Perfect 2-Lee error correcting codes over alpabets of size 5 or more do not exist for word length 5 and 6

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$(\pm 1\frac{1}{2}, \pm 1\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ are of type 1.
Hence, on a fixed radius 2-polytope every vertex of type 16 has 5 neighbours of type 16 and 5 neighbours of type 6. In the same way, every vertex of type 6 has 1 neighbour of type 16 and 9 neighbours of lower type, and a vertex of type 1 has no neighbours of type 16.

I. Introduction

Nonexistence theorems on perfect codes for the Lee metric have been found by ASTOLA ([1]), BASSALYGO ([2]), GOLOMB and WELCH ([3]), LENSTRA ([4]) and POST ([6]). The proofs of some of these theorems use only spherepacking arguments ([1]), other proofs combine sphere-packing arguments with Lloyd-like theorems ([2], [4]). A third class of proofs, especially suitable for large alphabets, is based on tilings of cubistic cross-polytopes in n-space ([3], [6]). In this paper, which is strongly related to [6], results by LENSTRA are generalized for arbitrary large alphabets. A combined result of [6] and this paper is, that perfect Lee codes do not exist for ($3 \le n \le 6$; $e \ge 2$; $q \ge 2e + 1$). This is a part of a conjecture by GOLOMB and WELCH ([3]).

II. The case (n,e) = (5,2)

Referring to the terminology and proof methods of [6] we observe that a cubistic cross-polytope of radius 2 in 5-space has vertices of type 1, 6 and 16, and we may assume that in a hypothetical periodic tiling of 5-space these types are grouped together up to 32 in the following combinations and frequencies pro period box.

combination	frequency	combination	frequency
$[16^2]$	A	$[6^3.1^{14}]$	E
$[16.6.1^{10}]$	B	$[6^2.1^{20}]$	F
$[16.1^{16}]$	C	$[6.1^{26}]$	G
$[6^4.1^8]$	D	$[1.^{32}]$	H

Remark. The combinations $[16.6^2.1^4]$ and $[6^5.1^2]$ are combinatorially impossible.

For a radius 2-polytope, centered at the origin, the vertices $(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2})$ are of type 16, and, apart from permutations of coordinates, the vertices $(\pm1\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2})$ are of type 6. The other vertices $(\pm2\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2})$ and $(\pm1\frac{1}{2},\pm1\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2})$ are of type 1.

Hence, on a fixed radius 2-polytope every vertex of type 16 has 5 neighbours of type 16 and 5 neighbours of type 6. In the same way, every vertex of type 6 has 1 neighbour of type 16 and 9 neighbours of lower type, and a vertex of type 1 has no neighbours of type 16.

Taking these arguments into account, we see that in a tiling of 5-space with radius 2-polytopes every combination $[16^2]$ must have 10 combinations $[16.6.1^{10}]$ as neighbours, and that every combination $[16.6.1^{10}]$ has at most 1 combination $[16^2]$ as neighbour. In other words, we must have

(1) $\mu := 10A - B \leq 0$.

Let the inventory of different types of vertices pro period box be denoted by t_1, t_6, t_{16}. Then we have the matrix equation

(2) $$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 & 3 & 2 & 1 & 0 \\ 0 & 10 & 16 & 8 & 14 & 20 & 26 & 32 \\ 10 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ \cdot \\ \cdot \\ H \end{bmatrix} = \begin{bmatrix} t_{16} \\ t_6 \\ t_1 \\ \mu \end{bmatrix} .$$

Left multiplication by [-20,-6,3,4] yields

(3) $28C + 24(E + 2F + 3G + 4H) = -20t_{16} - 6t_6 + 3t_1 + 4\mu$.

Since the left hand side of (3) is obviously nonnegative and $\mu \leq 0$ we must have

(4) $3t_1 - 6t_6 - 20t_{16} \geq 0$.

However, the numbers t_i are positively proportional to g_i, the numbers of vertices of different types pro cross-polytope pro orthant, i.e. $g_1 = 15$, $g_6 = 5$, $g_{16} = 1$, so that

$$3g_1 - 6g_6 - 20g_{16} = -5 < 0 ,$$

a contradiction. Hence, for $q \geq 5$ no perfect $(n,e) = (5,2)$-Lee code exists. In fact, for $(n,e) = (5,2)$ the only perfect Lee-code is the binary repetition code.

III. The case (n,e) = (6,2)

The types of vertices to be considered are 1, 7 and 22 (cf. [6]). With respect to the specification of adjacent combinations in a tiling of radius 2-polytopes, as we did in section II for $n = 5$, we must distinguish between combinations $[22^2.1^{20}]$ of two different kinds (cf. the Hamming isometry arguments in [6]).

a) The centers of the radius 2-spheres have distance 6. In this case all of the 12 neighbours are $[22.7^{\ell}.1^{42-7\ell}]$-combinations for some ℓ, $1 \le \ell \le 3$.

b) The centers of the radius 2-spheres have distance 5. In this case 1 neighbour is also a $[22^2.1^{20}]$-combination, 10 neighbours are $[22.7^{\ell}.1^{42-7\ell}]$-combinations, and 1 neighbour is a $[7^m,1^{64-7m}]$-combination.

On the other hand, in the same way as we saw in section II, every type 7-vertex on a fixed radius 2-polytope has a unique neighbour of type 22, and a type 1-vertex has no neighbours of type 22 at all, so that a combination $[22.7^{\ell}.1^{42-7\ell}]$ has at most ℓ combinations $[22^2.1^{20}]$ of either type a) or b) as neighbours.

Now let a periodic tiling of 6-space with radius 2-polytopes exist with the following combination-frequency pattern pro period box.

combination	frequency	combination	frequency
$[22^2.1^{20}]$(a)	A	$[7^6.1^{22}]$	K
$[22^2.1^{20}]$(b)	B	$[7^5.1^{29}]$	L
$[22.7^3.1^{21}]$	C	$[7^4.1^{36}]$	M
$[22.7^2.1^{28}]$	D	$[7^3.1^{43}]$	N
$[22.7.1^{35}]$	E	$[7^2.1^{50}]$	P
$[22.1^{42}]$	F	$[7.1^{57}]$	Q
$[7^8.1^8]$	G	$[1^{64}]$	R
$[7^7.1^{15}]$	H		

Then the arguments above imply that

$$\nu := 12A + 10B - 3C - 2D - E \le 0 \; .$$

For the inventory of different types of vertices pro period box we now have the matrix equation

$$(6) \quad \begin{bmatrix} 2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 0 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ 20 & 20 & 21 & 28 & 35 & 42 & 8 & 15 & 22 & 29 & 36 & 43 & 50 & 57 & 64 \\ 12 & 10 & -3 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ . \\ . \\ R \end{bmatrix} = \begin{bmatrix} t_{22} \\ t_7 \\ t_1 \\ \nu \end{bmatrix} .$$

Now we are looking for a row vector of the form $[\alpha,-1,1,\beta]$, left multiplication of (6) by which yields an obviously nonnegative left hand member, and such that $\beta \geq 0$ and α is as small as possible. We find $\alpha = -15$, $\beta = 1$, so that

$$(7) \qquad 2A + 9(D + 2E + 3F) + 8(H + 2K + 3L + 4M + 5N + 6P + 7Q + 8R) = \\ = t_1 - t_7 - 15t_{22} + \mu \ .$$

Bearing in mind, that t_i are proportional to g_i, the numbers of vertices of different types pro polytope pro orthant, and that for $(n,e) = (6,2)$ we have $g_1 = 21$, $g_7 = 6$, $g_{22} = 1$, we see that the right hand side of (7) reduces to the nonpositive number μ. Hence, we must have $\mu = 0$, $A = D = E = F = H = K = \ldots = R = 0$ and our tiling can only have the combinations

$$[22^2.1^{10}]^{(b)}, \ [22.7^3.1^{21}] \text{ and } [7^8.1^8] ,$$

with the frequencies B, C and G pro period box, respectively. These frequencies all turn out to be positive, as follows from (6) and the values of g_i. From the Hamming isometry arguments (cf. [6]) it follows that in the combination $[7^8.1^8]$ there are 4 radius 1-centers of even weight and 4 of odd weight. This implies that the radius 1-centers only have mutual distances 3 and 4.

On the other hand, a combination $[22^2.1^{20}]^{(b)}$ has a combination $[7^m.1^{64-7m}]$ as neighbour, in which a pair of radius 1-centers has distance 5. Contradiction. Hence, for large alphabet no perfect $(n,e) = (6,2)$-Lee code exists. In fact, no perfect $(n,e) = (6,2)$-Lee code at all exists, because of the sphere-packing condition for small alphabet.

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