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The calisson problem

N.G. de Bruijn

Technological University Eindhoven
Department of Mathematics and Computing Science
PO Box 513, 5600MB Eindhoven, The Netherlands
email `wsdwnb@win.tue.nl`

1. Introduction. Let ω be the complex third root of unity $\frac{1}{2}(-1 + i\sqrt{3})$, and let $\mathcal{Z}[\omega]$ be the set of all complex numbers $m + n\omega$, where m and n run through the integers.

We shall use words like “partition”, “disjoint union” in a sense that ignores sets of dimension lower than 2. So we can say that the complex plane is partitioned into what we shall call *grid triangles*: equilateral triangles with side length 1, having their vertices in the set $\mathcal{Z}[\omega]$.

There are two kinds of grid triangles, to be referred to as *upward* and *downward*. Each grid triangle has a horizontal side and an opposite vertex. In the upward triangles the opposite vertex is *above* the horizontal side, like in \triangle , in the downward ones it is *below*, like in ∇ .

The union of two adjacent grid triangles is a rhomb that will be called a *calisson* (it is the name of a French sweet of that shape). Note that a calisson is the union of an upward and a downward triangle. The calissons occur in three possible directions: the short diagonal is parallel either to the line from 0 to 1, or to the one from 0 to ω , or to the one from 0 to ω^2 . We shall say that these calissons have *direction* 1, ω , ω^2 , respectively.

The paper [1] presented the following property. Take a box in the form of a regular hexagon of arbitrary size, with vertices in $\mathcal{Z}[\omega]$ and with sides parallel to sides of the grid triangles. Such a box might have vertices $N, -N\omega^2, N\omega, -N, N\omega^2, -N\omega$, where N is a positive integer. The box can be filled with calissons in various ways. It is remarkable that for each partition of the hexagon into calissons the number of calissons in each of the three directions is exactly one third of the total. The proof sketched in [1] gives a very amusing

intuitive argument, interpreting the box with calissons as a two-dimensional drawing of a collection of unit cubes in three dimensions.

In the present note a more formal argument will be given, and a stronger result will be obtained. For any box, hexagonal or not, it will be shown that if it can be filled with calissons, then the number in each direction is uniquely determined by the box. These numbers can be found if we just know both the volume of the box and what we shall call the *weight sum* of the box. Moreover it will be shown that this weight sum can be expressed as a kind of discrete contour integral taken along the boundary of the box.

2. Sums of weights. The *sign* of a grid triangle is defined as $+1$ if it is upward, and as -1 if it is downward. The *weight* of a grid triangle is a complex number, defined as the product of its sign and its centre. For example, a downward triangle with vertices $\xi, \xi + 1, \xi - \omega$ has the centre $(3\xi + 1 - \omega)/3$ and weight $-(3\xi + 1 - \omega)/3$.

If S is a set of grid triangles then the *weight sum* of S is the sum of the weights of the elements of S .

Let S consist of two elements which together form a calisson with direction 1. If the leftmost vertex of the calisson is ξ , its downward triangle has vertices $\xi, \xi + 1, \xi - \omega$ and its upward triangle $\xi, \xi + 1, \xi - \omega^2$. So the weights are $-(3\xi + 1 - \omega)/3$ and $(3\xi + 1 - \omega^2)/3$, respectively. Therefore the weight sum of S equals $(\omega - \omega^2)/3$, which does not depend on ξ . Let us write

$$(\omega - \omega^2)/3 = \sigma.$$

If the set S consists of two triangles forming a calisson with direction ω we get a similar calculation (everything has to be multiplied by ω , which does not affect the sign of the triangles), and the weight sum turns out to be $\omega\sigma$. If the calisson has direction ω^2 the weight sum is $\omega^2\sigma$.

Theorem 1. Let S be a set of grid triangles. Assume that S can be partitioned into calissons, of which m_0, m_1, m_2 have direction 1, ω, ω^2 , respectively. Then these m_0, m_1, m_2 can be expressed in terms of W and N , where W is the weight sum of S and N the number of triangles in S :

$$(1) \quad m_j = W\omega^{-j}(3\sigma)^{-1} + \overline{W}\omega^j(3\overline{\sigma})^{-1} + N/6 \quad (j = 0, 1, 2)$$

(\overline{W} and $\overline{\sigma}$ denote the complex conjugates of W and σ , respectively).

Proof. The weight sum W of S is the sum of the weight sums of the calissons in the partition. Therefore

$$(2) \quad W = (m_0 + m_1\omega + m_2\omega^2)\sigma.$$

Taking complex conjugates we get

$$(3) \quad \overline{W} = (m_0 + m_1\omega^2 + m_2\omega)\overline{\sigma}.$$

The total number of triangles in S is twice the number of calissons in the partition, so

$$(4) \quad N = 2(m_0 + m_1 + m_2).$$

From the three equations (2), (3), (4) we can solve m_0, m_1, m_2 , which leads to (1). \square

3. Representing the weight sum by a kind of contour integral. Let Φ be a complex-valued function in the complex plane and let E be an oriented edge of some grid triangle. So E can run from a point ξ to η , where $\eta - \xi$ has one of the six values $1, -\omega^2, \omega, -1, \omega^2, -\omega$. The *discrete integral* of Φ along E , to be written as $I(E, \Phi)$, is defined as

$$I(E, \Phi) = (\eta - \xi)\Phi(\tfrac{1}{2}(\xi + \eta)).$$

In other words, it is the integral, taken along E , of the function whose values on E are all equal to the value of Φ at the mid-point of E .

Next we consider a closed oriented curve L consisting of a number of such oriented edges E_1, \dots, E_n . It is assumed that E_n ends where E_1 begins and that (for $k = 1, \dots, n - 1$) E_k ends where E_{k+1} begins. The discrete contour integral of Φ along L is defined as

$$I(L, \Phi) = I(E_1, \Phi) + \dots + I(E_n, \Phi).$$

Theorem 2. Assume that L has no double points, and that its orientation is counter-clockwise. Let S be the set of grid triangles lying inside L , and let Φ be the function defined by $\Phi(z) = -4z^3/3$. Then the discrete contour integral $I(L, \Phi)$ equals the weight sum of S .

Proof. Let the elements of S be the triangles t_1, \dots, t_M , and let (for $j = 1, \dots, M$) I_j be the discrete integral taken counter-clockwise around the circumference of t_j . We have

$$I(L, \Phi) = I_1 + \dots + I_M,$$

since the edges lying entirely *inside* L contribute twice to the right-hand side, with opposite signs. So all we have to show is that I_j equals the weight of t_j .

First we take an upward triangle. Let μ be its centre. The mid-point of the horizontal side is $\mu - \frac{1}{2}\sigma$, the mid-point of the side with direction ω is $\mu - \frac{1}{2}\omega\sigma$ and the mid-point of the third side is $I_j = \mu - \frac{1}{2}\omega^2\sigma$. The discrete integral around the triangle (oriented counter-clockwise) is therefore

$$\sum_{j=0}^2 \omega^j \Phi(\mu - \tfrac{1}{2}\omega^j\sigma).$$

The sum $1 + \omega^h + \omega^{2h}$ equals 3 if h is a multiple of 3, and 0 for all other integers. So with $\Phi(z) = -4z^3/3$ the value of the integral becomes $-(4/3) \cdot 3 \cdot 3 \cdot \mu(-\frac{1}{2}\sigma)^2 = \mu$, indeed the weight of the triangle.

The analogous argument for the case of the downward triangle gives $-\mu$ as the value of the (counter-clockwise) contour integral, and again that is the weight of the triangle. \square

Remark. In theorem 2 we only considered a simply-connected set of triangles. From the proof it is easy to see that how the result can be extended to the general case.

REFERENCE

- [1] G. David and C. Tomei. The problem of the calissons. American Mathematical Monthly, May 1989, 429-431.