

Partitioning and eigenvalues

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Partitioning and eigenvalues

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Let A be a complex hermitian matrix of size n , which is partitioned into block-matrices:

$$A = \begin{bmatrix} A_{11} & \dots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \dots & A_{mm} \end{bmatrix},$$

such that A_{ii} is a square matrix for all $1 \leq i \leq m$. Let B be the matrix of size m , any element b_{ij} of which equals the average rowsum of the block A_{ij} . Then the eigenvalues of A and B are real numbers, and it is known that the eigenvalues of B lie between the largest and the smallest eigenvalue of A , cf. [1], [3] where this fact is used under the name Higman-Sims technique. Here we prove a more general result:

Theorem. The eigenvalues $\alpha_1 \geq \dots \geq \alpha_n$ of A and the eigenvalue $\beta_1 \geq \dots \geq \beta_m$ of B satisfy

$$\alpha_{n-m+i} \leq \beta_i \leq \alpha_i, \quad \text{for all } 1 \leq i \leq m.$$

This property is often expressed as "the spectrum of B interlaces the spectrum of A ".

Proof. Let d_i be the size of A_{ii} . Consider the $m \times m$ matrix D , and the $m \times n$ matrix S defined by

$$D := \begin{bmatrix} \sqrt{d_1} & & & \\ & \circ & & \\ & & \ddots & \\ & & & \sqrt{d_m} \\ & \circ & & & \end{bmatrix}; \quad S := D^{-1} \begin{bmatrix} \overbrace{\text{11...11}}^{d_1} & \circ & \circ & & \\ & \overbrace{\text{11...11}}^{d_2} & & & \\ & & \overbrace{\text{11...11}}^{d_3} & & \\ & \circ & \circ & \circ & \\ & & & & \overbrace{\text{11...11}}^{d_m} \end{bmatrix}$$

Then we have $B = D^{-1} S A S^H D$, and $S S^H = I$, as can easily be verified.

Let T be a matrix of size $(n-m) \times n$, whose rows form an orthonormal basis of the orthogonal complement of the row-space of S , then $R := \begin{bmatrix} S \\ T \end{bmatrix}$ satisfies $R^H = R^{-1}$. Computing RAR^{-1} we obtain

$$RAR^{-1} = RAR^H = \begin{bmatrix} SAS^H & SAT^H \\ TAS^H & TAT^H \end{bmatrix}.$$

Now the theorem is proved, because the spectrum of any principal submatrix of a hermitian matrix interlaces the spectrum of that matrix, cf. [2], p. 119. Indeed, B is cospectral to SAS^H , which is a principal submatrix of the hermitian matrix RAR^{-1} , which is cospectral to A . \square

Remark 1. If any block A_{ij} has a constant rowsum then $AS^H D = S^H DB$, as can easily be verified. If in addition B has eigenvalue β , whose eigenspace is spanned by the columns of X , say, then we have $\lambda X = BX$, $\lambda S^H DX = S^H DBX = AS^H DX$. Hence the column-space of $S^H DX$ is an eigenspace of A belonging to the eigenvalue β . So in this case the spectrum of B is a sub(multi)set of the spectrum of A (note that in this case we do not need to take A hermitian).

Remark 2. Let \bar{B} , \bar{D} and \bar{S} be defined analogous to B , D and S , but with respect to another partition of A , which is a refinement of the above partitioning. Then the spectrum of B interlaces the spectrum of \bar{B} (note that in an extremal case we have $A = \bar{B}$). This can be proved in a similar way as above: first realize that $DBD^{-1} = \bar{S}\bar{S}^H\bar{D}\bar{D}^{-1}\bar{S}\bar{S}^H$, and $\bar{S}\bar{S}^H\bar{S}\bar{S}^H = I$, then let $\bar{S}\bar{S}^H$ do the job.

Remark 3. Of course everything remains valid if "rowsum" is replaced by "columnsum".

Literature

- [1] Hestenes, M.D. and D.G. Higman; Rank 3 groups and strongly regular graphs, Computers in Algebra and Number Theory, SIAM-AMS Proceedings, vol. IV, Amer. Math. Soc., (1971).
- [2] Marcus, M. and H. Minc; A survey of matrix theory and matrix inequalities, Allyn and Bacon, Boston (1964).
- [3] Payne, S.E.; Finite generalized quadrangles: a survey, proceedings of Washington State Univ. Conference on Proj. Planes (1973).