zeros of $A^*(z)$ lie in D_1, and since $|A(z)|=|A^*(z)|$ if z lies on C, it follows by application of the maximum modulus theorem, that $|A(z)|\leq|A^*(z)|$ in D_1. In the same way it can be shown that $|B(z)|\leq B^{**}(z)|$ in D_2. Hence the polynomials $A^*(z)-\lambda A(z)$ and $B^{**}(z)-\lambda B(z)$ have no zeros in D_1 and D_2 respectively, if λ is any number satisfying $|\lambda|<1$. By theorem 1 we now have $\{A^*-\lambda A, B^{**}-\lambda B\}\neq 0$ for $|\lambda|<1$. Thus the polynomial

$$\lambda^2\{A,B\}-\lambda(\{A,B^{**}\}+\{A^*,B\})+\{A^*,B^{**}\}$$

has no zeros in $|\lambda|<1$, and it follows that $|\{A,B\}|\leq|\{A^*,B^{**}\}|$.

In section 2 inequalities are derived for the distances of the zeros to a line. The following theorem shows the character of the results obtained in that section:

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[3]) J. H. GRACE, Proc. Cambridge Phil. Soc. **11**, 352—357 (1900—'02). — G. SZEGÖ, loc. cit. [2]). — G. PÓLYA and G. SZEGÖ, Aufgaben und Lehrsätze aus der Analysis, II, p. 64, Aufgabe 145.

[4]) A circle-domain is a closed part of the plane whose boundary is a circle or a straight line.

[5]) So if $A(z)\equiv A(z-\varrho_1)\ldots(z-\varrho_n)$, $\varrho_1,\ldots,\varrho_k$ in D_2, $\varrho_{k+1},\ldots,\varrho_n$ in D_1 but no on C, and if σ lies on C then

$$A^*(z)\equiv\varepsilon A\prod_{i=1}^{k}(z-\varrho_i)\prod_{i=k+1}^{n}(z-\varrho_i^*)\cdot\frac{\sigma-\varrho_i}{\sigma-\varrho_i^*},$$

where ϱ_i^* represents the image of ϱ_i with respect to C, and $|\varepsilon|=1$. The true value fo ε is irrelevant.

Theorem 2: *If all β_i are ≤ 0, then we have*

$$\sum_{\nu=1}^{n} |Im\, \gamma_\nu| \leqslant \frac{b_{n-1}}{b_n} \sum_{\nu=1}^{n} |Im\, \alpha_\nu|. \quad \ldots \ldots \quad (2)$$

This is a special case of the more general theorem 3.

Theorems 4 and 5, which are derived from theorem 3, may also be of some interest.

In section 3 we are dealing with the moduli of the zeros. There the main result is

Theorem 9: *With the same notations as before we have, if p and k are real numbers, $k \geq 1$, $\frac{1}{k} + \frac{1}{k'} = 1$,*

$$\sum_{\nu=1}^{n} |\gamma_\nu|^p \leqslant \left(\sum_{\nu=1}^{n} |\alpha_\nu|^{pk}\right)^{\frac{1}{k}} \cdot \left(\sum_{\nu=1}^{n} |\beta_\nu|^{pk'}\right)^{\frac{1}{k'}}. \quad \ldots \ldots \quad (3)$$

This inequality is deduced from a more general one, which contains an arbitrary convex function (Theorem 8).

2. We prove the following theorem on the distances of the zeros of a composition-polynomial to a straight line:

Theorem 3: *If $\psi(z, l)$ denotes the distance of the point z in the complex plane to a line l we have*

$$\text{a)} \quad \sum_{\nu=1}^{n} \psi(\gamma_\nu, l) \leqslant \frac{b_{n-1}}{b_n} \sum_{\nu=1}^{n} \psi(\alpha_\nu, l),$$

if the line l contains the point $z = 0$ and all zeros β_i satisfy $\beta_i \leq 0$ $(i = 1, 2, \ldots, n)$,

and

$$\text{b)} \quad \sum_{\nu=1}^{n} \{\psi(\gamma_\nu, l) - \psi(0, l)\} \leqslant \frac{b_{n-1}}{b_n} \sum_{\nu=1}^{n} \{\psi(\alpha_\nu, l) - \psi(0, l)\},$$

if l is arbitrary and all β_i satisfy $-1 \leq \beta_i \leq 0$ $(i = 1, 2, \ldots, n)$

Proof: Let H_1 and H_2 be the closed half-planes determined by the line l. If l does not contain the point $z = 0$ we suppose that this point lies in H_2. We introduce the auxiliary function $A^*(z)$ obtained from $A(z)$ by replacing the zeros of $A(z)$ which lie in H_1 by their images with respect to l, and which is normed such that $|A^*(z)| = |A(z)|$ on l.

Let y be a number on l. The polynomial

$$P(z) \equiv \binom{n}{0} b_0 z^n - \binom{n}{1} b_1 z^{n-1} y + \ldots + (-1)^n \binom{n}{n} b_n y^n$$

(considered as a polynomial in z) has the zeros $-\frac{y}{\beta_1}, \ldots, -\frac{y}{\beta_n}$. These numbers belong to H_1 or to l. This follows from the assumptions under a)

as well as from those under b). By application of the lemma [6]) to $A(z)$ and $P(z)$ we now obtain

$$\left|\binom{n}{0} a_0 b_0 + \binom{n}{1} a_1 b_1 y + \ldots + \binom{n}{n} a_n b_n y^n\right| \leqslant$$
$$\leqslant \left|\binom{n}{0} a_0 {}^*b_0 + \binom{n}{1} a_1 {}^*b_1 y + \ldots + \binom{n}{n} a_n {}^*b_n y^n\right|,$$

if y lies on l (the coefficient of z^k in $A^*(z)$ being denoted by $\binom{n}{k} a_k^*$).

Evidently $|a_n^*| = |a_n|$, and it follows, γ_ν^* ($\nu = 1, 2, \ldots, n$) being the zeros of $A^*B(z)$, that

$$\sum_{\nu=1}^{n} \log |y - \gamma_\nu| \leqslant \sum_{\nu=1}^{n} \log |y - \gamma_\nu^*|,$$

if y lies on l.

Now if y is a variable point on l we may write $y = xe^{i\varphi} + a$, where x is a real variable and φ and a are fixed real numbers. Thus

$$\sum_{\nu=1}^{n} \int_{-A}^{A} \log |xe^{i\varphi} + a - \gamma_\nu| \, dx \leqslant \sum_{\nu=1}^{n} \int_{-A}^{A} \log |xe^{i\varphi} + a - \gamma_\nu^*| \, dx. \quad (4)$$

By application of the formula

$$\int_{-A}^{A} \log |x - \alpha| \, dx = 2(A \log A - A) + \pi |Im\, \alpha| + O\left(\frac{1}{A}\right)$$

(cf. II, formula (9)), we obtain from (4) by making $A \to \infty$

$$\sum_{\nu=1}^{n} \psi(\gamma_\nu, l) \leqslant \sum_{\nu=1}^{n} \psi(\gamma_\nu^*, l).$$

It is easily deduced from GRACE's theorem [7]) that the γ_ν^* all lie in H_2; hence

$$\Sigma \psi(\gamma_\nu^*, l) = n\, \psi\left(\frac{1}{n} \Sigma \gamma_\nu^*, l\right)$$

and

$$\Sigma \psi(\gamma_\nu, l) \leqslant n\, \psi\left(-\frac{a_{n-1}^* b_{n-1}}{a_n^* b_n}, l\right). \quad \ldots\ldots \quad (5)$$

The numbers

$$-\frac{a_{n-1}^*}{a_n^*} = \frac{1}{n} \Sigma a_\nu^* \quad \text{and} \quad -\frac{a_{n-1}^* b_{n-1}}{a_n^* b_n} = \frac{1}{n} \Sigma \gamma_\nu^*$$

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[6]) Since the zeros of $P(z)$ lie in H_1, we have $P^{**}(z) \equiv \varepsilon P(z)$ ($|\varepsilon| = 1$).

[7]) Cf. SZEGÖ, loc. cit. [2]), Satz 2; PÓLYA–SZEGÖ II, p. 65, Aufg. 151.

both lie in H_2. Since b_{n-1}/b_n is positive, it is easily verified that

$$\psi\left(-\frac{a_{n-1}^* b_{n-1}}{a_n^* b_n}, l\right) - \psi(0, l) = \frac{b_{n-1}}{b_n}\left\{\psi\left(-\frac{a_{n-1}^*}{a_n^*}, l\right) - \psi(0, l)\right\}. \quad (6)$$

Finally using

$$\Sigma \psi(\alpha_\nu, l) = \Sigma \psi(\alpha_\nu^*, l) = n\, \psi\left(-\frac{a_{n-1}^*}{a_n^*}, l\right),$$

we obtain from (5) and (6)

$$\Sigma\{\psi(\gamma_\nu, l) - \psi(0, l)\} \leqslant \frac{b_{n-1}}{b_n}\, \Sigma\{\psi(\alpha_\nu, l) - \psi(0, l)\}.$$

This proves our theorem, for if l contains $z = 0$, we have $\psi(0, l) = 0$.

As to the validity of theorem 3 for other functions $\psi(z)$ we make the following remarks. b) is also true if $\psi(z, l)$ is replaced by a constant function of z, but a) is not. Both a) and b) remain true if $\psi(z, l)$ is replaced by an arbitrary homogeneous linear function of $Re\, z$ and $Im\, z$. It follows by an argument used in II that statement b) of theorem 3 holds true if $\psi(z, l)$ is replaced by a function that can be obtained by superposition of functions of the types $|Im\,(\alpha z + \beta)|$ and $Im\,(\alpha z + \beta)$. In II the class of these functions was called C*. Examples of such functions are $|Im\, z|^\lambda$ and $|z|^\lambda$ $(\lambda \geq 1)$. Thus we obtain

Theorem 4: *If $\psi(z)$ is a function of the class C* and $-1 \leq \beta_i \leq 0$ $(i = 1, 2, \ldots, n)$, we have*

$$\sum_{\nu=1}^{n}\{\psi(\gamma_\nu) - \psi(0)\} \leqslant \frac{b_{n-1}}{b_n} \sum_{\nu=1}^{n}\{\psi(\alpha_\nu) - \psi(0)\}.$$

By an integration-process, as carried out in the proof of theorem 4 in II, we can deduce from case a) of theorem 3 the following

Theorem 5: *If $\beta_i \leq 0$ $(i = 1, 2, \ldots, n)$, we have*

$$\sum_{\nu=1}^{n}|\gamma_\nu| \leqslant \frac{b_{n-1}}{b_n} \sum_{\nu=1}^{n}|\alpha_\nu|.$$

This also follows from Theorem 4 by a simple transformation.

Finally we remark that it is easy to infer theorems of the following type:

Theorem 6 [8]**:** *Let the polynomial*

$$f(z) \equiv a_n z^n + \ldots + a_1 z + a_0$$

have the zeros $\alpha_1, \ldots, \alpha_n$ and let $P(y)$ be a polynomial all of whose zeros are real and ≤ 0. Let *$\gamma_1, \ldots, \gamma_n$ be the zeros of*

$$a_n P(n) z^n + \ldots + a_1 P(1) z + a_0 P(0).$$

[8]) In I (theorem 4) this result was proved under the assumption that all a_i are real.

Then

$$\sum_{\nu=1}^{n} |Im\, \beta_\nu| \leqslant \frac{P(n-1)}{P(n)} \sum_{\nu=1}^{n} |Im\, \alpha_\nu|.$$

This follows from theorem 3 (case *a*)) and from the fact that

$$\binom{n}{n} P(n) z^n + \ldots + \binom{n}{1} P(1) z + \binom{n}{0} P(0)$$

has real, non-positive zeros only [9]).

3. In the proof of theorem 3 we integrated along a straight line. In this section we prove, by integration along a circle, inequalities of another type.

We use a well-known formula of JENSEN: if

$$f(z) \equiv c_n z^n + c_{n-1} z^{n-1} + \ldots + c_1 z + c_0$$

has the zeros $\xi_1, \ldots, \xi_n$, then we have

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f(re^{i\theta})}{c_0} \right| d\theta = \sum_{|\xi_\nu| \leq r} \log \frac{r}{|\xi_\nu|}.$$

The sum is taken over those values of ν, which satisfy the condition indicated under the summation sign. We can also write it in the form

$$\sum_{|\xi_\nu| \leq r} \log \frac{r}{|\xi_\nu|} = \sum_{\nu=1}^{n} \log \left\{ \text{Max} \left(\frac{r}{|\xi_\nu|}, 1 \right) \right\}.$$

Theorem 7: *With the same notation as in the introduction we have*

$$\prod_{|\gamma_\nu| \leq r_1 r_2} \frac{r_1 r_2}{|\gamma_\nu|} \leqslant \prod_{|\alpha_\nu| \leq r_1} \frac{r_1}{|\alpha_\nu|} \cdot \prod_{|\beta_\nu| \leq r_2} \frac{r_2}{|\beta_\nu|} \; (r_1 > 0, r_2 > 0) \quad . \quad . \quad (7)$$

Proof. Without loss of generality we may take $r_1 = r_2 = 1$.

We will use the notation A^*, A^{**}, explained in the introduction, taking for D_1 the domain $|z| \leq 1$.

The polynomials $A^*(z)$ and $B^*(z)$ having no roots in $|z| < 1$, if follows from GRACE's theorem [10]) that their composition $A^*B^*(z)$ is also $\neq 0$ for $|z| < 1$. Furthermore

$$|A\, B(y)| \leqslant |A^*\, B^*(y)| \qquad (|y| = 1) \quad . \quad . \quad . \quad . \quad . \quad (8)$$

To prove this, take

$$A_1(z) = A(y\, z), \quad B_1(z) = z^n B\left(-\frac{1}{z}\right).$$

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[9]) Cf. G. PÓLYA and G. SZEGÖ, Aufgaben und Lehrsätze aus der Analysis, II, p. 47, Aufg. 67.

[10]) Cf. SZEGÖ, loc. cit. [2]), Satz 3[1]; PÓLYA–SZEGÖ II, p. 65, Aufg. 152.

Then we have

$$A_1^*(z)=A^*(y\,z),\quad B_1^{**}(z)=z^n\,B^*\left(-\frac{1}{z}\right)$$

and

$$\{A_1, B_1\}=A\,B\,(y),\quad \{A_1^*, B_1^{**}\}=A^*\,B^*\,(y)$$

Now (8) follows from (1).

Since A^*, B^* and A^*B^* are $\neq 0$ for $|z|<1$, JENSEN's formula gives

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{A^*(e^{i\varphi})}{A^*(0)}\right|d\varphi+$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{B^*(e^{i\varphi})}{B^*(0)}\right|d\varphi=0=\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{A^*\,B^*\,(e^{i\varphi})}{A^*\,B^*\,(0)}\right|d\varphi.$$

From the relations

$$A^*(0)\,B^*(0)=A^*\,B^*(0),\quad A(0)\,B(0)=A\,B\,(0),$$
$$|A(e^{i\varphi})|=|A^*(e^{i\varphi})|,\quad |B(e^{i\varphi})|=|B^*(e^{i\varphi})|$$

and from (8) we now infer:

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{A(e^{i\varphi})}{A(0)}\right|d\varphi+\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{B(e^{i\varphi})}{B(0)}\right|d\varphi\geqslant\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{A\,B\,(e^{i\varphi})}{A\,B\,(0)}\right|d\varphi.$$

Now application of JENSEN's theorem leads to (7).

More general inequalities can be deduced from theorem 7.

Theorem 8: *If $\psi(x)$ is a convex function of the real variable x, we have, if $\lambda>0$, $\mu>0$,*

$$(\lambda+\mu)\sum_{\nu=1}^{n}\psi\left(\frac{\log|\gamma_\nu|}{\lambda+\mu}\right)\leqslant\lambda\sum_{\nu=1}^{n}\psi\left(\frac{\log|\alpha_\nu|}{\lambda}\right)+\mu\sum_{\nu=1}^{n}\psi\left(\frac{\log|\beta_\nu|}{\mu}\right).$$

Proof: Putting $r_1=e^{\lambda t}$, $r_2=e^{\mu t}$ in (7), we find, if

$$h_t(x)=\text{Max}\,(t-x,\,0),$$

$$(\lambda+\mu)\sum_{\nu=1}^{n}h_t\left(\frac{\log|\gamma_\nu|}{\lambda+\mu}\right)\leqslant\lambda\sum_{\nu=1}^{n}h_t\left(\frac{\log|\alpha_\nu|}{\lambda}\right)+\mu\sum_{\nu=1}^{n}h_t\left(\frac{\log|\beta_\nu|}{\mu}\right).$$

This inequality is also true (with the sign of equality), if $h(x)$ is a linear function of x, owing to the relation

$$\prod_{\nu=1}^{n}|\alpha_\nu|\cdot\prod_{\nu=1}^{n}|\beta_\nu|=\prod_{\nu=1}^{n}|\gamma_\nu|.$$

We can construct (cf. the argument in I, theorem 7) a linear combination of a linear function and functions $h_t(x)$, with positive weights, which attains the same value as $\psi(x)$ in the $3n$ points

$$x=\frac{\log|\alpha_\nu|}{\lambda}, \quad x=\frac{\log|\beta_\nu|}{\mu}, \quad x=\frac{\log|\gamma_\nu|}{\lambda+\mu} \qquad (\nu=1,2,\ldots,n).$$

The theorem follows from this remark.

Theorem 9: *If* $p\geq 0$ [11], $k\geq 1, \frac{1}{k}+\frac{1}{k'}=1$, *we have*

$$\sum_{\nu=1}^{n}|\gamma_\nu|^p\leq\left(\sum_{\nu=1}^{n}|\alpha_\nu|^{pk}\right)^{\frac{1}{k}}\left(\sum_{\nu=1}^{n}|\beta_\nu|^{pk'}\right)^{\frac{1}{k'}}. \quad \ldots\ldots \quad (3)$$

Proof: For $\psi(x)=e^{px}$ the previous theorem yields

$$(\lambda+\mu)\sum_{\nu=1}^{n}|\gamma_\nu|^{\frac{p}{\lambda+\mu}}\leq\lambda\sum_{\nu=1}^{n}|\alpha_\nu|^{\frac{p}{\lambda}}+\mu\sum_{\nu=1}^{n}|\beta_\nu|^{\frac{p}{\mu}}. \quad \ldots\ldots \quad (9)$$

Let u be a positive parameter. The inequality (9) being true for any two polynomials $A(z)$ and $B(z)$, we can also apply it to $A(\frac{z}{u})$ and $B(uz)$, so that

$$(\lambda+\mu)\sum_{\nu=1}^{n}|\gamma_\nu|^{\frac{p}{\lambda+\mu}}\leq\lambda u^{\frac{p}{\lambda}}\sum_{\nu=1}^{n}|\alpha_\nu|^{\frac{p}{\lambda}}+\mu u^{-\frac{p}{\mu}}\sum_{\nu=1}^{n}|\beta_\nu|^{\frac{p}{\mu}}.$$

For shortness we write this as

$$(\lambda+\mu)\,c\leq\lambda a\cdot u^{\frac{p}{\lambda}}+\mu b\cdot u^{-\frac{p}{\mu}}.$$

The right-hand side of this inequality attains its minimum if $u=\left(\frac{b}{a}\right)^{\frac{\lambda\mu}{p(\lambda+\mu)}}$ and for this value of u the inequality becomes

$$c\leq a^{\frac{\lambda}{\lambda+\mu}}\cdot b^{\frac{\mu}{\lambda+\mu}}$$

or

$$\sum_{\nu=1}^{n}|\gamma_\nu|^{\frac{p}{\lambda+\mu}}\leq\left(\sum_{\nu=1}^{n}|\alpha_\nu|^{\frac{p}{\lambda}}\right)^{\frac{\lambda}{\lambda+\mu}}\cdot\left(\sum_{\nu=1}^{n}|\beta_\nu|^{\frac{p}{\mu}}\right)^{\frac{\mu}{\lambda+\mu}}.$$

On taking $\lambda=\frac{1}{k}$, $\mu=\frac{1}{k'}$ we obtain (3).

The inequalities obtained above are symmetrical in $A(z)$ and $B(z)$. There also are inequalities in which $B(z)$ does not occur explicitly. We state

Theorem 10: *Let* $\psi(x)$ *be a convex function of the real variable* x. *Then we have*

$$\sum_{\nu=1}^{n}\psi(\log|\gamma_\nu|)\leq\sum_{\nu=1}^{n}\psi(\log|\alpha_\nu|).$$

[11]) The theorem is also valid for $p<0$ but then we must suppose no α_ν or β_ν to be zero.

in the following cases:

a) all zeros of $B(z)$ lie on the circle $|z| = 1$;

b) all zeros of $B(z)$ lie in $|z| \geq 1$ and $\psi(x)$ is monotonically non-increasing;

c) all zeros of $B(z)$ lie in $|z| \leq 1$ and $\psi(x)$ is monotonically non-decreasing.

Proof: In case *a*) theorem 7 yields ($r_1 = r$, $r_2 = 1$)

$$\prod_{|\gamma_\nu| \leq r} \frac{r}{|\gamma_\nu|} \leqslant \prod_{|\alpha_\nu| \leq r} \frac{r}{|\alpha_\nu|} \qquad (r > 0)$$

or

$$\sum_{\nu=1}^{n} \varphi_r (\log |\gamma_\nu|) \leqslant \sum_{\nu=1}^{n} \varphi_r (\log |\alpha_\nu|) \quad . \; . \; . \; . \; . \; . \quad (10)$$

if

$$\varphi_r(t) = \begin{cases} \log r - t & t \leqslant \log r \\ 0 & t \geqslant \log r \end{cases}.$$

Furthermore, (10) holds also for linear functions $\varphi(t)$ with the sign of equality. By the argument of theorem 8 it then follows that (10) holds for any convex function.

In case *b*) (10) is also valid. Besides, it holds also for non-increasing functions $\varphi(t) \equiv at + b$, which follows from

$$\sum_{\nu=1}^{n} \log |\gamma_\nu| = \sum_{\nu=1}^{n} \log |\alpha_\nu| + \sum_{\nu=1}^{n} \log |\beta_\nu| \geqslant \sum_{\nu=1}^{n} \log |\alpha_\nu|.$$

Now, by linear superpositions with positive weights, the result again follows (cf. I, theorem 7).

Case *c*) can be derived from *b*) by the transformation $z_1 = \frac{1}{z}$.

On taking $B(z) \equiv nz\,(1 + r)^{n-1}$, we find $AB(z) \equiv zA'(z)$. Therefore we can apply the preceding theorems to deduce some new results concerning the derivative of a polynomial, which were not obtained in I and II.

Theorem 11: *If the polynomial $f(z)$ has the zeros $\xi_1, \ldots, \xi_n$ and if $\eta_1, \ldots, \eta_{n-1}$ are the zeros of its derivative, then*

$$\sum_{\nu=1}^{n-1} |\eta_\nu|^p \leqslant (n-1)^{1-\frac{1}{k}} \left(\sum_{\nu=1}^{n} |\xi_\nu|^{pk}\right)^{\frac{1}{k}} \qquad (k \geqslant 1, p \geqslant 0).$$

Theorem 12: *Under the assumptions of theorem 11 we have*

$$\sum_{\nu=1}^{n-1} \psi (\log |\eta_\nu|) \leqslant \sum_{\nu=1}^{n} \psi (\log |\xi_\nu|)$$

where $\psi(x)$ is a convex function of the real variable x which is non-decreasing and for which $\psi(-\infty) \geq 0$.

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