Finsler pp-waves

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In this work we present Finsler gravitational waves. These are a Finslerian version of the well-known *pp*-waves, generalizing the very special relativity line element. Our Finsler *pp*-waves are an exact solution of Finslerian Einstein's equations in vacuum and describe gravitational waves propagating in an anisotropic background.

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I. INTRODUCTION

Finsler geometry is a generalization of Riemannian geometry in which geometrical quantities are direction dependent. The main object is the so-called Finsler structure F, which defines the infinitesimal line element $ds = F$. Riemannian geometry is recovered when the square of the Finsler function is constrained to be quadratic in dx, $F(x, dx) = \sqrt{g_{ij}(x)dx^i dx^j}$. Finsler geometry has found applications in several fields of research where anisotropic media play a role, such as seismology [1], optics [2], and medical imaging [3–6]. Finsler geometry is also appealing in relativity and cosmology [7,8], in particular concerning scenarios violating full Lorentz invariance.

In this work we consider the very special relativity (VSR) framework proposed by Cohen and Glashow [9], where only a subgroup of the full Lorentz group is preserved. Gibbons *et al.* [10] pointed out the Finslerian character of the corresponding line element, an anisotropic generalization of Minkowski spacetime. The question arises whether it is possible to find anisotropic generalizations of curved spacetimes, which satisfy Finslerian Einstein's equations. Several approaches to Finslerian extensions of the general relativity equations of motion have been suggested [11–14]. Finslerian extensions of well-known exact solutions such as the Schwarzschild metric have been proposed [15]. In the context of cosmology, Finslerian versions of the FRW metric have been studied [16,17]. Finslerian linearized gravitational waves have also been explored [18,19].

In this work we choose to adopt the framework by Pfeifer and Wohlfarth [13,14], in which the Finslerian field equation is derived from a well-defined action and the geometry-related term obeys the same conservation law as the matter source term. In this context we propose a Finslerian version of the well-known *pp*-waves. The *pp*-waves belong to a wider class of spacetimes with the property that all curvature invariants of all orders vanish, the so-called vanishing scalar invariant (VSI) spacetimes [20]. These are relevant in some supergravity and string theory scenarios since they are, due to the VSI property, exact solutions to the corresponding equations of motion [21–23]. We show that our Finsler *pp*-waves are an exact solution of the Finslerian field equation in vacuum.

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II. THEORY

A. (Pseudo-)Finsler geometry in a nutshell

Let M be an n-dimensional C^∞ manifold. We denote the tangent bundle of M, the set of tangent spaces T_xM at each $x \in M$, by $TM \coloneqq \{T_xM | x \in M\}$. We can write each element of TM as (x, y), where $x \in M$ and $y \in T_xM$. A Finsler structure is a function defined on the tangent bundle TM

$$F\colon\ TM \to [0, \infty) \tag{1}$$

satisfying the following properties:

(1) *Regularity*: F is C^∞ on the slit tangent bundle $TM\backslash 0 = TM\backslash\{y = 0\}$.
(2) *Homogeneity*: $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda > 0$ and $(x, y) \in TM$.
(3) *Strong convexity*: The fundamental metric tensor

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} \tag{2}$$

with $i, j = 1, \ldots, n$, is positive definite for all $(x, y) \in TM\backslash 0$.

The pair (F, M) is called a Finsler manifold or Finsler space. A Finsler manifold is Riemannian when the fundamental tensor is independent of the tangent vector y, $g_{ij}(x) \equiv g_{ij}(x, y)$. A thorough treatment of Finsler geometry can be found in [24]. To be precise, in this work we consider pseudo-Finsler spaces, for which the regularity property may not hold on the null vectors and the fundamental tensor is *not* restricted to be positive definite.

B. Finslerian generalization of Einstein's equations

In this section we treat one of the approaches to construct a Finslerian version of Einstein's equations. In doing so we consider the particular case of vacuum and Berwald spaces, which we characterize later.

We introduce in what follows a number of Finslerian objects needed to state the field equation. The geodesic spray coefficients are defined as

$$G^i \coloneqq \gamma^i{}_{jk} y^j y^k \tag{3}$$

where $\gamma^i{}_{jk}$ are the formal Christoffel symbols of the fundamental metric tensor (2). These adopt the same form as in the Riemannian case, but in this setting we generally have $\gamma^i{}_{jk} = \gamma^i{}_{jk}(x, y)$. As their name suggests the geodesic spray coefficients play a role in the Finslerian geodesic equations, which in the simplest case take the form

$$\ddot{x}^i + G^i = 0 \tag{4}$$

where in G^i we set $y^i \coloneqq \dot{x}^i = dx^i/dt$, with t parametrizing the geodesic curve. A useful identity reads

$$G^i = \Gamma^i{}_{jk} y^j y^k \tag{5}$$

where $\Gamma^i{}_{jk}$ are the so-called Chern connection coefficients. These adopt the same form as the formal Christoffel symbols $\gamma^i{}_{jk}$, but the usual partial derivative is replaced by the following derivative:

$$\frac{\delta}{\delta x^i} \coloneqq \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j} \tag{6}$$

where $N_i^j = (1/2)(\partial G^j/\partial y^i)$ is known as the nonlinear connection. The Chern connection is one of the possible connections in Finsler geometry, and probably the most studied one. It has no torsion (i.e., $\Gamma^i{}_{jk} = \Gamma^i{}_{kj}$), and it is almost metric compatible.

Note that identity (5) does not imply that the Chern connection coefficients and the formal Christoffel symbols are equal. In fact, the contribution to the Chern connection coefficients which arises from the nonlinear connection in (6) vanishes always when contracted with the y-quadratic term in Eq. (5), due to homogeneity considerations.

From the geodesic spray coefficients the tensor known as the predecessor of the flag curvature can be constructed as follows:

$$R^i{}_k = \frac{1}{F^2}\left(\partial_{x^k} G^i - \frac{1}{4}\partial_{y^k} G^j \partial_{y^j} G^i - \frac{1}{2} y^j \partial_{x^j} \partial_{y^k} G^i + \frac{1}{2} G^j \partial_{y^j} \partial_{y^k} G^i\right) \tag{7}$$

where $\partial_{x^k} = \partial/\partial x^k$, $\partial_{y^k} = \partial/\partial y^k$. This curvature tensor is related to the Finslerian Riemann tensor. Note that it does *not* reduce to the Ricci tensor in the Riemannian limit. We are now ready to introduce the simplest curvature scalar in Finsler geometry, the Finslerian Ricci scalar, as the trace of the tensor $R^i{}_k$ [24]:

$$\mathrm{Ric} \coloneqq R^i{}_i. \tag{8}$$

Clearly, its construction is notably different from that of the Ricci scalar in Riemannian geometry.

The following Finslerian Ricci tensor has been proposed [25]:

$$\mathrm{Ric}_{ij} \coloneqq \frac{1}{2}\partial_{y^i}\partial_{y^j}(F^2 \mathrm{Ric}). \tag{9}$$

This tensor is manifestly covariant and symmetric. Moreover, if the Finsler structure F is Riemannian, it reduces to the usual Ricci tensor. It is therefore a natural generalization of the Ricci tensor.

In what follows we adopt the approach by Pfeifer and Wohlfarth [13,14]. Here, a Finslerian Einstein-Hilbert action is proposed and the following vacuum Finslerian scalar field equation is derived:

$$g^{ij} Ric_{ij} - 3\mathrm{Ric} + g^{ij}(\nabla_i^B \dot{A}^k{}_{kj} + F\partial_{y^i} \ddot{A}^k{}_{kj}) = 0. \tag{10}$$

Here g^{ij} is the inverse of the fundamental tensor (2), ∇^B is the Berwald covariant derivative, $\dot{A}_{ijk} = -(1/4) y_l (\partial_{y^i}\partial_{y^j}\partial_{y^k} G^l)$ and $\ddot{A}_{ijk} = \dot{A}_{ijk|s}(y^s/F)$, where $|$ denotes the horizontal covariant derivative (see [14,24] for the explicit form of the Berwald and horizontal covariant derivatives). If the Finsler structure F is Riemannian, Eq. (10) is equivalent to the Einstein vacuum equations $R_{ij} = 0$ [14].

From now on we only consider a particular class of Finsler spaces, the so-called *Berwald spaces*. These are defined by the property that the geodesic spray coefficients G^i, Eq. (3), are quadratic in y. Equivalently, the Chern connection coefficients $\Gamma^i{}_{jk}$ (in natural coordinates) are independent of y. A remarkable property of Berwald spaces is that the Chern connection coefficients coincide with the Christoffel symbols of a (non-unique) Riemannian metric on M, as proved by Szabó [26]. Berwald spacetimes have for example been considered in [27].

In the Berwald case the quantities $\dot{A}_{ijk}$ trivially vanish (recall that the geodesic spray coefficients are quadratic in y). Therefore, Eq. (10) reduces to

$$g^{ij}\mathrm{Ric}_{ij} - 3\mathrm{Ric} = 0 \tag{11}$$

which we regard as the Finslerian vacuum field equation for Berwald spaces. Note that this equation can also be reformulated as $R - 3\mathrm{Ric} = 0$, where $R = g^{ij}\mathrm{Ric}_{ij}$ is the Finslerian analog of the scalar curvature.

C. Very special relativity

Cohen and Glashow [9] pointed out that the local laws of physics do not need to be invariant under the full Lorentz group but only under a certain subgroup, ISIM(2). This is called very special relativity. Subsequently, Gibbons, Gomis and Pope [10] studied deformations of the subgroup ISIM(2), investigating possibilities to incorporate gravity into the theory. They showed that the 1-parameter family $\mathrm{DISIM}_b(2)$, where D stands for deformation and b is a dimensionless parameter, is the only physically acceptable deformation of the VSR subgroup. The line element which is invariant under $\mathrm{DISIM}_b(2)$ reads

$$ds = (-2dudv + (dx^1)^2 + (dx^2)^2)^{(1-b)/2}(-du)^b \tag{12}$$

where we use coordinates $x^\mu = (u, v, x^1, x^2)$, and $u = (1/\sqrt{2})(t - x^3)$, $v = (1/\sqrt{2})(t + x^3)$ are light-cone coordinates. Obviously, this reduces to the Minkowski line element for $b = 0$. As Gibbons *et al.* noted, (12) is a Finslerian line element, which had already been suggested by Bogoslovsky [28]. The associated Finsler structure is given by

$$F(x, y) = (-2y^u y^v + (y^1)^2 + (y^2)^2)^{(1-b)/2}(-y^u)^b \tag{13}$$

where $y^\mu = (y^u, y^v, y^1, y^2)$. This space is locally Minkowskian [i.e. the Finsler structure is independent of x, $F(x, y) \equiv F(y)$] and it has vanishing geodesic spray coefficients:

$$G^i = 0. \tag{14}$$

Therefore Finslerian curvature tensors are zero, and geodesics are defined by linear equations. Such a space is called projectively flat.

D. Finsler *pp*-waves

Let us now consider the spacetime:

$$ds = (g_{\mu\nu} dx^\mu dx^\nu)^{(1-b)/2}(-du)^b. \tag{15}$$

This is a modification of the Bogoslovsky line element given by Eq. (12), where the Minkowski metric is substituted by a general Riemannian metric $g_{\mu\nu}$. We propose the case where $g_{\mu\nu} \equiv pp$-waves. The well-known pp-wave metric is given by [29]

$$ds^2 = -2dudv - \Phi(u, x^1, x^2)du^2 + (dx^1)^2 + (dx^2)^2. \tag{16}$$

This metric is an exact vacuum solution of the Einstein's equations when the function Φ is harmonic, $\triangle_{(x^1,x^2)}\Phi = 0$, and it describes exact gravitational waves propagating in a Minkowski background. Thus we have the following ansatz:

$$\begin{aligned} ds = (&-2dudv - \Phi(u, x^1, x^2)du^2 \\ &+ (dx^1)^2 + (dx^2)^2)^{(1-b)/2}(-du)^b. \end{aligned} \tag{17}$$

Note that we recover the pp-wave metric in the case $b = 0$. The associated Finsler structure is of the form

$$\begin{aligned} F(x, y) = (&-2y^u y^v - \Phi(u, x^1, x^2)(y^u)^2 \\ &+ (y^1)^2 + (y^2)^2)^{(1-b)/2}(-y^u)^b. \end{aligned} \tag{18}$$

From $F(x, y)$ we derive the fundamental tensor (see Appendix), the Christoffel symbols, and the geodesic spray coefficients defined by Eq. (3):

$$\begin{aligned} G^u &= 0, \\ G^v &= y^u y^1 \partial_{x^1}\Phi + y^u y^2 \partial_{x^2}\Phi + \frac{(y^u)^2}{2}\partial_u \Phi, \\ G^1 &= \frac{(y^u)^2}{2}\partial_{x^1}\Phi, \\ G^2 &= \frac{(y^u)^2}{2}\partial_{x^2}\Phi. \end{aligned}$$

These are obviously quadratic in y, and thus the postulated Finslerian space is of Berwald type. We can now compute the Chern connection coefficients by $\Gamma^i{}_{jk} = (1/2)\partial_{y^j}\partial_{y^k}G^i$ (which holds for Berwald spaces):

$$\begin{aligned} &\Gamma^v{}_{uu} = \frac{1}{2}\partial_u\Phi, \qquad \Gamma^v{}_{u1} = \Gamma^1{}_{uu} = \frac{1}{2}\partial_{x^1}\Phi, \\ &\Gamma^v{}_{u2} = \Gamma^2{}_{uu} = \frac{1}{2}\partial_{x^2}\Phi. \end{aligned} \tag{19}$$

Remarkably, these are exactly the same as the Christoffel symbols of the metric (16), which also implies that geodesic equations adopt the same form. Therefore, the proposed Finslerian line element (17) is an example of a (pseudo-Finsler) Berwald space whose Chern connection coincides with the Levi-Civita connection of a certain pseudo-Riemannian metric on M, the pp-wave metric. This is an example of a nontrivial metric satisfying Szabó's theorem in the context of pseudo-Finsler spaces.

The Finslerian Ricci scalar can be computed to be

$$\mathrm{Ric} = \frac{(y^u)^2}{2F^2}(\partial^2_{x^1}\Phi + \partial^2_{x^2}\Phi). \tag{20}$$

Hence, the Finslerian Ricci tensor has only the nonzero component:

$$\mathrm{Ric}_{uu} = \frac{1}{2}(\partial^2_{x^1}\Phi + \partial^2_{x^2}\Phi) = \frac{1}{2}\triangle_{(x^1,x^2)}\Phi. \tag{21}$$

The Finslerian vacuum field equation (11) reduces to

$$g^{uu}\mathrm{Ric}_{uu} - 3\mathrm{Ric} = \left(\frac{1}{2}g^{uu} - \frac{3}{2}\frac{(y^u)^2}{F^2}\right)\triangle_{(x^1,x^2)}\Phi. \tag{22}$$

Therefore, we conclude that every harmonic function Φ leads to an exact solution of the Finslerian Einstein's equations in vacuum (for Berwald spaces). This is completely analogous to the *pp*-waves case. We call the line element (17), with Φ a harmonic function, Finsler *pp*-waves. It describes exact gravitational waves propagating in an anisotropic background.

As a curiosity, we point out that the Finslerian analog of the scalar curvature is of the form

$$R = g^{uu}\mathrm{Ric}_{uu} = \frac{2b}{1+b}\mathrm{Ric} \tag{23}$$

which vanishes as expected in the Riemannian limit $b = 0$.

Remark.—The fundamental tensor arising from line element (15) is ill defined in the subspaces $y^u = 0$ and $g_{\mu\nu}y^\mu y^\nu = 0$, the null cone of the Riemannian metric. In such cases the Finsler structure (18) vanishes and the fundamental tensor becomes singular. However, this is not specific to our Finsler *pp*-waves since it holds for the Bogoslovsky line element (12) as well, in the subspaces $y^u = 0$ and $\eta_{\mu\nu}y^\mu y^\nu = 0$, with $\eta_{\mu\nu}$ the Minkowski metric. This shortcoming of pseudo-Finslerian line elements of this type has already been discussed in [14]. In fact, the issues regarding the null cone of the Riemannian metric can seemingly be avoided by working with a smooth function L such that $F = |L|^{(1-b)/2}$, instead of the Finsler structure F [30].

III. DISCUSSION

In this work we present a Finslerian version of the well-known *pp*-waves, which generalizes the (deformed) VSR line element investigated by Gibbons *et al.* Our Finsler *pp*-waves are an exact solution of Finslerian Einstein's equations in vacuum (for Berwald spaces). This result shows that the considered Finslerian gravity theory permits, as Einstein's gravity, traveling gravitational waves. It should be noted that Finsler and Riemannian *pp*-waves lead to geodesics of the same form, which poses the question of whether they could actually be distinguished from each other by standard tests of gravity. The presented solution also has interesting mathematical properties, since it is an example of a nontrivial metric satisfying Szabó's theorem in the context of pseudo-Finsler spaces.

In future work we will consider a number of open questions. First of all, the physical implications of the singularities of the fundamental tensor should be investigated. It would also be interesting to study the curvature scalar invariants corresponding to the presented Finsler *pp*-waves solution. Recall that Riemannian *pp*-waves belong to the class of spaces with vanishing scalar invariants. However, it is not clear how such spaces would be defined in a Finsler setting since scalar quantities, such as the Finslerian Ricci scalar and the Cartan scalar, are in principle direction dependent. Last, an obvious next step would be to consider solutions of Finslerian VSI type, where generalizations of the *pp*-wave metric are employed, both in four and in higher dimensions.

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APPENDIX

The components of the fundamental tensor g_{ij} corresponding to ansatz (18) read as follows, with $\alpha = (-2y^u y^v - \Phi(u, x^1, x^2)(y^u)^2 + (y^1)^2 + (y^2)^2)$:

$$\begin{aligned}
g_{uu} &= \frac{F^2}{\alpha}\Bigg(-\Phi - b \\
&\quad \times \left(\frac{((y^1)^2+(y^2)^2)^2 - 2b((y^1)^2+(y^2)^2 - y^u y^v)^2}{\alpha (y^u)^2}\right. \\
&\quad \left. - \frac{2(y^v)^2 - (y^u)^2\Phi}{\alpha} + \Phi\right)\Bigg) \\
g_{uv} &= \frac{F^2}{\alpha}\left(-1 - b\left((1-2b) + 2(1-b)\frac{y^u y^v + (y^u)^2\Phi}{\alpha}\right)\right) \\
g_{vv} &= \frac{F^2}{\alpha}(-2b)(1-b)\frac{(y^u)^2}{\alpha} \\
g_{11} &= \frac{F^2}{\alpha}(1-b)\left(1 - 2b\frac{(y^1)^2}{\alpha}\right) \\
g_{22} &= \frac{F^2}{\alpha}(1-b)\left(1 - 2b\frac{(y^2)^2}{\alpha}\right) \\
g_{u1} &= \frac{F^2}{\alpha}2b(1-b)\left(\frac{y^1}{y^u} + \frac{y^1(y^v + y^u\Phi)}{\alpha}\right) \\
g_{u2} &= \frac{F^2}{\alpha}2b(1-b)\left(\frac{y^2}{y^u} + \frac{y^2(y^v + y^u\Phi)}{\alpha}\right) \\
g_{v1} &= \frac{F^2}{\alpha}2b(1-b)\frac{y^1 y^u}{\alpha} \\
g_{v2} &= \frac{F^2}{\alpha}2b(1-b)\frac{y^2 y^u}{\alpha}
\end{aligned}$$

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