

## Perfect 2-Lee error correcting codes over alphabets of size 5 or more do not exist for word length 5 and 6

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TECHNOLOGICAL UNIVERSITY EINDHOVEN

Department of Mathematics

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Perfect 2-Lee error correcting codes over alphabets  
of size 5 or more do not exist  
for word length 5 and 6

by

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$(\pm 1\frac{1}{2}, \pm 1\frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})$  are of type 1.

Hence, on a fixed radius 2-polytope every vertex of type 16 has 5 neighbours of type 16 and 5 neighbours of type 6. In the same way, every vertex of type 6 has 1 neighbour of type 16 and 9 neighbours of lower type, and a vertex of type 1 has no neighbours of type 16.

## I. Introduction

Nonexistence theorems on perfect codes for the Lee metric have been found by ASTOLA ([1]), BASSALYGO ([2]), GOLOMB and WELCH ([3]), LENSTRA ([4]) and POST ([6]). The proofs of some of these theorems use only spherepacking arguments ([1]), other proofs combine sphere-packing arguments with Lloyd-like theorems ([2], [4]). A third class of proofs, especially suitable for large alphabets, is based on tilings of cubistic cross-polytopes in n-space ([3], [6]). In this paper, which is strongly related to [6], results by LENSTRA are generalized for arbitrary large alphabets. A combined result of [6] and this paper is, that perfect Lee codes do not exist for  $(3 \leq n \leq 6; e \geq 2; q \geq 2e + 1)$ . This is a part of a conjecture by GOLOMB and WELCH ([3]).

## II. The case $(n,e) = (5,2)$

Referring to the terminology and proof methods of [6] we observe that a cubistic cross-polytope of radius 2 in 5-space has vertices of type 1, 6 and 16, and we may assume that in a hypothetical periodic tiling of 5-space these types are grouped together up to 32 in the following combinations and frequencies pro period box.

combination	frequency	combination	frequency
$[16^2]$	A	$[6^3 \cdot 1^{14}]$	E
$[16 \cdot 6 \cdot 1^{10}]$	B	$[6^2 \cdot 1^{20}]$	F
$[16 \cdot 1^{16}]$	C	$[6 \cdot 1^{26}]$	G
$[6^4 \cdot 1^8]$	D	$[1 \cdot 32]$	H

Remark. The combinations  $[16 \cdot 6^2 \cdot 1^4]$  and  $[6^5 \cdot 1^2]$  are combinatorially impossible.

For a radius 2-polytope, centered at the origin, the vertices  $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$  are of type 16, and, apart from permutations of coordinates, the vertices  $(\pm 1\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$  are of type 6. The other vertices  $(\pm 2\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$  and  $(\pm 1\frac{1}{2}, \pm 1\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$  are of type 1.

Hence, on a fixed radius 2-polytope every vertex of type 16 has 5 neighbours of type 16 and 5 neighbours of type 6. In the same way, every vertex of type 6 has 1 neighbour of type 16 and 9 neighbours of lower type, and a vertex of type 1 has no neighbours of type 16.

Taking these arguments into account, we see that in a tiling of 5-space with radius 2-polytopes every combination  $[16^2]$  must have 10 combinations  $[16.6.1^{10}]$  as neighbours, and that every combination  $[16.6.1^{10}]$  has at most 1 combination  $[16^2]$  as neighbour. In other words, we must have

$$(1) \quad \mu := 10A - B \leq 0 .$$

Let the inventory of different types of vertices pro period box be denoted by  $t_1, t_6, t_{16}$ . Then we have the matrix equation

$$(2) \quad \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 & 3 & 2 & 1 & 0 \\ 0 & 10 & 16 & 8 & 14 & 20 & 26 & 32 \\ 10 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ \cdot \\ \cdot \\ H \end{bmatrix} = \begin{bmatrix} t_{16} \\ t_6 \\ t_1 \\ \mu \end{bmatrix} .$$

Left multiplication by  $[-20, -6, 3, 4]$  yields

$$(3) \quad 28C + 24(E + 2F + 3G + 4H) = -20t_{16} - 6t_6 + 3t_1 + 4\mu .$$

Since the left hand side of (3) is obviously nonnegative and  $\mu \leq 0$  we must have

$$(4) \quad 3t_1 - 6t_6 - 20t_{16} \geq 0 .$$

However, the numbers  $t_i$  are positively proportional to  $g_i$ , the numbers of vertices of different types pro cross-polytope pro orthant, i.e.  $g_1 = 15$ ,  $g_6 = 5$ ,  $g_{16} = 1$ , so that

$$3g_1 - 6g_6 - 20g_{16} = -5 < 0 ,$$

a contradiction. Hence, for  $q \geq 5$  no perfect  $(n, e) = (5, 2)$ -Lee code exists. In fact, for  $(n, e) = (5, 2)$  the only perfect Lee-code is the binary repetition code.

### III. The case $(n, e) = (6, 2)$

The types of vertices to be considered are 1, 7 and 22 (cf. [6]). With respect to the specification of adjacent combinations in a tiling of radius 2-polytopes, as we did in section II for  $n = 5$ , we must distinguish between combinations  $[22^2.1^{20}]$  of two different kinds (cf. the Hamming isometry arguments in [6]).

- a) The centers of the radius 2-spheres have distance 6. In this case all of the 12 neighbours are  $[22.7^\ell.1^{42-7\ell}]$ -combinations for some  $\ell$ ,  $1 \leq \ell \leq 3$ .
- b) The centers of the radius 2-spheres have distance 5. In this case 1 neighbour is also a  $[22^2.1^{20}]$ -combination, 10 neighbours are  $[22.7^\ell.1^{42-7\ell}]$ -combinations, and 1 neighbour is a  $[7^m.1^{64-7m}]$ -combination.

On the other hand, in the same way as we saw in section II, every type 7-vertex on a fixed radius 2-polytope has a unique neighbour of type 22, and a type 1-vertex has no neighbours of type 22 at all, so that a combination  $[22.7^\ell.1^{42-7\ell}]$  has at most  $\ell$  combinations  $[22^2.1^{20}]$  of either type a) or b) as neighbours.

Now let a periodic tiling of 6-space with radius 2-polytopes exist with the following combination-frequency pattern pro period box.

combination	frequency	combination	frequency
$[22^2.1^{20}]$ (a)	A	$[7^6.1^{22}]$	K
$[22^2.1^{20}]$ (b)	B	$[7^5.1^{29}]$	L
$[22.7^3.1^{21}]$	C	$[7^4.1^{36}]$	M
$[22.7^2.1^{28}]$	D	$[7^3.1^{43}]$	N
$[22.7.1^{35}]$	E	$[7^2.1^{50}]$	P
$[22.1^{42}]$	F	$[7.1^{57}]$	Q
$[7^8.1^8]$	G	$[1^{64}]$	R
$[7^7.1^{15}]$	H		

Then the arguments above imply that

$$v := 12A + 10B - 3C - 2D - E \leq 0 .$$

For the inventory of different types of vertices pro period box we now have the matrix equation

$$(6) \begin{bmatrix} 2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 0 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ 20 & 20 & 21 & 28 & 35 & 42 & 8 & 15 & 22 & 29 & 36 & 43 & 50 & 57 & 64 \\ 12 & 10 & -3 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ \cdot \\ \cdot \\ R \end{bmatrix} = \begin{bmatrix} t_{22} \\ t_7 \\ t_1 \\ v \end{bmatrix} .$$

Now we are looking for a row vector of the form  $[\alpha, -1, 1, \beta]$ , left multiplication of (6) by which yields an obviously nonnegative left hand member, and such that  $\beta \geq 0$  and  $\alpha$  is as small as possible. We find  $\alpha = -15$ ,  $\beta = 1$ , so that

$$(7) \quad 2A + 9(D + 2E + 3F) + 8(H + 2K + 3L + 4M + 5N + 6P + 7Q + 8R) = \\ = t_1 - t_7 - 15t_{22} + \mu .$$

Bearing in mind, that  $t_i$  are proportional to  $g_i$ , the numbers of vertices of different types pro polytope pro orthant, and that for  $(n, e) = (6, 2)$  we have  $g_1 = 21$ ,  $g_7 = 6$ ,  $g_{22} = 1$ , we see that the right hand side of (7) reduces to the nonpositive number  $\mu$ . Hence, we must have  $\mu = 0$ ,  $A = D = E = F = H = K = \dots = R = 0$  and our tiling can only have the combinations

$$[22^2 . 1^{10}]^{(b)}, [22.7^3 . 1^{21}] \text{ and } [7^8 . 1^8],$$

with the frequencies B, C and G pro period box, respectively. These frequencies all turn out to be positive, as follows from (6) and the values of  $g_i$ . From the Hamming isometry arguments (cf. [6]) it follows that in the combination  $[7^8 . 1^8]$  there are 4 radius 1-centers of even weight and 4 of odd weight. This implies that the radius 1-centers only have mutual distances 3 and 4.

On the other hand, a combination  $[22^2 . 1^{20}]^{(b)}$  has a combination  $[7^m . 1^{64-7m}]$  as neighbour, in which a pair of radius 1-centers has distance 5. Contradiction. Hence, for large alphabet no perfect  $(n, e) = (6, 2)$ -Lee code exists. In fact, no perfect  $(n, e) = (6, 2)$ -Lee code at all exists, because of the sphere-packing condition for small alphabet.

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