Partitioning and eigenvalues

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Let A be a complex hermitian matrix of size n, which is partitioned into block-matrices:

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ & & \\ A_{m1} & \cdots & A_{mm} \end{bmatrix},$$

such that A_{ii} is a square matrix for all $1 \le i \le m$. Let B be the matrix of size m, any element b_{ij} of which equals the average rowsum of the block A_{ij}. Then the eigenvalues of A and B are real numbers, and it is known that the eigenvalues of B lie between the largest and the smallest eigenvalue of A, cf. [1], [3] where this fact is used under the name Higman-Sims technique. Here we prove a more general result:

<u>Theorem</u>. The eigenvalues $\alpha_1 \ge \ldots \ge \alpha_n$ of A and the eigenvalue $\beta_1 \ge \ldots \ge \beta_m$ of B satisfy

$$\alpha_{n-m+i} \le \beta_i \le \alpha_i, \quad \text{for all } 1 \le i \le m .$$

This property is often expressed as "the spectrum of B interlaces the spectrum of A".

<u>Proof</u>. Let d_i be the size of A_{ii}. Consider the $m \times m$ matrix D, and the $m \times n$ matrix S defined by

$$D := \begin{bmatrix} \sqrt{d_1} & & O \\ & \ddots & \\ O & & \sqrt{d_m} \end{bmatrix}; \; S := D^{-1} \begin{bmatrix} \overbrace{11\ldots11}^{d_1} & \overbrace{\quad O \quad}^{d_2} & \overbrace{\quad O \quad}^{d_3} & & \overbrace{\quad O \quad}^{d_m} \\ & 11\ldots11 & & & \\ & & 11\ldots11 & & \\ O & O & O & \ddots & \\ & & & & 11\ldots11 \end{bmatrix}$$

Then we have $B = D^{-1} S A S^H D$, and $SS^H = I$, as can easily be verified.

Let T be a matrix of size $(n-m) \times n$, whose rows form an orthonormal basis of the orthogonal complement of the row-space of S, then $R := \begin{vmatrix} S \\ T \end{vmatrix}$ satisfies $R^H = R^{-1}$. Computing RAR^{-1} we obtain

$$RAR^{-1} = RAR^H = \begin{bmatrix} SAS^H & SAT^H \\ TAS^H & TAT^H \end{bmatrix} .$$

Now the theorem is proved, because the spectrum of any principal submatrix of a hermitian matrix interlaces the spectrum of that matrix, cf. [2], p. 119. Indeed, B is cospectral to SAS^H, which is a principal submatrix of the hermitian matrix RAR^{-1}, which is cospectral to A. □

Remark 1. If any block A_{ij} has a constant rowsum then $AS^HD = S^HDB$, as can easily be verified. If in addition B has eigenvalue β, whose eigenspace is spanned by the columns of X, say, then we have $\lambda X = BX$, $\lambda S^HDX = S^HDBX = AS^HDX$. Hence the column-space of S^HDX is an eigenspace of A belonging to the eigenvalue β. So in this case the spectrum of B is a sub(multi)set of the spectrum of A (note that in this case we do not need to take A hermitian).

Remark 2. Let $\bar{B}$, $\bar{D}$ and $\bar{S}$ be defined analogous to B, D and S, but with respect to another partition of A, which is a refinement of the above partitioning. Then the spectrum of B interlaces the spectrum of $\bar{B}$ (note that in an extremal case we have $A = \bar{B}$). This can be proved in a similar way as above: first realize that $DBD^{-1} = S\bar{S}^H\bar{D}\bar{B}\bar{D}^{-1}\bar{S}S^H$, and $S\bar{S}^H\bar{S}S^H = I$, then let $S\bar{S}^H$ do the job.

Remark 3. Of course everything remains valid if "rowsum" is replaced by "columnsum".

Literature

[1] Hestenes, M.D. and D.G. Higman; Rank 3 groups and strongly regular graphs, Computers in Algebra and Number Theory, SIAM-AMS Proceedings, vol. IV, Amer. Math. Soc., (1971).

[2] Marcus, M. and H. Minc; A survey of matrix theory and matrix inequalities, Allyn and Bacon, Boston (1964).

[3] Payne, S.E.; Finite generalized quadrangles: a survey, proceedings of Washington State Univ. Conference on Proj. Planes (1973).