Partitioning and eigenvalues

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Partitioning and eigenvalues

by

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Let $A$ be a complex hermitian matrix of size $n$, which is partitioned into block-matrices:

$$
A = \begin{bmatrix}
A_{11} & \cdots & A_{1m} \\
A_{m1} & \cdots & A_{mm}
\end{bmatrix},
$$

such that $A_{ii}$ is a square matrix for all $1 \leq i \leq m$. Let $B$ be the matrix of size $m$, any element $b_{ij}$ of which equals the average rowsum of the block $A_{ij}$. Then the eigenvalues of $A$ and $B$ are real numbers, and it is known that the eigenvalues of $B$ lie between the largest and the smallest eigenvalue of $A$, cf. [1], [3] where this fact is used under the name Higman-Sims technique. Here we prove a more general result:

**Theorem.** The eigenvalues $\alpha_1 \geq \ldots \geq \alpha_n$ of $A$ and the eigenvalue $\beta_1 \geq \ldots \geq \beta_m$ of $B$ satisfy

$$
\alpha_{n-m+i} \leq \beta_i \leq \alpha_i, \quad \text{for all } 1 \leq i \leq m.
$$

This property is often expressed as "the spectrum of $B$ interlaces the spectrum of $A".

**Proof.** Let $d_i$ be the size of $A_{ii}$. Consider the $m \times m$ matrix $D$, and the $m \times n$ matrix $S$ defined by

$$
D := \begin{bmatrix}
\sqrt{d_1} & 0 \\
0 & \sqrt{d_m}
\end{bmatrix}; \quad S := D^{-1}
$$

Then we have $B = D^{-1}SAS^HD$, and $SS^H = I$, as can easily be verified.
Let $T$ be a matrix of size $(n-m) \times n$, whose rows form an orthonormal basis of the orthogonal complement of the row-space of $S$, then $R := \begin{bmatrix} S \\ T \end{bmatrix}$ satisfies $R^H = R^{-1}$. Computing $RAR^{-1}$ we obtain

$$RAR^{-1} = RAR^H = \begin{bmatrix} SAS^H & SAT^H \\ TAS^H & TAT^H \end{bmatrix}.$$

Now the theorem is proved, because the spectrum of any principal submatrix of a hermitian matrix interlaces the spectrum of that matrix, cf. [2], p. 119. Indeed, $B$ is cospectral to $SAS^H$, which is a principal submatrix of the hermitian matrix $RAR^{-1}$, which is cospectral to $A$.

\textbf{Remark 1.} If any block $A_{ij}$ has a constant rowsum then $AS^H D = S^H DB$, as can easily be verified. If in addition $B$ has eigenvalue $\beta$, whose eigenspace is spanned by the columns of $X$, say, then we have $\lambda X = BX$, $\lambda S^H DX = S^H DBX = AS^H DX$. Hence the column-space of $S^H DX$ is an eigenspace of $A$ belonging to the eigenvalue $\beta$. So in this case the spectrum of $B$ is a sub(multi)set of the spectrum of $A$ (note that in this case we do not need to take $A$ hermitian).

\textbf{Remark 2.} Let $\bar{B}$, $\bar{D}$ and $\bar{S}$ be defined analogous to $B$, $D$ and $S$, but with respect to another partition of $A$, which is a refinement of the above partitioning. Then the spectrum of $B$ interlaces the spectrum of $\bar{B}$ (note that in an extremal case we have $A = \bar{B}$). This can be proved in a similar way as above: first realize that $DBD^{-1} = SS^H DB - SS^H$, and $SS^H S^H = I$, then let $SS^H$ do the job.

\textbf{Remark 3.} Of course everything remains valid if "rowsum" is replaced by "columnsum".

\textbf{Literature}

