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1. Introduction and summary

Throughout this paper p_{2k+1} denotes the monic polynomial $p_{2k+1}(x) = x(x^2-\alpha_1) \dots (x^2-\alpha_k)$, where $\alpha_1,\dots,\alpha_k$ are real numbers such that $0 \le \alpha_1 \le \dots \le \alpha_k$. The linear differential operator having p_{2k+1} as its characteristic polynomial is denoted by L_{2k+1}, i.e., $L_{2k+1}(D) = p_{2k+1}(D)$, where D is the ordinary first order differentiation operator.

A complex-valued function s is called a *cardinal L-spline* with respect to L_{2k+1} if it satisfies the conditions

1.1 $$\begin{cases} \text{(i)} & s \in C^{(2k-1)}(\mathbb{R}) \, , \\ \text{(ii)} & L_{2k+1}\, s(t) = 0 \qquad (\nu < t < \nu + 1 \, , \nu = 0, \pm 1, \pm 2, \dots) \, . \end{cases}$$

The set of cardinal L-splines with respect to L_{2k+1} is denoted by S_{2k+1}. Obviously, S_{2k+1} depends on $\alpha_1,\dots,\alpha_k$; this, however, is suppressed in our notation. The following interpolation property holds.

Lemma 1.1 (Michelli [3])

Let $(y_\nu)_{-\infty}^{\infty}$ *be a bounded sequence of complex numbers. Then a unique bounded function* $s \in S_{2k+1}$ *exists such that*

1.2 $$s(\nu+\tfrac{1}{2}) = y_\nu \qquad (\nu = 0, \pm 1, \pm 2, \dots) .$$

The boundedness of the interpolant s in Lemma 1.1 is required to ensure the unicity of s.

Let $\mathcal{S}_{2k+1}$ be the linear operator mapping the set of bounded sequences $\underline{y} = (y_\nu)_{-\infty}^{\infty}$ onto the set of bounded functions in S_{2k+1} by way of interpolation according to 1.2. The purpose of this paper is to study the asymptotic behaviour of the operator norm

1.3 $$\|\mathcal{S}_{2k+1}\| = \sup_{\underline{y}\neq\underline{0}} \frac{\|\mathcal{S}_{2k+1}\,\underline{y}\|_\infty}{\|\underline{y}\|_\infty}$$

as $k \to \infty$.

Taking in particular the sequence $(y_\nu) = (\delta_{\nu,0})$ in 1.2 we obtain the so-called fundamental solution L_{2k+1} of the interpolation problem. In Schoenberg [7] it is shown that $|L_{2k+1}(t)| < A\,e^{-\alpha|t|}$ $(t \in \mathbb{R})$ for appropriate positive constants A and α. Hence, for any bounded sequence $\underline{y} = (y_\nu)_{-\infty}^{\infty}$, the corresponding bounded interpolant $\mathcal{S}_{2k+1}\,\underline{y}$ may be written in the form

1.4 $$S_{2k+1}\,\underline{y}(t) = \sum_{\nu=-\infty}^{\infty} y_\nu\, L_{2k+1}(t-\nu) \qquad (-\infty < t < \infty)\,.$$

It immediately follows from 1.4 that

$$\|S_{2k+1}\| \le \sup_{t\in\mathbb{R}} \overline{L}_{2k+1}(t)\,,$$

where

1.5 $$\overline{L}_{2k+1}(t) = \sum_{\nu=-\infty}^{\infty} |L_{2k+1}(t-\nu)|$$

is the Lebesgue function associated with the given cardinal interpolation problem.

In Section 3 it is proved that on $[-\frac{1}{2},\frac{1}{2}]$ the function $\overline{L}_{k+1}$ coincides with the cardinal *L*-spline

1.6 $$\tilde{L}_{2k+1}(t) = \sum_{\nu=-\infty}^{\infty} \tilde{y}_\nu\, L_{2k+1}(t-\nu) \qquad (-\infty < t < \infty)\,,$$

where

1.7 $$\tilde{y}_\nu = \begin{cases} (-1)^\nu & (\nu = 0,1,2,\dots)\,, \\ (-1)^{\nu+1} & (\nu = -1,-2,\dots)\,. \end{cases}$$

We also show that

1.8 $$\|S_{2k+1}\| = \tilde{L}_{2k+1}(0)\,.$$

In view of this the operatornorm $\|S_{2k+1}\|$ (cf. 1.3) is also called the Lebesgue constant for the interpolation problem. Our study of the asymptotic behaviour of $\|S_{2k+1}\|$ $(k\to\infty)$ is based on an integral representation of $\|S_{2k+1}\|$; cf. also Section 3. In order to derive this representation, some known results in the theory of cardinal L-splines are needed; these are collected in Section 2. Finally, the asymptotic behaviour of $\|S_{2k+1}\|$ is studied in Section 4. The following result is obtained.

Let

$$\beta_k = \frac{2}{\pi} + 4\pi \sum_{j=1}^{k} \frac{1}{\alpha_j + \pi^2} ,$$

and let γ denote Euler's constant. It is shown that

$$\|S_{2k+1}\| = \frac{2}{\pi} (\ln \beta_k + 3 \ln 2 - \ln \pi + \gamma) + O(\beta_k^{-2}) \qquad (k\to\infty) ,$$

as $\beta_k \to \infty$ $(k\to\infty)$. If the sequence (β_k) converges then it is proved that $\|S_{2k+1}\|$ converges as well.

2. Preliminaries

Let the polynomial $\tilde{p}_{2k+1}$ be defined by

2.1 $$\tilde{p}_{2k+1}(z) = (z-1)(z-e^{-\sqrt{\alpha_1}})(z-e^{\sqrt{\alpha_1}}) \ldots (z-e^{-\sqrt{\alpha_k}})(z-e^{\sqrt{\alpha_k}}) ,$$

where $z \in \mathbb{C}$.

For all $z \in \mathbb{C}$ with $\tilde{p}_{2k+1}(z) \neq 0$ and for all $t \in \mathbb{R}$ the function $\psi(z,t)$ is then defined by

2.2 $$\psi(z,t) = \frac{\tilde{p}_{k+1}(z)}{2\pi i} \oint_C \frac{e^{t\zeta}}{(z-e^{\zeta})\, p_{2k+1}(\zeta)} d\zeta ,$$

where p_{2k+1} is given in Section 1, and where C is any contour in the complex plane surrounding the zeros of p_{2k+1} but excluding the zeros of $\zeta \mapsto z - e^{\zeta}$.

In the sequel the following properties of $\psi(z,t)$ are needed; they are contained in Ter Morsche [5] as well as in Michelli [3], where, apart from a normalisation factor, $\psi(z,t)$ is also used.
One has

2.3 $t \mapsto \psi(z,t) \in \mathrm{Ker}(L_{2k+1})$, the kernel of L_{2k+1} ,

2.4 $\left(\frac{\partial}{\partial t}\right)^j \psi(z,t)\Big|_{t=1} = z\left(\frac{\partial}{\partial t}\right)^j \psi(z,t)\Big|_{t=0} \qquad (j = 0,1,\ldots,2k-1)$,

2.5 $\psi(z,1-t) = z^{2k}\,\psi(z^{-1},t)$,

2.6 $\psi(z,t) = \sum_{j=0}^{2k} A_j(t) z^j$,

with $A_j \in \mathrm{Ker}(L_{2k+1})$, $A_{2k}(t) > 0 \quad (t \neq 0)$, $A_{2k}(0) = 0$.

Apart from these relations the following property of $\psi(z,t)$ is of interest.

<u>Lemma 1.2</u> (Michelli [3])

If $z < 0$ *the function* $t \mapsto \psi(z,t)$ *has precisely one zero in* $(0,1]$.
Furthermore, if $t \in [0,1)$ *then the polynomial* $z \mapsto \psi(z,t)$ *has only real zeros; these zeros are negative and simple.*

The polynomial $z \mapsto \psi(z,t)$ is usually called the exponential L-polynomial, and in case $\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$ it is the well-known Euler-Frobenius polynomial of degree at most 2k (cf. Ter Morsche [5, p.62]). From 2.4 it follows that $\psi(z,1) = z\psi(z,0)$. Therefore, by Lemma 2.1, $\psi(z,1)$ has $2k-1$ negative simple zeros and, in addition, $z = 0$ is also a zero.

Let the zeros of $z \mapsto \psi(z,t)$ $(t \in (0,1])$ be denoted by $\lambda_1(t),\ldots,\lambda_{2k}(t)$ with

$$-\infty < \lambda_1(t) < \lambda_2(t) < \ldots < \lambda_{2k}(t) \le 0 .$$

In Schoenberg [7] it is shown that the functions $t \mapsto \lambda_i(t)$ $(i = 1,\ldots,2k)$ are increasing on (0,1], satisfying the inequalities

2.7 $$\lambda_{i-1}(1) < \lambda_i(t_1) < \lambda_i(t_2) < \lambda_i(1) \le 0 ,$$

where $0 < t_1 < t_2 < 1$ and, by definition, $\lambda_0(1) = -\infty$.

In the polynomial case, i.e., the case $\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$, the inequalities 2.7 are already contained in Ter Morsche [4].

In view of 2.5 the zeros of $\psi(z,\frac{1}{2})$ are ordered as follows

2.8 $$\begin{cases} \lambda_1(\frac{1}{2}) < \ldots < \lambda_k(\frac{1}{2}) < -1 < \lambda_{k+1}(\frac{1}{2}) < \ldots < \lambda_{2k}(\frac{1}{2}) < 0 , \\ \lambda_{k+i}(\frac{1}{2})\, \lambda_{k-i+1}(\frac{1}{2}) = 1 \qquad (i = 0,1,\ldots,k) . \end{cases}$$

According to Ter Morsche [4, p. 68] the relation

2.9 $$\sum_{j=0}^{2k} A_j(\tfrac{1}{2}) s(\mu + j + t) = \sum_{j=0}^{2k} A_j(t) y_{\mu+j} \qquad (0 \le t < 1,\ \mu = 0, \pm 1, \ldots)$$

holds for all functions $s \in S_{2k+1}$ satisfying 1.2; here the functions A_j are given by 2.6.

Relation 2.9 may be considered as a linear difference equation for the unknown sequence $(s(\mu+t))_{\mu=-\infty}^{\infty}$ having $\psi(z,\frac{1}{2})$ as its characteristic polynomial.

We know, however, that $\psi(z,\frac{1}{2})$ is a polynomial of degree 2k with 2k distinct negative zeros. Since, in view of 2.8, $\psi(-1,\frac{1}{2}) \neq 0$, the polynomial $\psi(z,\frac{1}{2})$ has no zeros on the unit circle in the complex plane, and therefore Lemma 3.4.1 of Ter Morsche [5, p.74] may be applied to 2.9. This yields the following result.

Lemma 2.1 *Let* $(y_\nu)_{-\infty}^{\infty}$ *be a bounded sequence of complex numbers. Then a unique bounded function* $s \in S_{2k+1}$ *exists satisfying* 1.2. *Moreover, this interpolating function* s *can be written in the form*

$$2.10 \quad s(\mu+t) = \sum_{j=-\infty}^{\infty} \omega_j(t) y_{\mu+j} \qquad (0 \le t < 1,\ \mu = 0, \pm 1, \ldots)$$

where $\omega_j(t)$ *is given by the contour integral*

$$2.11 \quad \omega_j(t) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{\psi(z,t)}{z^{j+1}\,\psi(z,\frac{1}{2})} dz \qquad (j = 0, \pm 1, \pm 2, \ldots) .$$

3. The Lebesgue function and an integral representation of $\|S_{2k+1}\|$

An application of Formula 2.10 to the particular sequence $(y_\nu) = (\delta_{\nu,0})$ yields the fundamental solution L_{2k+1} as introduced in Section 1. In view of

Lemma 2.1 one has

3.1 $$L_{2k+1}(t-\mu) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{\psi(z,t)}{z^{\mu+1}\,\psi(z,\frac{1}{2})}\,dz \qquad (0 \le t < 1,\ \mu = 0, \pm 1, \pm 2, \ldots)\ .$$

Using the residue theorem and 2.8, we obtain the representation

3.2 $$L_{2k+1}(t-\mu) = \sum_{\ell=k+1}^{2k} \frac{\psi(\lambda_\ell(\frac{1}{2}),t)}{(\lambda_\ell(\frac{1}{2}))^{\mu+1}\ \psi_z(\lambda_\ell(\frac{1}{2}),\frac{1}{2})} \qquad (0 \le t < 1,\ \mu = -1,-2,\ldots)\ ,$$

here ψ_z denotes the partial derivative of $\psi(z,t)$ with respect to z. It follows from 2.7 that

3.3 $$\operatorname{sgn}\left(\frac{\psi(\lambda_\ell(\frac{1}{2}),t)}{\psi_z(\lambda_\ell(\frac{1}{2}),\frac{1}{2})}\right) = \begin{cases} -1 & (\frac{1}{2} < t \le 1)\ , \\ 0 & (t = \frac{1}{2})\ , \\ 1 & (0 \le t < \frac{1}{2})\ . \end{cases}$$

Consequently,

3.4 $$\operatorname{sgn} L_{2k+1}(t-\mu) = (-1)^\mu \operatorname{sgn}(t-\tfrac{1}{2}) \qquad (0 \le t < 1,\ \mu = -1,-2,\ldots)\ .$$

Since, by Lemma 2.1, the function L_{2k+1} is uniquely determined, one has

3.5 $$L_{2k+1}(\tfrac{1}{2}+t) = L_{2k+1}(\tfrac{1}{2}-t) \qquad (-\infty < t < \infty)\ .$$

Therefore

3.6 $$\operatorname{sgn} L_{2k+1}(t-\mu) = (-1)^\mu \operatorname{sgn}(\tfrac{1}{2}-t) \qquad (0 < t \le 1,\ \mu = 1,2,\ldots)\ .$$

Taking $\mu = 0$ and applying the residue theorem, we obtain

3.7 $$L_{2k+1}(t) = \frac{\psi(0,t)}{\psi(0,\frac{1}{2})} + \sum_{\ell=k+1}^{2k} \frac{\psi(\lambda_\ell(\frac{1}{2}),t)}{\lambda_\ell(\frac{1}{2})\,\psi_z(\lambda_\ell(\frac{1}{2}),\frac{1}{2})} \qquad (0 \le t < 1)\ .$$

From 2.6 it follows that $\psi(0,t)\psi^{-1}(0,\frac{1}{2}) > 0 \quad (t \in [0,1))$.
Using this and Formulae 2.8, 3.3, we conclude that $L_{2k+1}(t) > 0 \quad (t \in [\frac{1}{2},1))$.
Hence, in view of 3.5,

3.8 $$\operatorname{sgn}(L_{2k+1}(t)) = 1 \qquad (0 < t < 1)\ .$$

The fundamental solution L_{2k+1} thus changes sign at the points $\nu + \frac{1}{2}$ $(\nu = \pm 1, \pm 2, \ldots)$, and these points are the only zeros of L_{2k+1}.

Therefore, on the interval $[-\frac{1}{2},\frac{1}{2}]$ the Lebesgue function $\overline{L}_{2k+1}$ as given by 1.5 coincides with the function $\tilde{L}_{2k+1}$ defined by 1.6.
Having established this, our next goal is to show that $\| L_{2k+1} \| = \tilde{L}_{2k+1}(0)$ holds. To this end we introduce the function $L_{2k+1}^{[n]}$ $(n \in \mathbb{N})$, being the unique bounded cardinal L-spline in S_{2k+1} interpolating the periodic sequence

3.9 $$\begin{cases} y_\nu^{[n]} = (-1)^\nu & (\nu = 0,1,\ldots,2n)\ , \\ \\ y_{\nu+2n+1}^{[n]} = \tilde{y}_\nu^{[n]} & (\nu = 0, \pm 1, \pm 2, \ldots)\ . \end{cases}$$

We emphasize that $y_\nu^{[n]} = y_{-\nu}^{[n]}$ $(\nu \in \mathbb{Z})$. Consequently, the unicity of $L_{2k+1}^{[n]}$ implies that $L_{2k+1}^{[n]}$ is an even and periodic function with period $2n+1$.
Since (cf. 1.7)

$$y_\nu^{[n]} = \tilde{y}_\nu \qquad (\nu = -2n, -2n+1, \ldots 2n)$$

one has

$$\lim_{n\to\infty} L_{2k+1}^{[n]}(t) = \tilde{L}_{2k+1}(t) ,$$

uniformly on every compact interval of $\mathbb{R}$.

Therefore 1.8 will be established if it is shown that

3.10 $$L_{2k+1}^{[n]}(0) = \max_{0\le t\le \frac{1}{2}} L_{2k+1}^{[n]}(t) .$$

This assertion may be proved as follows. Since $L_{2k+1}^{[n]}$ is an even function having at least 2n zeros in $(\frac{1}{2}, 2n+\frac{1}{2})$, its derivative $L_{2k+1}^{'[n]}$ has at least $2n-1$ zeros in $(\frac{1}{2}, 2n+\frac{1}{2})$, where, in addition,

$$L_{2k+1}^{'[n]}(0) = L_{2k+1}^{'[n]}(2n+1) = 0 .$$

In order to prove that these zeros are the only zeros of $L_{2k+1}^{'[n]}$ on $[0,2n+1]$, we use a generalized version of Rolle's theorem (cf. Ter Morsche [5, Lemma 1.4.11]). Also taking into account that the functions involved, together with their $(2k-1)$-st derivatives, are periodic with period $2n+1$, and the fact that

$$(D - \sqrt{\alpha_k} I)(D^2 - \alpha_{k+1} I) \ldots (D^2 - \alpha_1 I) L_{2k+1}^{'[n]}$$

has at most 2n sign changes in $(0, 2n+1)$, imply that $L_{2k+1}^{'[n]}$ has at most $2n-1$ zeros in $(0, 2n+1)$. It follows that $L_{2k+1}^{'[n]}$ has precisely $2n-1$ zeros in $(0, 2n+1)$, all of which are contained in the subinterval $(\frac{1}{2}, 2n+\frac{1}{2})$.

In view of $L_{2k+1}^{[n]}(\nu+\frac{1}{2}) = (-1)^{\nu}$ $(\nu = 0,1,2,\ldots,2n)$ we obtain that $L_{2k+1}^{'[n]}(t) \le 0$ in $(0,\frac{1}{2}]$. Hence 3.10 holds, which implies that $\|S_{2k+1}\| = \tilde{L}_{2k+1}(0)$.

An integral representation of $\|S_{2k+1}\|$ is now obtained as follows. We recall (cf. 1.6, 1.7) that $\tilde{L}_{2k+1}$ is the unique bounded cardinal L-spline interpolating the sequence $(\tilde{y}_\nu)$. Formula 2.10 combined with 2.11 yields

$$\tilde{L}_{2k+1}(0) = \frac{1}{2\pi i}\left(\sum_{j=-\infty}^{-1} \oint_{|z|=1-\varepsilon} \frac{(-1)^{j+1}}{z^{j+1}} \frac{\psi(z,0)}{\psi(z,\frac{1}{2})} dz + \sum_{j=0}^{\infty} \oint_{|z|=1+\varepsilon} \frac{(-1)^{j}}{z^{j+1}} \frac{\psi(z,0)}{\psi(z,\frac{1}{2})} dz \right),$$

where ε is chosen so small that $\psi(z,\frac{1}{2})$ has no zeros in the ring $1 - 2\varepsilon < |z| < 1 + 2\varepsilon$.

Consequently,

$$\tilde{L}_{2k+1}(0) = \frac{1}{2\pi i}\left(\oint_{|z|=1-\varepsilon} \frac{\psi(z,0)}{(1+z)\,\psi(z,\frac{1}{2})} dz + \oint_{|z|=1+\varepsilon} \frac{\psi(z,0)}{(1+z)\,\psi(z,\frac{1}{2})} dz \right).$$

It easily follows from 2.4 and 2.5 that $\psi(-1,0) = 0$.

Hence, by 1.8, we obtain an integral representation of the form

3.11 $$\|S_{2k+1}\| = \frac{1}{\pi i} \oint_{|z|=1} \frac{\psi(z,0)}{(1+z)\,\psi(z,\frac{1}{2})} dz .$$

This formula will now be used to study the asymptotic behaviour of $\|S_{2k+1}\|$.

4. The asymptotic behaviour of $\|S_{2k+1}\|$

We first observe that the sum of the residues of the function

$$\zeta \longmapsto \frac{e^{t\zeta}}{(z - e^{\zeta})\, p_{2k+1}(\zeta)}$$

is zero in case $0 \le t \le 1$ as can be shown rather easily.

Consequently, if $\varphi \neq 0 \pmod{2\pi}$, (2.2) yields

$$\psi(e^{i\varphi},t) = \tilde{p}_{2k+1}(e^{i\varphi}) \sum_{m=-\infty}^{\infty} \frac{e^{i(t-1)(2m\pi+\varphi)}}{p_{2k+1}(2m\, i\pi + i\varphi)} .$$

Recalling that $0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_k$ (cf. 2.1), we define the polynomial q_{2k+1} by

$$q_{2k+1}(z) = z(z^2+\alpha_1) \ldots (z^2+\alpha_k) .$$

Since

$$p_{2k+1}(iz) = (-1)^k \, i \, q_{2k+1}(z)$$

one has

$$\frac{\psi(e^{i\varphi},0)}{\psi(e^{i\varphi},\frac{1}{2})} = e^{-\frac{1}{2}i\varphi} \frac{\sum_{m=-\infty}^{\infty} q_{2k+1}^{-1}(\varphi+2m\pi)}{\sum_{m=-\infty}^{\infty} (-1)^m \, q_{2k+1}^{-1}(\varphi+2m\pi)} .$$

Substituting $z = e^{i(\pi-\tau)}$ in 3.11, we then obtain

$$4.1 \qquad \|S_{2k+1}\| = \frac{1}{\pi} \int_0^{\pi} \frac{\sum_{m=-\infty}^{\infty} q_{2k+1}^{-1}((2m+1)\pi-\tau)}{\sum_{m=-\infty}^{\infty} (-1)^m \, q_{2k+1}^{-1}((2m+1)\pi-\tau)} \frac{d\tau}{\sin(\tau/2)} .$$

Now let $u_{m,k}^{\pm}$ $(m = 0,1,\ldots)$ be defined by

$$u_{m,k}^{\pm} = q_{2k+1}^{-1}((2m+1)\pi-\tau) \pm q_{2k+1}^{-1}((2m+1)\pi+\tau) \qquad (0 \leq \tau < \pi) .$$

One easily verifies that

$$\sum_{m=-\infty}^{\infty} q_{2k+1}^{-1}((2m+1)\pi-\tau) = \sum_{m=0}^{\infty} u_{m,k}^{-}(\tau) ,$$

$$\sum_{m=-\infty}^{\infty} (-1)^m q_{2k+1}^{-1}((2m+1)\pi-\tau) = \sum_{m=0}^{\infty} (-1)^m u_{m,k}^{+}(\tau) .$$

Define the functions R_k^+, R_k^{-1}, and v_k on $[0,\pi]$ by

4.2
$$\begin{cases} R_k^+(\tau) = q_{2k+1}(\pi-\tau) \sum_{m=1}^{\infty} (-1)^m u_{m,k}^{+}(\tau) , \\ R_k^-(\tau) = q_{2k+1}(\pi-\tau) \sum_{m=1}^{\infty} u_{m,k}^{-}(\tau) , \\ v_k(\tau) = q_{2k+1}(\pi-\tau)\, q_{2k+1}^{-1}(\pi+\tau) . \end{cases}$$

In view of 4.1 we then have

4.3
$$\|S_{2k+1}\| = \frac{1}{\pi} \int_0^{\pi} \frac{1 - v_k(\tau) + R_k^-(\tau)}{1 + v_k(\tau) + R_k^+(\tau)} \frac{d\tau}{\sin(\tau/2)} .$$

Let the increasing sequence $(\omega_k)_1^{\infty}$ be defined by

4.4
$$\omega_k = \sum_{j=1}^{k} (\alpha_j + 1)^{-1} .$$

From now on we distinguish between two cases, i.e.,

$$\lim_{k\to\infty} \omega_k = \infty \text{ and } \lim_{k\to\infty} \omega_k < \infty .$$

I. $\lim_{k\to\infty} \omega_k = \infty$.

We first give a couple of assertions concerning the behaviour of the functions $u^-_{m,k}$ and $u^+_{m,k}$ as $k \to \infty$. Their verification involves straightforward, but rather tedious computations, which are omitted here. The two relations are: a positive constant c exists such that for all $m \in \mathbb{N}$ and all $\tau \in [0,\pi]$

4.5 $$\begin{cases} q_{2k+1}(\pi - \tau)\, u^-_{m,k}(\tau) = \tau m^{-2} O(e^{-c\omega_k}) , \\ \\ q_{2k+1}(\pi - \tau)\, u^+_{m,k}(\tau) = m^{-3} O(e^{-c\omega_k}) , \end{cases} \qquad (k \to \infty)$$

uniformly in m and τ.

From 4.2 and 4.5 it immediately follows that

4.6 $$\begin{cases} R^-_k(\tau) = \tau O(e^{-c\omega_k}) , \\ \\ R^+_k(\tau) = O(e^{-c\omega_k}) , \end{cases} \qquad (k \to \infty)$$

uniformly in τ.

Since in view of 4.2 one has $v_k(\tau) \geq 0$ on $[0,\pi]$, it follows from 4.3 and 4.6 that

4.7 $$\|S_{2k+1}\| = \frac{1}{\pi} \int_0^\pi \frac{1 - v_k(\tau)}{1 + v_k(\tau)} \, \frac{d\tau}{\sin(\tau/2)} \,(1 + O(e^{-c\omega_k})) + O(e^{-c\omega_k})$$

as $k \to \infty$.

In order to analyze 4.7, it is convenient to write v_k in the form

$$v_k(\tau) = \exp\left[\ln\left(\frac{\pi - \tau}{\pi + \tau}\right) + \sum_{j=1}^{k} \ln\left(\frac{\alpha_j + (\pi - \tau)^2}{\alpha_j + (\pi + \tau)^2}\right)\right] .$$

Hence,

$$\ln v_k(\tau) = \ln\left(\frac{1 - \tau/\pi}{1 + \tau/\pi}\right) + \sum_{j=1}^{k} \ln\left(\frac{1 - 2\pi\tau(\alpha_j + \pi^2 + \tau^2)^{-1}}{1 + 2\pi\tau(\alpha_j + \pi^2 + \tau^2)^{-1}}\right).$$

We observe that $0 < \tau < \pi$ implies

$$0 \leq 2\pi\tau(\alpha_j + \pi^2 + \tau^2)^{-1} \leq 2\pi\tau(\pi^2 + \tau^2)^{-1} < 1 \ .$$

An application of the Taylor expansion

$$\ln\left(\frac{1 - t}{1 + t}\right) = -2 \sum_{\ell=0}^{\infty} \frac{t^{2\ell+1}}{2\ell + 1} \qquad (-1 \leq t < 1)$$

now yields

4.8 $$v_k(\tau) = \exp(-\tau g_k(\tau) - \tau^3 h_k(\tau)) \qquad (0 \leq \tau < \pi) \ ,$$

where

4.9 $$\begin{cases} g_k(\tau) = \dfrac{2}{\pi} + \displaystyle\sum_{j=1}^{k} \frac{4\pi}{\alpha_j + \pi^2 + \tau^2} \ , \\ \\ h_k(\tau) = 2\left(\displaystyle\sum_{\ell=1}^{\infty} \pi^{-2\ell-1} \frac{\tau^{2\ell-2}}{2\ell + 1} + \sum_{j=1}^{k} \sum_{\ell=1}^{\infty} \left(\frac{2\pi}{\alpha_j + \pi^2 + \tau^2}\right)^{2\ell+1} \frac{\tau^{2\ell-2}}{2\ell + 1} \right). \end{cases}$$

Apparently, the function g_k satisfies on $[0,\pi)$ the inequalities

4.10 $$g_k(\tau) > \sum_{j=1}^{k} \frac{4\pi}{\alpha_j + \pi^2 + \tau^2} \geq \sum_{j=1}^{k} \frac{4\pi}{\alpha_j + 2\pi^2} \geq \frac{2}{\pi}\omega_k \ .$$

Since

$$\sum_{j=1}^{k}\sum_{\ell=1}^{\infty}\left(\frac{2\pi}{\alpha_j+\pi^2+\tau^2}\right)^{2\ell+1}\frac{\tau^{2\ell-2}}{2\ell+1}=\sum_{j=1}^{k}\left(\frac{2\pi}{\alpha_j+\pi^2+\tau^2}\right)^{3}\sum_{\ell=1}^{\infty}\frac{1}{2\ell+1}\left(\frac{2\pi\tau}{\alpha_j+\pi^2+\tau^2}\right)^{2\ell-2}\le$$

$$\le\sum_{j=1}^{k}\frac{2\pi}{\alpha_j+\pi^2}\sum_{\ell=1}^{\infty}\frac{1}{2\ell+1}\left(\frac{2\pi\tau}{\alpha_j+\pi^2+\tau^2}\right)^{2\ell-2}\le\omega_k\sum_{\ell=1}^{\infty}\frac{2\pi}{2\ell+1}\left(\frac{2\pi\tau}{\pi^2+\tau^2}\right)^{2\ell-2},$$

one has (cf. 4.9)

4.11 $\quad 0\le h_k(\tau)\le g(\tau)+\omega_k h(\tau)\qquad (0\le\tau<\pi)$,

where the functions g and h are given by

4.12
$$\begin{cases} g(\tau)=2\displaystyle\sum_{\ell=1}^{\infty}\left(\frac{1}{\pi}\right)^{2\ell+1}\frac{\tau^{2\ell-2}}{2\ell+1}, \\ h(\tau)=4\pi\displaystyle\sum_{\ell=1}^{\infty}\frac{1}{2\ell+1}\left(\frac{2\pi\tau}{\pi^2+\tau^2}\right)^{2\ell-2}. \end{cases}$$

Obviously, g and h are positive on $[0,\pi)$ and, moreover, $g(\tau)\to\infty$ and $h(\tau)\to\infty$ as $\tau\to\pi$.

Let

$$\int_0^{\pi}\frac{1-v_k(\tau)}{1+v_k(\tau)}\,\frac{d\tau}{\sin(\tau/2)}=I_1+I_2,$$

where

$$I_1=\int_0^{\pi}\frac{e^{-\tau g_k(\tau)}}{1+v_k(\tau)}\left(\frac{1-e^{-\tau^3 h_k(\tau)}}{\tau}\right)\frac{\tau\,d\tau}{\sin(\tau/2)},$$

$$I_2=\int_0^{\pi}\frac{1-e^{-\tau g_k(\tau)}}{1+v_k(\tau)}\,\frac{d\tau}{\sin(\tau/2)}.$$

Using 4.10, the inequality $1 - e^{-t} \le 2t(t+1)^{-1}$ $(t \ge 0)$ and the observation that

$$\frac{h_k(\tau)}{1 + \tau^3 h_k(\tau)} = O(\omega_k) \qquad (k \to \infty) ,$$

uniformly on $[0,\pi)$, we may conclude that

$$I_1 = O\left(\int_0^\pi \omega_k \tau^2 e^{-\frac{2}{\pi}\omega_k \tau} d\tau \right).$$

Hence

4.13 $$I_1 = O\left(\omega_k^{-2}\right) \qquad (k \to \infty) .$$

In a similar way one can prove that

4.14 $$I_2 = \int_0^\pi \frac{1 - e^{-\tau g_k(\tau)}}{1 + e^{-\tau g_k(\tau)}} \frac{d\tau}{\sin(\tau/2)} + O\left(\omega_k^{-2}\right) .$$

In view of 4.7 this leads to

4.15 $$\| S_{2k+1} \| = \frac{1}{\pi} \int_0^\pi \frac{1 - e^{-\tau g_k(\tau)}}{1 + e^{-\tau g_k(\tau)}} \frac{d\tau}{\sin(\tau/2)} \left(1 + O(e^{-c\omega_k})\right) + O(\omega_k^{-2}) .$$

On account of 4.9 the function g_k may be written in the form

4.16 $$g_k(\tau) = \beta_k - \tau^2 r_k(\tau) \qquad (0 \le \tau < \pi) ,$$

where

4.17 $$\beta_k = \frac{2}{\pi} + 4\pi \sum_{j=1}^{k} \frac{1}{\alpha_j + \pi^2} ,$$

and

$$4.18 \quad r_k(\tau) = 4\pi \sum_{j=1}^{k} \frac{1}{(\alpha_j + \pi^2 + \tau^2)(\alpha_j + \pi^2)} .$$

We observe that positive constants c_1 and c_2 exist such that $c_1\omega_k \le \beta_k \le c_2\omega_k$ $(k \in \mathbb{N})$. Therefore $O(\omega_k^{-2})$ may be replaced by $O(\beta_k^{-2})$, and vice versa.

From 4.18 it easily follows that

$$4.19 \quad 0 < r_k(\tau) < \frac{\beta_k}{\pi^2 + \tau^2} \qquad (k = 1,2,\dots;\ 0 \le \tau < \pi) .$$

Now, let

$$\int_0^\pi \frac{1 - e^{-\tau g_k(\tau)}}{1 + e^{-\tau g_k(\tau)}} \frac{d\tau}{\sin(\tau/2)} = J_1 + J_2 ,$$

where

$$J_1 = \int_0^\pi \frac{e^{-\beta_k\tau}(1 - e^{\tau^3 r_k(\tau)})}{1 + e^{-\tau g_k(\tau)}} \frac{d\tau}{\sin(\tau/2)} ,$$

$$J_2 = \int_0^\pi \frac{1 - e^{-\beta_k\tau}}{1 + e^{-\tau g_k(\tau)}} \frac{d\tau}{\sin(\tau/2)} .$$

Using 4.19 together with the inequality $e^t - 1 \le te^t$ $(t \ge 0)$, we conclude that

$$J_1 = O\left(\int_0^\pi \tau^2 e^{-\beta_k\tau} e^{\tau^3 r_k(\tau)} r_k(\tau)d\tau \right) =$$

$$= O\left(\int_0^{\pi} \tau^2 e^{-\beta_k \tau (1-\tau^2(\pi^2+\tau^2)^{-1})} r_k(\tau) d\tau \right) =$$

$$= O\left(\beta_k \int_0^{\pi} \tau^2 e^{-\beta_k \tau/2} d\tau \right) = O\left(\beta_k^{-2}\right)$$

as $k \to \infty$.

Similarly one has

$$J_2 = \int_0^{\pi} \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\beta_k \tau}} \frac{d\tau}{\sin(\tau/2)} + O\left(\beta_k^{-2}\right) \qquad (k \to \infty) .$$

These relations for J_1 and J_2 yield (cf. 4.15)

$$4.20 \quad \| S_{2k+1} \| = \frac{1}{\pi} \int_0^{\pi} \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\beta_k \tau}} \frac{d\tau}{\sin(\tau/2)} \left(1 + O\left(e^{-c\omega_k}\right)\right) + O\left(\beta_k^{-2}\right) .$$

The integral in the right-hand side of 4.20 can be written as follows.

$$\int_0^{\pi} \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\beta_k \tau}} \frac{d\tau}{\sin(\tau/2)} =$$

$$= \int_0^{\pi} \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\beta_k \tau}} \frac{2}{\tau} d\tau + \int_0^{\pi} \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\beta_k \tau}} \left(\frac{1}{\sin(\tau/2)} - \frac{2}{\tau}\right) d\tau =$$

$$= \int_0^{\pi} \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\beta_k \tau}} \frac{2}{\tau} d\tau + \int_0^{\pi} \left(\frac{1}{\sin(\tau/2)} - \frac{2}{\tau}\right) d\tau + O\left(\beta_k^{-2}\right) .$$

The second integral can be evaluated quite easily; in fact

$$\int_0^{\pi} \left(\frac{1}{\sin(\tau/2)} - \frac{2}{\tau}\right) d\tau = 4 \ln 2 - 2 \ln \pi .$$

With respect to the first integral one has

$$\int_0^{\pi} \frac{1 - e^{-\beta_k \tau}}{1 + e^{-\beta_k \tau}} \frac{d\tau}{\tau} = \int_0^{\beta_k \pi} \frac{1 - e^{-\tau}}{1 + e^{-\tau}} \frac{d\tau}{\tau} =$$

$$= \int_0^{\beta_k \pi} \tanh(\tau/2) d \ln\tau = \tanh(\beta_k \pi/2) \ln(\beta_k \pi) - \frac{1}{2} \int_0^{\beta_k \pi} \frac{\ln \tau}{\cosh^2(\tau/2)} d\tau =$$

$$= \ln(\beta_k \pi) - \frac{1}{2} \int_0^{\infty} \frac{\ln \tau}{\cosh^2(\tau/2)} d\tau + \mathcal{O}\left(e^{-d\omega_k}\right) ,$$

where d is a positive constant.

Using Formula 4.371(3) in Gradshteyn and Ryzhik [1, p. 580], we obtain

$$\int_0^{\infty} \frac{\ln \tau}{\cosh^2(\tau/2)} d\tau = 2(\ln \pi - \ln 2 - \gamma) ,$$

where γ denotes Euler's constant.

We thus arrive at the following theorem.

Theorem 4.1 *If* $\beta_k = 2/\pi + 4\pi \sum_{j=1}^{k} (\alpha_j + \pi^2)^{-1} \to \infty$ *as* $k \to \infty$, *then*

4.21 $$\|S_{2k+1}\| = \frac{2}{\pi} (\ln \beta_k + 3 \ln 2 - \ln \pi + \gamma) + \mathcal{O}(\beta_k^{-2}) \qquad (k \to \infty) .$$

Having dealt with $\omega_k \to \infty$ as $k \to \infty$, we now consider the case that the sequence (ω_k) is convergent.

II. $\lim_{k\to\infty} \omega_k < \infty$.

The convergence of the sequence (ω_k) implies that $\lim_{j\to\infty} \alpha_j = \infty$, so a positive integer k_0 exists such that $\alpha_j > 0$ for $j \geq k_0$. The polynomial q_{2k+1}, introduced at the beginning of this section, may therefore be written in the form

$$q_{2k+1}(z) = \gamma_k \, z^{2k_0-1} \left(1 + \frac{z^2}{\alpha_{k_0}}\right) \cdots \left(1 + \frac{z^2}{\alpha_k}\right) \qquad (k \geq k_0) ,$$

where

$$\gamma_k = \alpha_{k_0} \, \alpha_{k_0+1} \cdots \alpha_k .$$

Since $\sum_{j=k_0}^{\infty} \alpha_j^{-1}$ is finite, the product $\prod_{j=k_0}^{k} (1 + z^2 \alpha_j^{-1})$ converges uniformly in z on every bounded set of $\mathbb{C}$. As a consequence its limit function, which we denote by q, is a holomorphic function.

Taking into account 4.1 we obtain

<u>Theorem 4.2</u> *If* $\sum_{j=1}^{\infty} (\alpha_j + 1)^{-1} < \infty$ *then*

$$4.22 \quad \lim_{k\to\infty} \| S_{2k+1} \| = \frac{1}{\pi} \int_0^{\pi} \frac{\sum_{m=-\infty}^{\infty} q^{-1}((2m+1)\pi - \tau)}{\sum_{m=-\infty}^{\infty} (-1)^m q^{-1}((2m+1)\pi - \tau)} \, \frac{d\tau}{\sin(\tau/2)} ,$$

where

$$q(z) = \prod_{j=k_0}^{\infty} (1 + z^2 \alpha_j^{-1}) .$$

Finally, we examine a few particular cases.

(a) *The polynomial case:* $\alpha_j = 0 \quad (j = 1,2,\dots)$.

Since $\beta_k = \pi^{-1}(2+4k) \to \infty$, Theorem 4.1 may be applied. A simple computation yields

$$\|S_{2k+1}\| = \frac{2}{\pi}\left(\ln k + \ln \frac{32}{\pi^2}\right) + O(k^{-1}) \qquad (k \to \infty) ,$$

which is in agreement with results obtained by Meinardus [2], and Richards [6].

(b) *The hyperbolic case:* $\alpha_j = j^2 \quad (j = 1,2,\dots)$.

Obviously $\sum_{j=1}^{\infty} (\alpha_j + 1)^{-1} < \infty$, and thus Theorem 4.2 may be applied. Using the well-known relation

$$\frac{\sinh(\pi z)}{\pi} = z \prod_{j=1}^{\infty}\left(1 + \frac{z^2}{j^2}\right) ,$$

we conclude from 4.22 that

$$\lim_{k\to\infty} \|S_{2k+1}\| = \frac{1}{\pi}\int_0^{\pi} \frac{\sum_{m=-\infty}^{\infty} \sinh^{-1}(\pi((2m+1)\pi - \tau))}{\sum_{m=-\infty}^{\infty} (-1)^m \sinh^{-1}(\pi((2m+1)\pi - \tau))} \; \frac{d\tau}{\sin(\tau/2)} .$$

A numerical computation of the integral yields

$$\lim_{k\to\infty} \|S_{2k+1}\| \approx 2.1314.$$

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