

## MASTER

### Identification of multivariable systems and representation in canonical and pseudo-canonical form

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DEPARTMENT OF ELECTRICAL ENGINEERING  
EINDHOVEN UNIVERSITY OF TECHNOLOGY  
Group Measurement and Control

IDENTIFICATION OF MULTIVARIABLE SYSTEMS  
AND REPRESENTATION IN CANONICAL AND  
PSEUDO-CANONICAL FORM.

by P.M. Carrière

This report is submitted in partial fulfillment of the requirements for the degree of electrical engineer (M.Sc.) at the Eindhoven University of Technology.

The work was carried out from dec. 1983 until okt. 1984

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De afdeling der elektrotechniek van de Technische Hogeschool aanvaardt geen verantwoordelijkheid voor de inhoud van Stage- en Afstudeerverslagen.

## Summary

In this report the structural and parametric identification of stable, linear, time-invariant, time-discrete, observable (or controllable) multivariable systems will be treated.

The structural identification will be performed following the method proposed by Guidorzi and similar methods making use of the determinants ratio and the singular value decomposition. Two methods, the Compensated Least Squares method and the Instrumental Variable method, will be used to accomplish the parametric identification.

The processes will be identified in an Input/Output model and from there transformations will be performed in order to represent them in the canonical and pseudo-canonical form of the state space model.

Inspired by and making use of a part of the available software developed on a PDP 11-60 computer at the group Measurement and Control of the Electrical Engineering department of the THE, some of these identification methods were implemented on a VAX 11/750 computer of the PICOS department of Philips Glass (Eindhoven). The software was written in standard FORTRAN 77 (without using VAX extensions) in order to be compatible with other computer systems.

In the second part of this report the behaviour of the algorithms will be tested with simulated data and the usefulness of these methods will be studied with the aid of practical data recorded in the field.

## Samenvatting

In dit verslag worden diverse methodes behandeld die de structuur en de parameters van stabiele, lineaire, tijd-invariante, tijd-discrete, observeerbare (of controleerbare) multivariabele systemen kunnen identificeren. Wat betreft de structurele identificatie zullen we drie methodes behandelen. Deze zijn: de determinant methode, de kleinste singuliere waarde methode en de Range Error test methode. Vervolgens zullen twee methodes die de parametrische identificatie verrichten (Instrumental Variable en Compensated Least Squares methode) toegelicht worden.

Tijdens de schattingsprocedure wordt het proces als een equivalent Input/Output model geïdentificeerd, wat daarna getransformeerd wordt naar de gewenste canonieke of pseudo-canonieke vorm van de state space representatie.

M.b.v. een deel van de bestaande programmatuur, ontwikkeld op een PDP 11/60 van de vakgroep Meten en Regelen (afdeling Elektrotechniek, THE) zijn sommige van deze identificatiemethoden op een VAX 11/750 computer van de PICOS groep (Philips Glass) geïmplementeerd. De software werd geschreven in standaard FORTRAN 77 (zonder VAX extensies) om de uitwisselbaarheid van de programmatuur te garanderen.

In tweede instantie worden de geïmplementeerde methodes getest m.b.v. gesimuleerde en praktische data.

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## 1.0 Introduction

The PICOS group of the Philips Glass division has been working for several years in a project trying to apply modern system control theories in order to improve the quality of the production of different industrial plants. Although these theories are quite general and can be applied to different fields ( industry, economy, medicine, biology..), the PICOS group is concentrating its efforts in optimising glass production processes.

Until nowadays, most of the production processes have been controlled in a quite empirical way. The results of many years of experience were used to control "as well as possible" a plant in order to obtain a "reasonable" product quality. What the modern control theories pretend is to obtain a better knowledge about the process, so that an optimal (for a given criterion) control strategy can be derived.

The work of the PICOS group can be divided into 3 main parts, those are:

- 1 - Measurement
- 2 - Modeling or Identification
- 3 - Control

Using measurements of the inputs and outputs of a process a model that describes the behaviour of the process can be built. Then, this model will be used to find a control strategy that will optimise a certain criterion.

During my project I have been working on the identification of the process. In its most general form the problem of system identification can be loosely formulated as follows: for a given finite number of input/-output measurements of a system, find the PARAMETERS of a MODEL which will describe the relation between input and output "as well as possible" according to a given criterion (see fig. 1.1). It is obvious that to determine an optimal control strategy, the engineer must be provided with a "good" model.

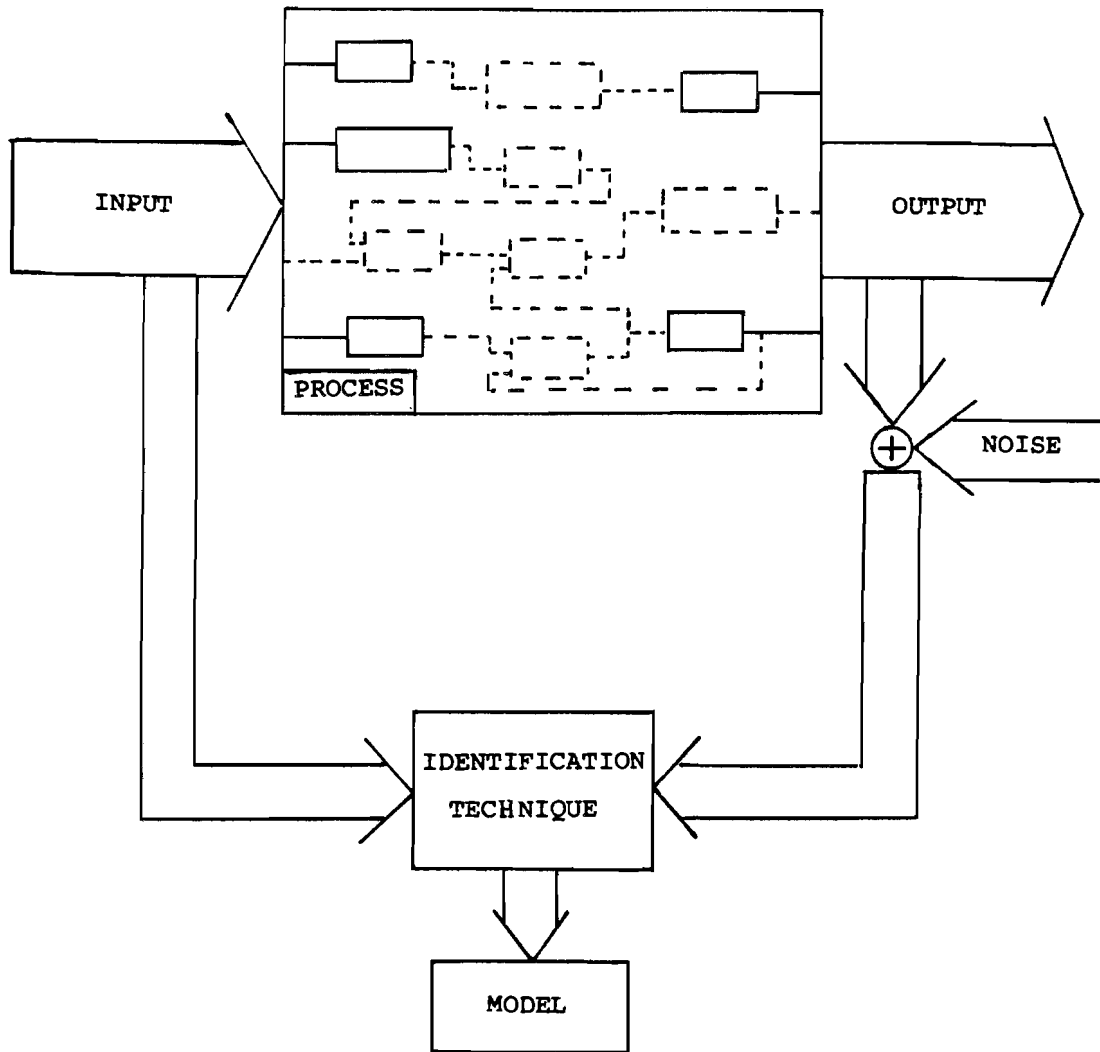


Fig.1.1 System identification problem

Although the subject of SISO (Single Input, Single Output) system identification has reached a significant level of maturity and perfection, relatively little has been done on the identification of MIMO (Multiple Input, Multiple Output) systems. To get a general idea about what has been done in this field the reader can refer to El-Sherief (1982).

An important factor in identifying multivariable systems is the choice of the model. We can distinguish four main types of models used for multivariable system representation; those are the transfer function matrix, the impulse response, the input/output model and the state space representation. These models are equivalent and transformations can be performed between them. They will be described in chapter 2.



Among these models the state space representation is popular among control engineers. This representation however is not unique and as a result many canonical forms have been suggested. Those reduce the number of unknown parameters to a minimum. One of these canonical forms, the canonical observable form used by Guidorzi (see Guidorzi (1975), (1981), (1982), Guidorzi and Losito (1982)) and a pseudo-canonical form (see Guidorzi and Beghelli (1982) and V.Overbeek (1982)) which leads to overlapping parametrisations will be discussed in chapter 3.

Because the state space representation introduces an intermediate state space vector between the input and the output of the process, this model is not very convenient for identification purposes. Therefore the identification will be performed using an input/output model, the Matrix Fraction Description (MFD). A transformation between this representation and the canonical and pseudo-canonical form in the state space model will be derived and described in chapter 4.

To identify the system, the MIMO problem will be divided in several MISO (Multiple In, Single Out) problems. The estimation of the order of each of these MISO subsystems, also called the structural identification, following the method of Guidorzi will be described in chapter 5. As it will be remarked, this is only necessary in the case that the process is represented in the canonical form.

In chapter 6 two methods, the (Compensated) Least Squares and the Instrumental Variable method, for estimating the parameters of each of the MISO subsystems will be discussed. Also an extension to estimate systems with offset and some model validity tests will be treated in this chapter.

The results of these methods implemented on a VAX 11/750 computer in FORTRAN language will be presented in chapter 7.

Finally I will present some conclusions on these methods and their implementation.

## 2.0 Different representations of multivariable systems

### 2.1 Introduction

One of the important factors involved in the identification of MIMO systems is the choice of the model in which the system will be represented. Each model affects the complexity of the identification algorithm in a different way.

We can distinguish four types of models used for system representation. Those are the transfer function matrix, the impulse response, the Input/-Output model and the state space formulation. These representations are equivalent and transformations between them are possible.

In this chapter I will give a brief description and some of the most important properties of these models. For a deeper insight in this question the reader can refer to El-Sherief and Sinha (1979), Hajdasinski (1980) and Hajdasinski, Eykhoff, Damen and van den Boom (1981).

In the following we will assume that the system to be represented has:

- p inputs
- q outputs
- order n

The following notations will be used:

- $\underline{u}(k)$  : p-dimensional input vector at time k.
- $\underline{y}(k)$  : q-dimensional output vector at time k.
- $z$  : unit advance operator (  $y(k+1)=zy(k)$  ) .

### 2.2 Transfer function matrix representation

A system can be described by a transfer function matrix  $G(z)$  as follows:

$$\underline{y}(k) = G(z) \underline{u}(k) \quad (2.1)$$

$$\text{with : } G(z) = \begin{bmatrix} \frac{A_{11}(z)}{B_{11}(z)} & \dots & \frac{A_{1p}(z)}{B_{1p}(z)} \\ \dots & \dots & \dots \\ \frac{A_{q1}(z)}{B_{q1}(z)} & \dots & \frac{A_{qp}(z)}{B_{qp}(z)} \end{bmatrix} \quad (2.2)$$

where  $A_{ij}(z)$  and  $B_{ij}(z)$ ,  $i=1,q$  and  $j=1,p$  are polynomials in  $z$  of degree less than or equal to  $n$ .

The structural parameters required to characterise  $G(z)$  are the orders of the numerator and the denominator of each entry of  $G(z)$ . The number of parameters to estimate depends on these orders.

The transfer function matrix gives a unique representation of a multi-variable dynamical system. It has a simple physical interpretation because each element  $G_{ij}(z)$  of  $G(z)$  represents the relation between input  $i$  and output  $j$ .

However this description has a lot of drawbacks like:

- complexity of numerical calculations
- difficulties with handling the noise
- possibility of not encountering all dynamical properties of the system
- inconvenient for digital modelling.

### 2.3 Impulse response representation

A linear time-discrete system can also be represented by the following infinite series:

$$\underline{y}(k) = [ M_0 + M_1 z^{-1} + M_2 z^{-2} + \dots ] \underline{u}(k) \quad (2.3)$$

where  $M_0, M_1, M_2, \dots$  are constant ( $q \times p$ ) matrices called the Markov parameters of the system. This theoretically infinite number of parameters defines the impulse response of the system. Hence it is possible to

truncate the series after  $\ell$  terms if the system is stable. Physically it means that the Markov parameters  $M_j$  can be neglected for  $j > \ell$ . The system can then be approximated by :

$$\underline{y}(k) = [ M_0 + M_1 z^{-1} + M_2 z^{-2} + \dots + M_\ell z^{-\ell} ] \underline{u}(k) \quad (2.4)$$

The structural parameter required to characterise the system in the model of equation (2.4) is the value of  $\ell$ , i.e.  $\ell+1$  is the minimum number of Markov parameters required to describe the system completely. This description of a multivariable system is unique. The number of required parameters to describe the system is  $\ell \times p \times q$ .

#### 2.4 Input/Output model: the Matrix Fraction Description (MFD)

Another form to represent the system is by means of the difference equation:

$$P(z) \underline{y}(k) = Q(z) \underline{u}(k) \quad (2.5)$$

The I/O model we will use in this report is the Matrix Fraction Description (2.5). This representation is not unique. The  $P(z)$  and  $Q(z)$  polynomial matrices can have different forms (see El-Sherief and Sinha (1979)). The one we will consider is the input/output difference equation presented by Guidorzi (see Guidorzi (1975) and (1979)):

$$P(z) = \begin{bmatrix} p_{11}(z) & \dots & p_{1q}(z) \\ \vdots & \vdots & \vdots \\ p_{q1}(z) & \dots & p_{qq}(z) \end{bmatrix} \quad (2.6)$$

$$Q(z) = \begin{bmatrix} q_{11}(z) & \dots & q_{1p}(z) \\ \vdots & \vdots & \vdots \\ q_{q1}(z) & \dots & q_{qp}(z) \end{bmatrix} \quad (2.7)$$

where  $p_{ij}(z)$  and  $q_{ij}(z)$  are polynomials in  $z$  defined as follows:

$$\begin{aligned}
 p_{ii}(z) &= z^{v_i} + \alpha_{ii, v_i} z^{v_i-1} + \alpha_{ii, v_i-1} z^{v_i-2} + \dots + \alpha_{ii, 1} \\
 p_{ij}(z) &= \alpha_{ij, v_j} z^{v_j-1} + \alpha_{ij, v_j-1} z^{v_j-2} + \dots + \alpha_{ij, 1} \quad (2.8) \\
 q_{ij}(z) &= \beta_{i, v_i} z^{v_i-1} + \beta_{i, (v_i-1)} z^{v_i-2} + \dots + \beta_{i, 1}
 \end{aligned}$$

The structural parameters characterising this model are the observability sub-indices of the system ( $v_i, i=1, q$ ) and the number of parameters to identify is  $nx(p+q)$ , where  $n = \sum v_i$ .

### 2.5 State space representation

In this representation a state space vector is introduced as being the link between input and output of the system. This vector "buffers" information about the influence of past events on the present and the future. For the time-discrete system the state space representation is:

$$\left\{ \begin{array}{l} \underline{x}(k+1) = \underline{F}\underline{x}(k) + \underline{G}\underline{u}(k) \\ \underline{y}(k) = \underline{H}\underline{x}(k) + \underline{K}\underline{u}(k) \end{array} \right. \quad (2.9)$$

where :

- $\underline{x}(k)$  : state space vector at time  $k$  ( $n$ )
- $\underline{F}$  : system matrix ( $n \times n$ )
- $\underline{G}$  : distribution matrix ( $n \times p$ )
- $\underline{H}$  : output or measurement matrix ( $q \times n$ )
- $\underline{K}$  : input/output matrix ( $q \times p$ )

This  $\{\underline{F}, \underline{G}, \underline{H}, \underline{K}\}$  state space representation is not unique. Because of the introduction of the "intermediate" state space vector it has not a direct physical interpretation so that it gives little insight into the process. This representation is economic in the sense of the number of parameters

to estimate which can be minimised by choosing a correct set  $\{F,G,H,K\}$ . It is also very convenient for digital modelling and encounters all dynamical properties of the system being modelled. It is quite easy to perform transformations from state space to both the transfer function matrix and to the impulse response model.

These advantages have made the state space model very popular among control engineers. During this work we will use this model to represent the identified process.

### 3.0 Canonical and pseudo-canonical representation

#### 3.1 Introduction

In the previous paragraph we have remarked that the state space representation is not unique. Different representations can describe the same process. In fact there is an infinite number of representations as there is an infinite number of bases in which the same model (and thus the state space vector) can be written. This degree of freedom allows the control engineer to minimise the number of parameters necessary to describe the system by choosing an appropriate basis.

Two possible ways of choosing this basis will be described in this chapter. The first way, called the Kronecker selection (see Guidorzi (1975), (1981), Meertens (1983) and Van Buuren (1982)), leads to a unique basis and therefore to a unique "canonical representation".

A second way of choosing a basis is the nice selection (see Guidorzi (1982), van Overbeek (1982), Pfennings (1983)) leading to a finite number of vector-sets, each constituting a basis (in general). This results in a finite number of different "pseudo-canonical representations". Because these bases will have some vectors in common, these second types of representations will overlap each other. By performing simple transformations one representation can be derived from another. Therefore they are called in the literature by the name of overlapping parametrisation. Because several structures are allowed, we have to deal in this case with a multistructural identification problem.

For both cases we will treat the observable (pseudo-)canonical form assuming that the system is observable. It is also possible to obtain a controllable (pseudo-)canonical form assuming that the system is controllable. This leads to similar results (see Hajdasinski (1980)) .

#### 3.2 Observable canonical form (form of Guidorzi)

Consider the time-discrete observable system in its state space representation ( see section 2.5) :

$$\begin{cases} \underline{x}'(k+1) = F\underline{x}'(k) + G\underline{u}(k) \\ \underline{y}(k) = H\underline{x}'(k) + K\underline{u}(k) \end{cases} \quad (3.1)$$

where :

$$H = \begin{bmatrix} \underline{h}_1^T \\ \underline{h}_2^T \\ \vdots \\ \underline{h}_q^T \end{bmatrix} \quad (3.2)$$

It will be assumed that: Rank(H) = q (no. of outputs)

Construct the following vector sequence :

$$\begin{array}{ccccccccc} \underline{h}_1 & \uparrow & (F^T)\underline{h}_1 & \uparrow & (F^T)^2\underline{h}_1 & \uparrow & (F^T)^3\underline{h}_1 & \uparrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \underline{h}_2 & & (F^T)\underline{h}_2 & & (F^T)^2\underline{h}_2 & & (F^T)^3\underline{h}_2 & & \dots \\ \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \underline{h}_q & \rightarrow \uparrow & (F^T)\underline{h}_q & \rightarrow \uparrow & (F^T)^2\underline{h}_q & \rightarrow \uparrow & (F^T)^3\underline{h}_q & \rightarrow \uparrow & \dots \end{array} \quad (3.3)$$

The Kronecker selection procedure is as follows:

Consider the vector sequence (3.3) in the order indicated by the arrows and retain vector  $(F^T)^S \underline{h}_i$  if and only if it is independent from all previously selected vectors.

It is easy to prove that if  $(F^T)^S \underline{h}_i$  is dependent of previously selected vectors, then all vectors  $(F^T)^j \underline{h}_i$ ,  $S < j$  are also dependent vectors. This means that from the moment that a dependent vector has been found at row  $i$  of (3.3), this and the remaining vectors of this row can be skipped during the further selection procedure.

The Kronecker selection ends when a dependent vector has been found in



each row of (3.3). From the definition of this selection procedure we can derive that it leads to a unique set of independent vectors.

Let  $v_1, v_2, \dots, v_q$  be the number of vectors selected from respectively rows 1, 2, ..., q of (3.3). The obtained set of independent vectors is:

$$\begin{aligned} \underline{h}_1, (F^T)\underline{h}_1, \dots, (F^T)^{v_1-1}\underline{h}_1, \underline{h}_2, (F^T)\underline{h}_2, \dots, (F^T)^{v_2-1}\underline{h}_2, \\ \dots, \underline{h}_q, (F^T)\underline{h}_q, \dots, (F^T)^{v_q-1}\underline{h}_q \end{aligned} \quad (3.4)$$

The dependent vectors  $(F^T)^{v_i}\underline{h}_i, i=1, q$  can be written as a linear combination of the previously selected vectors:

$$(F^T)^{v_i}\underline{h}_i = \sum_{j=1}^q \left( \sum_{\ell=1}^{v_{ij}} \alpha_{ij,\ell} (F^T)^{\ell-1} \underline{h}_j \right) \quad (3.5)$$

$$\begin{aligned} \text{with : } v_{ij} &= v_i && \text{for } i=j \\ v_{ij} &= \min(v_i+1, v_j) && \text{for } i>j \\ v_{ij} &= \min(v_i, v_j) && \text{for } i<j \end{aligned}$$

The complete observability of the system implies that:

$$v_1 + v_2 + \dots + v_q = n \quad (3.6)$$

where  $n$  is the order of the system. The  $v_i$ 's are called the Kronecker invariants of the system and we will see later how they define the structure of the system.

In order to represent the system in the basis obtained by the Kronecker selection, we build the following transformation matrix:

$$T^T = \left[ \begin{array}{cccccccc} \underline{h}_1 & (F^T)\underline{h}_1 & \dots & (F^T)^{v_1-1}\underline{h}_1 & \dots & \underline{h}_q & (F^T)\underline{h}_q & \dots & (F^T)^{v_q-1}\underline{h}_q \end{array} \right] \quad (3.7)$$

$T$  is composed by independent vectors and is therefore a nonsingular ma-

trix. By applying the similarity transformation

$$\underline{x}(k) = T\underline{x}'(k) \quad (3.8)$$

we will obtain an equivalent description  $\{A,B,C,D\}$  of the system. The new system equations are:

$$\left| \begin{array}{l} \underline{x}(k+1) = A\underline{x}(k) + B\underline{u}(k) \\ \underline{y}(k) = C\underline{x}(k) + D\underline{u}(k) \end{array} \right. \quad (3.9)$$

$$\begin{array}{l} \text{with :} \\ A = TFT^{-1} \\ B = TG \\ C = HT^{-1} \\ D = K \end{array} \quad (3.10)$$

It can be proved that this new representation has a particular structure .

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1q} \\ A_{21} & A_{22} & \dots & A_{2q} \\ \vdots & \vdots & \vdots & \vdots \\ A_{q1} & A_{q2} & \dots & A_{qq} \end{bmatrix} \quad (3.11)$$

with :

$$A_{ii} = \begin{bmatrix} 0 & & & & \\ 0 & & & & \\ \vdots & & I_{v_i-1} & & \\ 0 & & & & \\ \alpha_{ii,1} & \alpha_{ii,2} & \alpha_{ii,3} & \dots & \alpha_{ii,v_i} \end{bmatrix} \quad (3.12 a)$$

←  $v_i$  →

$$A_{ij} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \dots & \dots & \dots & \dots \\ \alpha_{ij,1} & \alpha_{ij,2} & \alpha_{ij,3} & \dots & \alpha_{ij,v_{ij}} & 0 & \dots & 0 & 0 \end{bmatrix} \begin{matrix} \uparrow \\ \\ \\ \\ \downarrow \end{matrix} \quad (3.12 \text{ b})$$

$\leftarrow$   $v_j$   $\rightarrow$

with  $v_{ij}$  defined as:

$$\begin{aligned} v_{ij} &= v_i && \text{for } i=j \\ v_{ij} &= \min(v_i+1, v_j) && \text{for } i>j \\ v_{ij} &= \min(v_i, v_j) && \text{for } i<j \end{aligned}$$

The A matrix (see also 4.42) has a block companion structure. An example will be treated in section 3.4 to illustrate this point. The number,  $v_{ij}$ , of non-zero elements in submatrix  $A_{ij}$ , because of the order followed in the selection of vectors (3.3) and of the consequent structure of the matrix T, is at most equal to  $v_i+1$  if  $i>j$ , to  $v_i$  if  $i<j$ .

Furthermore :

$$C=HT^{-1} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 1 & 0 & \dots & 0 \end{bmatrix} \begin{matrix} \uparrow \\ \\ \\ \\ \downarrow \end{matrix} \quad (3.13)$$

$\leftarrow$   $n$   $\rightarrow$

column  $\uparrow$   $1$   $\uparrow$   $1+v_1$   $\uparrow$   $1+v_1+\dots+v_{q-1}$

The matrix C is completely described by the (structural) Kronecker invariants. The system matrices B and D do not have any special structure.

$$B = TG = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ \cdot & \cdot & \dots & \cdot \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix} \begin{matrix} \uparrow \\ \\ \downarrow \end{matrix} \quad (3.14)$$

$\leftarrow$   $p$   $\rightarrow$

$$D = K = \begin{bmatrix} d_{11} & d_{12} & \cdot & \cdot & \cdot & d_{1p} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ d_{n1} & d_{n2} & \cdot & \cdot & \cdot & d_{np} \end{bmatrix} \begin{matrix} \uparrow \\ q \\ \downarrow \end{matrix} \quad (3.15)$$

← p →

Because of the special properties of the Kronecker selection, this particular state space representation is unique and one can describe the system by a reduced amount of parameters (see matrices A and C). This canonical form is known in the literature by the name of form of Guidorzi.

### 3.3 Observable pseudo-canonical form (overlapping parametrisation)

A second way (nice selection) to choose a vector set in which we will write the system equations will now be introduced. Starting with the same state space representation (3.1), let us consider the vector sequence (3.3) but now without any order:

$$\begin{array}{cccccc} \underline{h}_1 & (F^T)\underline{h}_1 & (F^T)^2\underline{h}_1 & (F^T)^3\underline{h}_1 & \dots & \\ \underline{h}_2 & (F^T)\underline{h}_2 & (F^T)^2\underline{h}_2 & (F^T)^3\underline{h}_2 & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \underline{h}_q & (F^T)\underline{h}_q & (F^T)^2\underline{h}_q & (F^T)^3\underline{h}_q & \dots & \end{array} \quad (3.16)$$

The nice selection procedure runs as follows:

- select vectors  $\underline{h}_1, \underline{h}_2, \dots, \underline{h}_q$
- vector  $(F^T)^S \underline{h}_i$  can be selected if all vectors  $(F^T)^j \underline{h}_i, j=1, \dots, S-1$  had been selected previously.

Opposite to the Kronecker selection, the nice selection does not prescribe an order in which the vectors sequence (3.16) has to be considered. This results in the fact that the nice selection does not provide a unique set of vectors. The number of different Nice selections is (see

Pfennings (1983)) :

$$\begin{bmatrix} n-1 \\ q-1 \end{bmatrix} = \frac{(n-1)!}{(q-1)!(n-q)!} \quad (3.17)$$

where :

$n$  = order of system

$q$  = number of outputs of system.

Let  $\mu_1, \mu_2, \dots, \mu_q$  be the number of vectors selected from row 1, 2, ...,  $q$  of (3.16). Those indices are called Kronecker numbers and define the following vector set:

$$\begin{aligned} \underline{h}_1, (F^T)\underline{h}_1, \dots, (F^T)^{\mu_1-1}\underline{h}_1, \underline{h}_2, (F^T)\underline{h}_2, \dots, (F^T)^{\mu_2-1}\underline{h}_2 \\ , \dots, \underline{h}_q, (F^T)\underline{h}_q, \dots, (F^T)^{\mu_q-1}\underline{h}_q \end{aligned} \quad (3.18)$$

We will assume in the following that the selected vectors are independent. In the case that one of the vectors is dependent, the dimension of the space generated by the vector set (3.18) will be less than the dimension of the system itself. This would mean that the system can not be represented by the pseudo-canonical form corresponding to this nice selection.

By applying the similarity transformation  $T'$  defined by the vectors (3.18) we will obtain, in a similar way as in section 3.2, an equivalent state space representation defined by:

$$\left| \begin{array}{l} \underline{x}(k+1) = A\underline{x}(k) + B\underline{u}(k) \\ \underline{y}(k) = C\underline{x}(k) + D\underline{u}(k) \end{array} \right. \quad (3.19)$$

with :

$$\begin{aligned} A &= T' F T'^{-1} \\ B &= T' G \\ C &= H T'^{-1} \\ D &= K \end{aligned} \quad (3.20)$$

This new representation has the following structure .

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdot & \cdot & \cdot & A_{1q} \\ A_{21} & A_{22} & \cdot & \cdot & \cdot & A_{2q} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{q1} & A_{q2} & \cdot & \cdot & \cdot & A_{qj} \end{bmatrix} \quad (3.21)$$

with :

$$A_{ii} = \begin{bmatrix} 0 & & & & & \\ 0 & & & & & \\ \vdots & & I & & & \\ & & \mu_i^{-1} & & & \\ 0 & & & & & \\ \alpha_{ii,1} & \alpha_{ii,2} & \alpha_{ii,3} & \cdot & \cdot & \cdot & \alpha_{ii,\mu_i} \end{bmatrix} \begin{matrix} \uparrow \\ \\ \mu_i \\ \downarrow \end{matrix} \quad (3.22 \text{ a})$$

←  $\mu_i$  →

$$A_{ij} = \begin{bmatrix} 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ \alpha_{ij,1} & \alpha_{ij,2} & \alpha_{ij,3} & \cdot & \cdot & \cdot & \alpha_{ij,\mu_j} \end{bmatrix} \begin{matrix} \uparrow \\ \\ \mu_i \\ \downarrow \end{matrix} \quad (3.22 \text{ b})$$

←  $\mu_j$  →

The matrix A (see also (4.7)) has a block companion structure. An example will be treated in section 3.4 to illustrate this point.

The matrix C is completely described by the (structural) Kronecker numbers  $\mu_i$ .

$$C=HT^{-1} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 1 & 0 & \dots & 0 \end{bmatrix} \quad (3.23)$$

$\leftarrow$   $\uparrow$   $\uparrow$   $\uparrow$   $\rightarrow$   
 column 1  $1+\mu_1$   $1+\mu_1+\dots+\mu_{q-1}$

The system matrices B and D do not have any special structure. They can be represented by the same matrices (3.14) and (3.15).

The pseudo-canonical representation ( or overlapping parametrisation) is very close to the canonical representation of Guidorzi. In the next section we will illustrate both forms and compare them by treating an example.

#### 3.4 Canonical versus pseudo-canonical form

The selection of a parametrisation in the canonical set may lead to numerical and statistical problems. This is essentially due to the fact that a given system only can be represented by one canonical representation which can be "ill-conditioned".

To alleviate this problem the use of a pseudo-canonical set of parametrisations has been suggested. When this set is used the problem of trying to chose the correct set of structural indices (structural identification) does not appear as such. Instead, the problem that arises is which parametrisation is most suitable to represent a certain model, according to statistical and numerical considerations (see Gevers (1982), van Overbeek (1982)).

The fact that the parametrisations in the pseudo-canonical set overlap makes it generally possible to move from one to another (in order to avoid "ill-conditioned" representations) by performing a simple transformation without having to perform the entire parametric identification (see Guidorzi (1982)).

The difference between the canonical and the pseudo-canonical form resides in the way the basis in which they are represented is chosen.

The stronger constraints imposed on the vectors during the Kronecker selection will result in general in a smaller number of model parameters in the system matrix A.

To get a better insight in this question, lets consider the following example (for more examples the reader can refer to Van Buuren (1982)):

Example

system of order  $n = 7$   
 number of outputs  $q = 3$   
 Kronecker invariants  $v_1 = 2, v_2 = 4, v_3 = 1$   
 Kronecker numbers  $\mu_1 = 2, \mu_2 = 4, \mu_3 = 1$

The structural indices indicates that both canonical and pseudo-canonical form are written in the following basis:

$$\underline{c}_1, (A^T)\underline{c}_1, \underline{c}_2, (A^T)\underline{c}_2, (A^T)^2\underline{c}_2, (A^T)^3\underline{c}_2, \underline{c}_3 \quad (3.24)$$

$$\text{with: } C = \begin{bmatrix} \underline{c}_1^T \\ \underline{c}_2^T \\ \underline{c}_3^T \end{bmatrix} \quad (3.25)$$

Each row of the A matrix expresses the coordinates of the image by the transformation defined by A of each vector of the basis in which the equations are written. According to the vector dependencies (3.5) the matrix A can be written as follows:



$$\begin{bmatrix} \underline{C}_1^T A \\ \underline{C}_1^T A^2 \\ \underline{C}_2^T A \\ \underline{C}_2^T A^2 \\ \underline{C}_2^T A^3 \\ \underline{C}_2^T A^4 \\ \underline{C}_3^T A \end{bmatrix} = \begin{bmatrix} \underline{C}_1^T \\ \underline{C}_1^T A \\ \underline{C}_2^T \\ \underline{C}_2^T A \\ \underline{C}_2^T A^2 \\ \underline{C}_2^T A^3 \\ \underline{C}_3^T \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha_{11,1} & \alpha_{11,2} & \alpha_{12,1} & \alpha_{12,2} & \alpha_{12,3} & \alpha_{12,4} & \alpha_{13,1} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \alpha_{21,1} & \alpha_{21,2} & \alpha_{22,1} & \alpha_{22,2} & \alpha_{22,3} & \alpha_{22,4} & \alpha_{23,1} \\ \alpha_{31,1} & \alpha_{31,2} & \alpha_{32,1} & \alpha_{32,2} & \alpha_{32,3} & \alpha_{32,4} & \alpha_{33,1} \end{bmatrix}$$

(3.26)

This illustrates the fact that the choice of the basis by Kronecker or nice selection confers the system matrix A a block-companion structure.

In the case that the basis has been obtained by a Kronecker selection procedure, some supplementary information about the basis is available (see section 3.2):

a) - vector  $(A^T)^2 \underline{C}_1$  is linearly dependent on the vectors  $\underline{C}_1$ ,  $\underline{C}_2$ ,  $\underline{C}_3$ ,  $A^T \underline{C}_1$ ,  $A^T \underline{C}_2$  and  $A^T \underline{C}_3$ .

b) - vector  $(A^T)^4 \underline{C}_2$  is linearly dependent on all vectors of the basis (obviously).

c) - vector  $(A^T) \underline{C}_3$  is linearly dependent on vectors  $\underline{C}_1$ ,  $\underline{C}_2$ ,  $\underline{C}_3$ ,  $(A^T) \underline{C}_1$ , and  $(A^T) \underline{C}_2$ .

The consequences of a), b) and c) is that in the canonical representation some of the  $\alpha_{ij,l}$  parameters of the system matrix A will be "structurally" equal to zero. In our example :

$$a) \quad \implies \quad \alpha_{12,3} = 0 \text{ and } \alpha_{12,4} = 0 .$$

$$b) \quad \text{has no special consequences on the } \alpha_{2j,l} .$$

$$c) \quad \implies \quad \alpha_{32,3} = 0 \text{ and } \alpha_{32,4} = 0 .$$

The parameters that are structurally zero ( in the case of a canonical representation ) are underlined in matrix A (3.26).

The nice selection procedure doesn't tell us anything about the dependence between vectors, so nothing can be concluded about the parameters  $\alpha_{ij,l}$ . They will have to be identified. In general the pseudo-canonical form needs more parameters to represent a process than the canonical form.

In the canonical case we can also obtain the parameters that are structurally zero by using property (3.12). The number of non-zero parameters in submatrix  $A_{ij}$  is given by:

$$v_{ij} = \begin{cases} v_i & \text{for } i=j \\ \min(v_i+1, v_j) & \text{for } i>j \\ \min(v_i, v_j) & \text{for } i<j \end{cases} \quad (3.27)$$

So that :

a) - submatrices  $A_{11}$ ,  $A_{12}$ ,  $A_{13}$  have respectively

$$v_{11} = v_1 = 2$$

$$v_{12} = \min(v_1, v_2) = 2$$

$$v_{13} = \min(v_1, v_3) = 1$$

non-zero parameters. The dimension of this submatrices being respectively 2, 4 and 1, this implies that parameters  $\alpha_{12,3}$

and  $\alpha_{12,4}$  are structurally equal to zero.

b) - submatrices  $A_{21}$ ,  $A_{22}$ ,  $A_{23}$  have respectively

$$v_{21} = \min(v_2 + 1, v_1) = 2$$

$$v_{22} = v_2 = 4$$

$$v_{23} = \min(v_2, v_3) = 1$$

non-zero parameters. The  $v_{2j}$   $j=1,3$  are equal to the dimension of the relevant submatrix so that we can affirm that all  $\alpha$ -parameters of this sub-matrix are non-zero.

c) - submatrices  $A_{31}$ ,  $A_{32}$ ,  $A_{33}$  have respectively

$$v_{31} = \min(v_2 + 1, v_1) = 2$$

$$v_{32} = \min(v_3 + 1, v_2) = 2$$

$$v_{33} = v_3 = 1$$

which implies that parameters  $v_{32,3}$  and  $v_{32,4}$  are structurally equal to zero.

## 4.0 Transformation from (pseudo-)canonical form to I/O model

### 4.1 Introduction

The parametric identification will be done in the I/O model (see par. 2.3) rather than in the state space model which does not link directly input to output. The MFD I/O representation describes the system in the following way:

$$P(z) \underline{y}(z) = Q(z) \underline{u}(z) \quad (4.1)$$

In a second step the parametrisation obtained during the identification will be transformed into the desired (pseudo-)canonical form. In this chapter we will derive and describe this transformation.

In the first place we will treat the transformation between the pseudo-canonical form and the I/O model (section 4.1) and from there we will derive the transformation between the canonical form and the I/O model (section 4.2). Similar calculations have been done by Pfenning (1983) but only for the strictly proper case (systems without simultaneous interaction between input and output; in state space this corresponds to matrix  $D=0$ ). We will treat in the following the more general case of the proper systems ( $D \neq 0$ ).

### 4.2 From pseudo-canonical form to I/O model

Let's consider the pseudo-canonical form represented by

$$\begin{cases} \underline{x}(k+1) = \underline{A}\underline{x}(k) + \underline{B}\underline{u}(k) \\ \underline{y}(k) = \underline{C}\underline{x}(k) + \underline{D}\underline{u}(k) \end{cases} \quad (4.2)$$

The vectors  $\underline{u}(k)$ ,  $\underline{y}(k)$ ,  $\underline{x}(k)$ ,  $\underline{a}_i$  and the matrices  $A$ ,  $B$ ,  $D$  as they have already been defined in paragraph 3.2 are explicitly written in (4.3) to (4.9).

$$\underline{u}(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \\ \vdots \\ u_i(k) \\ \vdots \\ u_p(k) \end{bmatrix} \begin{matrix} + \\ \\ \\ p \\ \\ + \end{matrix}$$

#### 4.3 Input vector

$$\underline{y}(k) = \begin{bmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_i(k) \\ \vdots \\ y_q(k) \end{bmatrix} \begin{matrix} + \\ \\ \\ q \\ \\ + \end{matrix}$$

#### 4.4 Output vector

$$\underline{x}(k) = \begin{bmatrix} x_{1,1}(k) \\ x_{1,2}(k) \\ \vdots \\ x_{1,\mu_1}(k) \\ \hline x_{2,1}(k) \\ x_{2,2}(k) \\ \vdots \\ x_{2,\mu_2}(k) \\ \vdots \\ x_{i,1}(k) \\ x_{i,2}(k) \\ \vdots \\ x_{i,\mu_i}(k) \\ \hline \vdots \\ \hline x_{q,1}(k) \\ x_{q,2}(k) \\ \vdots \\ x_{q,\mu_q}(k) \end{bmatrix} \begin{matrix} + \\ \\ \mu_1 \\ + \\ + \\ \mu_2 \\ + \\ \vdots \\ + \\ \mu_i \\ + \\ \hline + \\ \mu_q \\ + \end{matrix}$$

#### 4.5 State space vector

$$\underline{\alpha}_i = \begin{bmatrix} \alpha_{i1,1} \\ \alpha_{i1,2} \\ \vdots \\ \alpha_{i1,\mu_1} \\ \hline \alpha_{i2,1} \\ \alpha_{i2,2} \\ \vdots \\ \alpha_{i2,\mu_2} \\ \hline \vdots \\ \hline \alpha_{ii,1} \\ \alpha_{ii,2} \\ \vdots \\ \alpha_{ii,\mu_i} \\ \hline \vdots \\ \hline \alpha_{iq,1} \\ \alpha_{iq,2} \\ \vdots \\ \alpha_{iq,\mu_q} \end{bmatrix} \begin{matrix} + \\ \mu_1 \\ + \\ + \\ \mu_2 \\ + \\ \vdots \\ + \\ \mu_i \\ + \\ \hline + \\ \mu_q \\ + \end{matrix}$$

#### 4.6 $\alpha$ -parameter vector

|   |            |            |         |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
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|   | $\nu_1$    | $\nu_2$    | $\nu_1$ | $\nu_q$        |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| <table border="1" style="border-collapse: collapse; width: 100%; text-align: center;"> <tr><td>0</td><td>1</td><td>0</td><td>...</td><td>0</td></tr> <tr><td>0</td><td>0</td><td>1</td><td>...</td><td>0</td></tr> <tr><td>⋮</td><td>⋮</td><td>⋮</td><td>...</td><td>⋮</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>...</td><td>1</td></tr> <tr><td><math>a_{11,1}</math></td><td><math>a_{11,2}</math></td><td><math>a_{11,3}</math></td><td>...</td><td><math>a_{11,\nu_1}</math></td></tr> <tr><td>0</td><td>0</td><td>0</td><td>...</td><td>0</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>...</td><td>0</td></tr> <tr><td>⋮</td><td>⋮</td><td>⋮</td><td>...</td><td>⋮</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>...</td><td>0</td></tr> <tr><td><math>a_{21,1}</math></td><td><math>a_{21,2}</math></td><td><math>a_{21,3}</math></td><td>...</td><td><math>a_{21,\nu_1}</math></td></tr> <tr><td>⋮</td><td>⋮</td><td>⋮</td><td>...</td><td>⋮</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>...</td><td>0</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>...</td><td>0</td></tr> <tr><td>⋮</td><td>⋮</td><td>⋮</td><td>...</td><td>⋮</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>...</td><td>0</td></tr> <tr><td><math>a_{11,1}</math></td><td><math>a_{11,2}</math></td><td><math>a_{11,3}</math></td><td>...</td><td><math>a_{11,\nu_1}</math></td></tr> <tr><td>⋮</td><td>⋮</td><td>⋮</td><td>...</td><td>⋮</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>...</td><td>0</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>...</td><td>0</td></tr> <tr><td>⋮</td><td>⋮</td><td>⋮</td><td>...</td><td>⋮</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>...</td><td>0</td></tr> <tr><td><math>a_{q1,1}</math></td><td><math>a_{q1,2}</math></td><td><math>a_{q1,3}</math></td><td>...</td><td><math>a_{q1,\nu_1}</math></td></tr> </table> | 0          | 1          | 0       | ...            | 0 | 0 | 0 | 1 | ... | 0 | ⋮ | ⋮ | ⋮ | ... | ⋮ | 0 | 0 | 0 | ... | 1 | $a_{11,1}$ | $a_{11,2}$ | $a_{11,3}$ | ... | $a_{11,\nu_1}$ | 0 | 0 | 0 | ... | 0 | 0 | 0 | 0 | ... | 0 | ⋮ | ⋮ | ⋮ | ... | ⋮ | 0 | 0 | 0 | ... | 0 | $a_{21,1}$ | $a_{21,2}$ | $a_{21,3}$ | ... | $a_{21,\nu_1}$ | ⋮ | ⋮ | ⋮ | ... | ⋮ | 0 | 0 | 0 | ... | 0 | 0 | 0 | 0 | ... | 0 | ⋮ | ⋮ | ⋮ | ... | ⋮ | 0 | 0 | 0 | ... | 0 | $a_{11,1}$ | $a_{11,2}$ | $a_{11,3}$ | ... | $a_{11,\nu_1}$ | ⋮ | ⋮ | ⋮ | ... | ⋮ | 0 | 0 | 0 | ... | 0 | 0 | 0 | 0 | ... | 0 | ⋮ | ⋮ | ⋮ | ... | ⋮ | 0 | 0 | 0 | ... | 0 | $a_{q1,1}$ | $a_{q1,2}$ | $a_{q1,3}$ | ... | $a_{q1,\nu_1}$ | <table border="1" style="border-collapse: collapse; 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width: 100%; text-align: center;"> <tr><td>+</td><td></td><td></td><td></td><td></td><td>+</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>...</td><td>0</td><td>+</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>...</td><td>0</td><td>+</td></tr> <tr><td>⋮</td><td>⋮</td><td>⋮</td><td>...</td><td>⋮</td><td>+</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>...</td><td>0</td><td>+</td></tr> <tr><td><math>a_{1q,1}</math></td><td><math>a_{1q,2}</math></td><td><math>a_{1q,3}</math></td><td>...</td><td><math>a_{1q,\nu_q}</math></td><td>+</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>...</td><td>0</td><td>+</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>...</td><td>0</td><td>+</td></tr> <tr><td>⋮</td><td>⋮</td><td>⋮</td><td>...</td><td>⋮</td><td>+</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>...</td><td>0</td><td>+</td></tr> <tr><td><math>a_{2q,1}</math></td><td><math>a_{2q,2}</math></td><td><math>a_{2q,3}</math></td><td>...</td><td><math>a_{2q,\nu_q}</math></td><td>+</td></tr> <tr><td>⋮</td><td>⋮</td><td>⋮</td><td>...</td><td>⋮</td><td>+</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>...</td><td>0</td><td>+</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>...</td><td>0</td><td>+</td></tr> <tr><td>⋮</td><td>⋮</td><td>⋮</td><td>...</td><td>⋮</td><td>+</td></tr> <tr><td>0</td><td>0</td><td>0</td><td>...</td><td>0</td><td>+</td></tr> <tr><td><math>a_{q1,1}</math></td><td><math>a_{q1,2}</math></td><td><math>a_{q1,3}</math></td><td>...</td><td><math>a_{q1,\nu_1}</math></td><td>+</td></tr> <tr><td><math>a_{qq,1}</math></td><td><math>a_{qq,2}</math></td><td><math>a_{qq,3}</math></td><td>...</td><td><math>a_{qq,\nu_q}</math></td><td>+</td></tr> </table> | + |  |  |  |  | + | 0 | 0 | 0 | ... | 0 | + | 0 | 0 | 0 | ... | 0 | + | ⋮ | ⋮ | ⋮ | ... | ⋮ | + | 0 | 0 | 0 | ... | 0 | + | $a_{1q,1}$ | $a_{1q,2}$ | $a_{1q,3}$ | ... | $a_{1q,\nu_q}$ | + | 0 | 0 | 0 | ... | 0 | + | 0 | 0 | 0 | ... | 0 | + | ⋮ | ⋮ | ⋮ | ... | ⋮ | + | 0 | 0 | 0 | ... | 0 | + | $a_{2q,1}$ | $a_{2q,2}$ | $a_{2q,3}$ | ... | $a_{2q,\nu_q}$ | + | ⋮ | ⋮ | ⋮ | ... | ⋮ | + | 0 | 0 | 0 | ... | 0 | + | 0 | 0 | 0 | ... | 0 | + | ⋮ | ⋮ | ⋮ | ... | ⋮ | + | 0 | 0 | 0 | ... | 0 | + | $a_{q1,1}$ | $a_{q1,2}$ | $a_{q1,3}$ | ... | $a_{q1,\nu_1}$ | + | $a_{qq,1}$ | $a_{qq,2}$ | $a_{qq,3}$ | ... | $a_{qq,\nu_q}$ | + |
| 0   | 1          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 1          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 1              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{11,1}$  | $a_{11,2}$ | $a_{11,3}$ | ...     | $a_{11,\nu_1}$ |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{21,1}$  | $a_{21,2}$ | $a_{21,3}$ | ...     | $a_{21,\nu_1}$ |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{11,1}$  | $a_{11,2}$ | $a_{11,3}$ | ...     | $a_{11,\nu_1}$ |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{q1,1}$  | $a_{q1,2}$ | $a_{q1,3}$ | ...     | $a_{q1,\nu_1}$ |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{12,1}$  | $a_{12,2}$ | $a_{12,3}$ | ...     | $a_{12,\nu_2}$ |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 1          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 1          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 1              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{22,1}$  | $a_{22,2}$ | $a_{22,3}$ | ...     | $a_{22,\nu_2}$ |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{12,1}$  | $a_{12,2}$ | $a_{12,3}$ | ...     | $a_{12,\nu_2}$ |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{q2,1}$  | $a_{q2,2}$ | $a_{q2,3}$ | ...     | $a_{q2,\nu_2}$ |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{11,1}$  | $a_{11,2}$ | $a_{11,3}$ | ...     | $a_{11,\nu_1}$ |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{21,1}$  | $a_{21,2}$ | $a_{21,3}$ | ...     | $a_{21,\nu_1}$ |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 1          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 1          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 1              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{11,1}$  | $a_{11,2}$ | $a_{11,3}$ | ...     | $a_{11,\nu_1}$ |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{q1,1}$  | $a_{q1,2}$ | $a_{q1,3}$ | ...     | $a_{q1,\nu_1}$ |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{1q,1}$  | $a_{1q,2}$ | $a_{1q,3}$ | ...     | $a_{1q,\nu_q}$ |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{2q,1}$  | $a_{2q,2}$ | $a_{2q,3}$ | ...     | $a_{2q,\nu_q}$ |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{1q,1}$  | $a_{1q,2}$ | $a_{1q,3}$ | ...     | $a_{1q,\nu_q}$ |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 1          | 0          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 1          | ...     | 0              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 1              |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{qq,1}$  | $a_{qq,2}$ | $a_{qq,3}$ | ...     | $a_{qq,\nu_q}$ |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| +   |            |            |         |                | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{1q,1}$  | $a_{1q,2}$ | $a_{1q,3}$ | ...     | $a_{1q,\nu_q}$ | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{2q,1}$  | $a_{2q,2}$ | $a_{2q,3}$ | ...     | $a_{2q,\nu_q}$ | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{q1,1}$  | $a_{q1,2}$ | $a_{q1,3}$ | ...     | $a_{q1,\nu_1}$ | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{qq,1}$  | $a_{qq,2}$ | $a_{qq,3}$ | ...     | $a_{qq,\nu_q}$ | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| +   |            |            |         |                | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{1q,1}$  | $a_{1q,2}$ | $a_{1q,3}$ | ...     | $a_{1q,\nu_q}$ | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{2q,1}$  | $a_{2q,2}$ | $a_{2q,3}$ | ...     | $a_{2q,\nu_q}$ | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| ⋮   | ⋮          | ⋮          | ...     | ⋮              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| 0   | 0          | 0          | ...     | 0              | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{q1,1}$  | $a_{q1,2}$ | $a_{q1,3}$ | ...     | $a_{q1,\nu_1}$ | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |
| $a_{qq,1}$  | $a_{qq,2}$ | $a_{qq,3}$ | ...     | $a_{qq,\nu_q}$ | + |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |   |   |   |     |   |            |            |            |     |                |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |   |   |  |  |  |  |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |   |   |   |     |   |   |            |            |            |     |                |   |            |            |            |     |                |   |

4.7 System matrix A (pseudo-canonical form)

$$B = \begin{matrix} & + & & p & & + \\ \left[ \begin{array}{cccccc} b_{1,11} & b_{1,12} & b_{1,13} & \cdot & \cdot & \cdot & b_{1,1p} \\ b_{1,21} & b_{1,22} & b_{1,23} & \cdot & \cdot & \cdot & b_{1,2p} \\ \vdots & \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & \vdots & & & & \vdots \\ \hline b_{1,\mu_1 1} & b_{1,\mu_1 2} & b_{1,\mu_1 3} & \cdot & \cdot & \cdot & b_{1,\mu_1 p} \\ \hline b_{2,11} & b_{2,12} & b_{2,13} & \cdot & \cdot & \cdot & b_{2,1p} \\ b_{2,21} & b_{2,22} & b_{2,23} & \cdot & \cdot & \cdot & b_{2,2p} \\ \vdots & \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & \vdots & & & & \vdots \\ \hline b_{2,\mu_2 1} & b_{2,\mu_2 2} & b_{2,\mu_2 3} & \cdot & \cdot & \cdot & b_{2,\mu_2 p} \\ \hline \vdots & & & & & & \\ \hline b_{i,11} & b_{i,12} & b_{i,13} & \cdot & \cdot & \cdot & b_{i,1p} \\ b_{i,21} & b_{i,22} & b_{i,23} & \cdot & \cdot & \cdot & b_{i,2p} \\ \vdots & \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & \vdots & & & & \vdots \\ \hline b_{i,\mu_i 1} & b_{i,\mu_i 2} & b_{i,\mu_i 3} & \cdot & \cdot & \cdot & b_{i,\mu_i p} \\ \hline \vdots & & & & & & \\ \hline b_{q,11} & b_{q,12} & b_{q,13} & \cdot & \cdot & \cdot & b_{q,1p} \\ b_{q,21} & b_{q,22} & b_{q,23} & \cdot & \cdot & \cdot & b_{q,2p} \\ \vdots & \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & \vdots & & & & \vdots \\ \hline b_{q,\mu_q 1} & b_{q,\mu_q 2} & b_{q,\mu_q 3} & \cdot & \cdot & \cdot & b_{q,\mu_q p} \end{array} \right] & = & \begin{matrix} \left[ \begin{array}{c} b_{1,1}^T \\ b_{1,2}^T \\ \vdots \\ b_{1,\mu_1}^T \\ \hline b_{2,1}^T \\ b_{2,2}^T \\ \vdots \\ b_{2,\mu_2}^T \\ \hline \vdots \\ \hline b_{i,1}^T \\ b_{i,2}^T \\ \vdots \\ b_{i,\mu_i}^T \\ \hline \vdots \\ \hline b_{q,1}^T \\ b_{q,2}^T \\ \vdots \\ b_{q,\mu_q}^T \end{array} \right] & \begin{matrix} + \\ \mu_1 \\ + \\ \mu_2 \\ + \\ \mu_i \\ + \\ \mu_q \\ + \end{matrix} \end{matrix}
 \end{matrix}$$

4.8 Distribution matrix B

$$D = \begin{matrix} & + & & p & & + \\ \left[ \begin{array}{cccccc} d_{11} & d_{12} & d_{13} & \cdot & \cdot & \cdot & d_{1p} \\ d_{21} & d_{22} & d_{23} & \cdot & \cdot & \cdot & d_{2p} \\ \vdots & \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & \vdots & & & & \vdots \\ \hline d_{q1} & d_{q2} & d_{q3} & \cdot & \cdot & \cdot & d_{qp} \end{array} \right] & = & \begin{matrix} \left[ \begin{array}{c} d_1^T \\ d_2^T \\ \vdots \\ d_q^T \end{array} \right] & \begin{matrix} + \\ q \\ + \end{matrix} \end{matrix}
 \end{matrix}$$

4.9 Input/Output matrix D

If we consider the  $i$ -th subsystem (MISO-subsystem corresponding to output  $i$ ) then:

$$\begin{aligned}
 y_i(k) &= x_{i,1}(k) + \underline{d}_i^T \underline{u}(k) \\
 x_{i,1}(k+1) &= x_{i,2}(k) + \underline{b}_{i,1}^T \underline{u}(k) \\
 x_{i,2}(k+1) &= x_{i,3}(k) + \underline{b}_{i,2}^T \underline{u}(k) \\
 &\vdots \\
 x_{i,j}(k+1) &= x_{i,j+1}(k) + \underline{b}_{i,j}^T \underline{u}(k) \\
 &\vdots \\
 x_{i,\mu_i-1}(k+1) &= x_{i,\mu_i}(k) + \underline{b}_{i,\mu_i-1}^T \underline{u}(k)
 \end{aligned} \tag{4.10a}$$

$$x_{i,\mu_i}(k+1) = \underline{\alpha}_i x(k) + \underline{b}_{i,\mu_i}^T \underline{u}(k) \tag{4.10b}$$

If we use the delay operator  $z^{-1}$  ( $x(k-1) = z^{-1}x(k)$ ), we can derive from equations (4.10a):

$$\begin{aligned}
 x_{i,\mu_i}(k) &= z^{\mu_i-1} y_i(k) - z^{\mu_i-2} \underline{b}_{i,1}^T \underline{u}(k) - \dots - \underline{b}_{i,\mu_i-1}^T \underline{u}(k) \\
 &\quad - z^{\mu_i-1} \underline{d}_i \underline{u}(k)
 \end{aligned} \tag{4.11}$$

The  $\mu_i$  elements of the  $i$ -th block of the state space vector have now been written as a function of the  $i$ -th output  $y_i(k)$ , the input vector  $\underline{u}(k)$ , the  $b$ -parameters of the  $i$ -th bloc of matrix  $B$  and the  $d$ -parameters of the the  $i$ -th row of matrix  $D$ . From (4.10) and (4.11) we obtain a general form for the state space vector :



$$\underline{x}(k) = V(z) \underline{y}(k) - X Z(z) \underline{u}(k) \tag{4.12}$$

where  $V(z)$ ,  $Z(z)$  and  $X$  are given by resp. (4.13), (4.14) and (4.15).

$$V(z) = \begin{bmatrix} \begin{matrix} + & & & & + \\ 1 & 0 & 0 & \dots & 0 \\ z & 0 & 0 & \dots & 0 \\ z^2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z^{\mu_1-1} & 0 & 0 & \dots & 0 \\ \hline 0 & 1 & 0 & \dots & 0 \\ 0 & z & 0 & \dots & 0 \\ 0 & z^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & z^{\mu_2-1} & 0 & \dots & 0 \\ \hline \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 \\ 0 & 0 & \dots & z & \dots & 0 \\ 0 & 0 & \dots & z^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & z^{\mu_1-1} & \dots & 0 \\ \hline \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & z \\ 0 & 0 & 0 & \dots & z^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & z^{\mu_q-1} \end{matrix} \end{bmatrix} \begin{matrix} + \\ \mu_1 \\ + \\ \mu_2 \\ + \\ \mu_1 \\ + \\ \mu_q \\ + \end{matrix}$$

4.14 Matrix Z(z)

4.13 Matrix V(z)

$$Z(z) = \begin{bmatrix} \begin{matrix} + & & & & + \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \hline z & 0 & 0 & \dots & 0 \\ 0 & z & 0 & \dots & 0 \\ 0 & 0 & z & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & z \\ \hline \vdots \\ z^{i-1} & 0 & 0 & \dots & 0 \\ 0 & z^{i-1} & 0 & \dots & 0 \\ 0 & 0 & z^{i-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & z^{i-1} \\ \hline \vdots \\ z^{\mu_m-2} & 0 & 0 & \dots & 0 \\ 0 & z^{\mu_m-2} & 0 & \dots & 0 \\ 0 & 0 & z^{\mu_m-2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & z^{\mu_m-2} \\ \hline z^{\mu_m-1} & 0 & 0 & \dots & 0 \\ 0 & z^{\mu_m-1} & 0 & \dots & 0 \\ 0 & 0 & z^{\mu_m-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & z^{\mu_m-1} \end{matrix} \end{bmatrix} \begin{matrix} + \\ + \\ P \\ + \\ + \\ P \\ + \\ + \\ P \\ + \\ + \\ P \\ + \\ + \\ P \\ + \\ + \end{matrix}$$

x =

|                    |                    |         |                    |                    |                    |         |                    |         |          |          |         |          |   |         |         |   |
|--------------------|--------------------|---------|--------------------|--------------------|--------------------|---------|--------------------|---------|----------|----------|---------|----------|---|---------|---------|---|
| $d_{11}$           | $d_{12}$           | $\dots$ | $d_{1p}$           | 0                  | 0                  | $\dots$ | 0                  | $\dots$ | 0        | 0        | $\dots$ | 0        | 0 | 0       | $\dots$ | 0 |
| $b_{1,11}$         | $b_{1,12}$         | $\dots$ | $b_{1,1p}$         | $d_{11}$           | $d_{12}$           | $\dots$ | $d_{1p}$           | $\dots$ | 0        | 0        | $\dots$ | 0        | 0 | $\dots$ | 0       | 0 |
| $b_{1,21}$         | $b_{1,22}$         | $\dots$ | $b_{1,2p}$         | $b_{1,11}$         | $b_{1,12}$         | $\dots$ | $b_{1,1p}$         | $\dots$ | :        | :        | $\dots$ | :        | : | $\dots$ | :       | : |
| :                  | :                  | :       | :                  | :                  | :                  | :       | :                  | :       | :        | :        | :       | :        | : | :       | :       | : |
| $b_{1,(\mu_1-1)1}$ | $b_{1,(\mu_1-1)2}$ | $\dots$ | $b_{1,(\mu_1-1)p}$ | $b_{1,(\mu_1-2)1}$ | $b_{1,(\mu_1-2)2}$ | $\dots$ | $b_{1,(\mu_1-2)p}$ | $\dots$ | $d_{11}$ | $d_{12}$ | $\dots$ | $d_{1p}$ | 0 | 0       | $\dots$ | 0 |
| :                  |                    |         |                    |                    |                    |         |                    |         |          |          |         |          |   |         |         |   |
| $d_{21}$           | $d_{22}$           | $\dots$ | $d_{2p}$           | 0                  | 0                  | $\dots$ | 0                  | $\dots$ | 0        | 0        | $\dots$ | 0        | 0 | 0       | $\dots$ | 0 |
| $b_{2,11}$         | $b_{2,12}$         | $\dots$ | $b_{2,1p}$         | $d_{21}$           | $d_{22}$           | $\dots$ | $d_{2p}$           | $\dots$ | 0        | 0        | $\dots$ | 0        | 0 | $\dots$ | 0       | 0 |
| $b_{2,21}$         | $b_{2,22}$         | $\dots$ | $b_{2,2p}$         | $b_{2,11}$         | $b_{2,12}$         | $\dots$ | $b_{2,1p}$         | $\dots$ | :        | :        | $\dots$ | :        | : | $\dots$ | :       | : |
| :                  | :                  | :       | :                  | :                  | :                  | :       | :                  | :       | :        | :        | :       | :        | : | :       | :       | : |
| $b_{2,(\mu_2-1)1}$ | $b_{2,(\mu_2-1)2}$ | $\dots$ | $b_{2,(\mu_2-1)p}$ | $b_{2,(\mu_2-2)1}$ | $b_{2,(\mu_2-2)2}$ | $\dots$ | $b_{2,(\mu_2-2)p}$ | $\dots$ | $d_{21}$ | $d_{22}$ | $\dots$ | $d_{2p}$ | 0 | 0       | $\dots$ | 0 |
| :                  |                    |         |                    |                    |                    |         |                    |         |          |          |         |          |   |         |         |   |
| $d_{11}$           | $d_{12}$           | $\dots$ | $d_{1p}$           | 0                  | 0                  | $\dots$ | 0                  | $\dots$ | 0        | 0        | $\dots$ | 0        | 0 | 0       | $\dots$ | 0 |
| $b_{1,11}$         | $b_{1,12}$         | $\dots$ | $b_{1,1p}$         | $d_{11}$           | $d_{12}$           | $\dots$ | $d_{1p}$           | $\dots$ | 0        | 0        | $\dots$ | 0        | 0 | $\dots$ | 0       | 0 |
| $b_{1,21}$         | $b_{1,22}$         | $\dots$ | $b_{1,2p}$         | $b_{1,11}$         | $b_{1,12}$         | $\dots$ | $b_{1,1p}$         | $\dots$ | :        | :        | $\dots$ | :        | : | $\dots$ | :       | : |
| :                  | :                  | :       | :                  | :                  | :                  | :       | :                  | :       | :        | :        | :       | :        | : | :       | :       | : |
| $b_{1,(\mu_1-1)1}$ | $b_{1,(\mu_1-1)2}$ | $\dots$ | $b_{1,(\mu_1-1)p}$ | $b_{1,(\mu_1-2)1}$ | $b_{1,(\mu_1-2)2}$ | $\dots$ | $b_{1,(\mu_1-2)p}$ | $\dots$ | $d_{11}$ | $d_{12}$ | $\dots$ | $d_{1p}$ | 0 | 0       | $\dots$ | 0 |
| :                  |                    |         |                    |                    |                    |         |                    |         |          |          |         |          |   |         |         |   |
| $d_{q1}$           | $d_{q2}$           | $\dots$ | $d_{qp}$           | 0                  | 0                  | $\dots$ | 0                  | $\dots$ | 0        | 0        | $\dots$ | 0        | 0 | 0       | $\dots$ | 0 |
| $b_{q,11}$         | $b_{q,12}$         | $\dots$ | $b_{q,1p}$         | $d_{q1}$           | $d_{q2}$           | $\dots$ | $d_{qp}$           | $\dots$ | 0        | 0        | $\dots$ | 0        | 0 | $\dots$ | 0       | 0 |
| $b_{q,21}$         | $b_{q,22}$         | $\dots$ | $b_{q,2p}$         | $b_{q,11}$         | $b_{q,12}$         | $\dots$ | $b_{q,1p}$         | $\dots$ | :        | :        | $\dots$ | :        | : | $\dots$ | :       | : |
| :                  | :                  | :       | :                  | :                  | :                  | :       | :                  | :       | :        | :        | :       | :        | : | :       | :       | : |
| $b_{q,(\mu_q-1)1}$ | $b_{q,(\mu_q-1)2}$ | $\dots$ | $b_{q,(\mu_q-1)p}$ | $b_{q,(\mu_q-2)1}$ | $b_{q,(\mu_q-2)2}$ | $\dots$ | $b_{q,(\mu_q-2)p}$ | $\dots$ | $d_{q1}$ | $d_{q2}$ | $\dots$ | $d_{qp}$ | 0 | 0       | $\dots$ | 0 |

4.15 Matrix X

The dimensions of these matrices are :

$$\begin{aligned} \text{matrix } V(z) &: n \times q \\ \text{matrix } Z(z) &: p_{\mu_m} \times p \\ \text{matrix } X &: n \times p_{\mu_m} \end{aligned}$$

$$\text{where } \mu_m = \max (\mu_1, \mu_2, \dots, \mu_q)$$

But we still do not have a relation linking the input and output of the system. This can be obtained by combining equations (4.2) and (4.12):

$$\left\{ \begin{array}{l} (zI - A) \underline{x}(k) = B \underline{u}(k) \\ \underline{x}(k) = V(z) \underline{y}(k) - X Z(z) \underline{u}(k) \end{array} \right. \quad (4.16)$$

which leads to the following input/output relation:

$$(zI - A) V(z) \underline{y}(k) = [(zI - A) X Z(z) + B] \underline{u}(k) \quad (4.17)$$

This matrix equation consists of  $n$  equations but only the  $q$  equations  $\mu_1, \mu_1 + \mu_2, \dots, \mu_1 + \mu_2 + \dots + \mu_q (=n)$  are relevant. The other  $(n-q)$  equations are identities.

Now the  $q$  relevant equations of (4.17) can be written in the form of an MFD input/output representation :

$$P(z) \underline{y}(k) = Q(z) \underline{u}(k) \quad (4.18)$$

with:

$$P(z) = \begin{bmatrix} p_{11}(z) & \dots & p_{1q}(z) \\ \vdots & \vdots & \vdots \\ p_{q1}(z) & \dots & p_{qq}(z) \end{bmatrix}$$

$$Q(z) = \begin{bmatrix} q_{11}(z) & \dots & q_{1p}(z) \\ \vdots & \vdots & \vdots \\ q_{q1}(z) & \dots & q_{qp}(z) \end{bmatrix}$$

where  $p_{ij}(z)$   $i=1,q$ ,  $j=1,q$  and  $q_{ij}(z)$   $i=1,q$ ,  $j=1,p$  are polynomials in  $z$ . To determine the degree and the coefficients of all these polynomials we will have to write explicitly equation (4.14) and identify it with equation (4.18).

Knowing that the  $i$ -th equation of (4.18) corresponds to the  $(\mu_1 + \mu_2 + \dots + \mu_i)$  equation of (4.17) we can write the following two equalities:

$$p_{i1}(z)y_1 + p_{i2}(z)y_2 + \dots + p_{iq}(z)y_q = \quad (4.19)$$

$$(zI-A) \Big|_{\text{row}(\mu_1+\mu_2+\dots+\mu_i)} V(z)\underline{y}(k)$$

$$q_{k1}(z)u_1 + q_{k2}(z)u_2 + \dots + q_{kp}(z)u_p = \quad (4.20)$$

$$\{(zI-A) \Big|_{\text{row}(\mu_1+\mu_2+\dots+\mu_k)} XZ(z)+B \Big|_{\text{row}(\mu_1+\mu_2+\dots+\mu_k)}\} \underline{u}(k)$$

If we consider the right hand side of equation (4.19) we find :

$$(zI-A) \Big|_{\text{row}(\mu_1+\mu_2+\dots+\mu_i)} V(z)\underline{y}(k) = \quad (4.21)$$

$$\left\{ \begin{aligned} & (-\alpha_{i1,1} - z\alpha_{i1,2} - z^2\alpha_{i1,3} - \dots - z^{\mu_1-1}\alpha_{i1,\mu_1})y_1(k) + \\ & (-\alpha_{i2,1} - z\alpha_{i2,2} - z^2\alpha_{i2,3} - \dots - z^{\mu_2-1}\alpha_{i2,\mu_2})y_2(k) + \\ & \vdots \\ & (-\alpha_{ii,1} - z\alpha_{ii,2} - z^2\alpha_{ii,3} - \dots - z^{\mu_i-1}\alpha_{ii,\mu_i} + z^{\mu_i})y_i(k) + \\ & \vdots \\ & (-\alpha_{iq,1} - z\alpha_{iq,2} - z^2\alpha_{iq,3} - \dots - z^{\mu_q-1}\alpha_{iq,\mu_q})y_q(k) \end{aligned} \right\}$$

Identifying (4.21) with the left hand side of equation (4.19) we can

obtain the  $p_{ij}(z)$  polynomials:

$$\begin{aligned}
 p_{ii}(z) &= z^{\mu_i} - \alpha_{ii,\mu_i} z^{\mu_i-1} - \dots - \alpha_{ii,2} z - \alpha_{ii,1} \\
 p_{ij}(z) &= -\alpha_{ij,\mu_j} z^{\mu_j-1} - \dots - \alpha_{ij,2} z - \alpha_{ij,1} \\
 & \quad i \neq j
 \end{aligned} \tag{4.22}$$

The order of the  $p_{ij}(z)$  polynomials is given by the Kronecker numbers  $\mu_i$ ,  $i=1,q$  and the parameters can be obtained directly from the entries of the system matrix A.

A more complex expression is the right hand side of equation (4.20) which will give us the  $q_{ij}(z)$  polynomials .

Lets define:

$$\Gamma = XZ(z) = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & \dots & Y_{1p} \\ Y_{21} & Y_{22} & Y_{23} & \dots & Y_{2p} \\ Y_{31} & Y_{32} & Y_{33} & \dots & Y_{3p} \\ \vdots & \vdots & \vdots & & \vdots \\ Y_{q1} & Y_{q2} & Y_{q3} & \dots & Y_{qp} \end{bmatrix} \tag{4.23}$$

with:

$$Y_{ij} = \begin{bmatrix} d_{ij} \\ b_{i,1j} + z d_{ij} \\ b_{i,2j} + z b_{i,1j} + z^2 d_{ij} \\ \vdots \\ b_{i,(\mu_i-1)j} + z b_{i,(\mu_i-2)j} + \dots + z^{\mu_i-2} b_{i,1j} + z^{\mu_i-1} d_{ij} \end{bmatrix} \tag{4.24}$$

The dimension of:  $\Gamma = [n \times p(\mu_m)] \times [p(\mu_m) \times p] = n \times p$

$$Y_{ij} = \mu_i \times 1$$

If we only consider the input 1 in equation (4.17), we found:

$$q_{k\ell}(z)u_\ell = \{(zI-A) \Big|_{\text{row}(\mu_1+\mu_2+\dots+\mu_k)} \Gamma \Big|_{\text{column } \ell} + b_{k,\mu_k \ell}\} u_\ell \quad (4.25)$$

If we develop this expression and rearrange it to the powers of  $z$ , we obtain the polynomial  $q_{k\ell}(z)$

$$\begin{aligned} q_{k\ell}(z) = & \\ & -\alpha_{k1,2} b_{1,1\ell} -\alpha_{k1,3} b_{1,2\ell} -\dots -\alpha_{k1,\mu_1} b_{1,(\mu_1-1)\ell} -\alpha_{k1,1} d_{1\ell} \\ & -\alpha_{k2,2} b_{2,1\ell} -\alpha_{k2,3} b_{2,2\ell} -\dots -\alpha_{k2,\mu_2} b_{2,(\mu_2-1)\ell} -\alpha_{k2,1} d_{2\ell} \\ & \quad : \\ & -\alpha_{kk,2} b_{k,1\ell} -\alpha_{kk,3} b_{k,2\ell} -\dots -\alpha_{kk,\mu_k} b_{k,(\mu_k-1)\ell} + b_{k,\mu_k \ell} -\alpha_{kk,1} d_{k\ell} \\ & \quad : \\ & -\alpha_{kq,2} b_{q,1\ell} -\alpha_{kq,3} b_{q,2\ell} -\dots -\alpha_{kq,\mu_q} b_{q,(\mu_q-1)\ell} -\alpha_{kq,1} d_{q\ell} \\ +z \{ & -\alpha_{k1,3} b_{1,1\ell} -\alpha_{k1,4} b_{1,2\ell} -\dots -\alpha_{k1,\mu_1} b_{1,(\mu_1-2)\ell} -\alpha_{k1,2} d_{1\ell} \\ & -\alpha_{k2,3} b_{2,1\ell} -\alpha_{k2,4} b_{2,2\ell} -\dots -\alpha_{k2,\mu_2} b_{2,(\mu_2-2)\ell} -\alpha_{k2,2} d_{2\ell} \\ & \quad : \\ & -\alpha_{kk,3} b_{k,1\ell} -\dots -\alpha_{kk,\mu_k} b_{k,(\mu_k-2)\ell} + b_{k,(\mu_k-1)\ell} -\alpha_{kk,2} d_{k\ell} \\ & \quad : \\ & -\alpha_{kq,3} b_{q,1\ell} -\alpha_{kq,4} b_{q,2\ell} -\dots -\alpha_{kq,\mu_q} b_{q,(\mu_q-2)\ell} -\alpha_{kq,2} d_{q\ell} \} \end{aligned}$$

$$\begin{aligned}
& z^2 \{ -\alpha_{k1,4} b_{1,1\ell} - \alpha_{k1,5} b_{1,2\ell} - \dots - \alpha_{k1,\mu} b_{1,(\mu-3)\ell} - \alpha_{k1,3} d_{1\ell} \\
& \quad - \alpha_{k2,4} b_{2,1\ell} - \alpha_{k2,5} b_{2,2\ell} - \dots - \alpha_{k2,\mu_2} b_{2,(\mu_2-3)\ell} - \alpha_{k2,3} d_{2\ell} \\
& \quad \vdots \\
& \quad - \alpha_{kk,4} b_{k,1\ell} - \dots - \alpha_{kk,\mu_k} b_{k,(\mu_k-3)\ell} + b_{k,(\mu_k-2)\ell} - \alpha_{kk,3} d_{k\ell} \\
& \quad \vdots \\
& \quad - \alpha_{kq,4} b_{q,1\ell} - \alpha_{kq,5} b_{q,2\ell} - \dots - \alpha_{kq,\mu_q} b_{q,(\mu_q-3)\ell} - \alpha_{kq,3} d_{q\ell} \\
& \quad \vdots \\
& + z^{\mu_1-2} \{ -\alpha_{k1,\mu_1} b_{1,1\ell} - \alpha_{k1,\mu_1-1} d_{1\ell} \} \\
& \quad \vdots \\
& + z^{\mu_2-2} \{ -\alpha_{k2,\mu_2} b_{2,1\ell} - \alpha_{k2,\mu_2-1} d_{2\ell} \} \\
& \quad \vdots \\
& + z^{\mu_k-2} \{ -\alpha_{kk,\mu_k} b_{k,1\ell} + b_{k,2\ell} - \alpha_{kk,\mu_k-1} d_{k\ell} \} \\
& \quad \vdots \\
& + z^{\mu_q-2} \{ -\alpha_{kq,\mu_q} b_{q,1\ell} - \alpha_{kq,\mu_q-1} d_{q\ell} \} \\
& + z^{\mu_1-1} \{ \quad \quad \quad -\alpha_{k1,\mu_2} d_{1\ell} \} \\
& + z^{\mu_2-1} \{ \quad \quad \quad -\alpha_{k2,\mu_2} d_{2\ell} \} \\
& \quad \vdots \\
& + z^{\mu_k-1} \{ \quad b_{k,1\ell} \quad -\alpha_{kk,\mu} d_{k\ell} \quad + z d_{k\ell} \} \\
& \quad \vdots \\
& + z^{\mu_q-1} \{ \quad \quad \quad -\alpha_{kq,\mu_q} d_{q\ell} \}
\end{aligned} \tag{4.26}$$

By inspecting (4.26) we can make some general remarks about the  $q_{k\ell}$  polynomials:

- all  $q_{k\ell}(z)$ ,  $\ell=1,p$  have the same degree.

- the degree of the  $q_{k\ell}(z)$  polynomials are completely determined by the structural indices  $\mu_1, \mu_2, \dots, \mu_q$ .

If  $\mu_m = \max \{\mu_1, \mu_2, \dots, \mu_q\}$  then the degree of polynomial  $q_{k\ell}(z)$  is given by :

$$\text{degree} \{q_{k\ell}(z)\} = \begin{cases} \mu_m & \text{if } \mu_k = \mu_m \\ \mu_m - 1 & \text{if } \mu_k < \mu_m \end{cases} \quad (4.27)$$

The  $q_{k\ell}(z)$  polynomials can be written as follows:

$$q_{k\ell}(z) = \beta_{k, \mu_m} \ell^z \mu_m^{-1} + \beta_{k, (\mu_m - 1)} \ell^z \mu_m^{-2} + \beta_{k, (\mu_m - 2)} \ell^z \mu_m^{-3} + \dots + \beta_{k, 1} \ell^z \mu_m^{-\mu_m} \quad (4.28)$$

$\mu_k < \mu_m$

$$q_{k\ell}(z) = \beta_{k, (\mu_m + 1)} \ell^z \mu_m + \beta_{k, \mu_m} \ell^z \mu_m^{-1} + \beta_{k, (\mu_m - 1)} \ell^z \mu_m^{-2} + \dots + \beta_{k, 1} \ell^z \mu_m^{-\mu_m} \quad (4.28)$$

$\mu_k = \mu_m$

The coefficients  $\beta$  are then obtained by the following matrix multiplication:

$$\bar{B} = M\Phi \quad (4.29)$$

The dimensions are: matrix  $\bar{B}$  :  $q(\mu_m + 1) \times n$

matrix  $M$  :  $q(\mu_m + 1) \times (n+q)$

matrix  $\Phi$  :  $(n+p) \times p$

The matrices  $M$ ,  $\bar{B}$  and  $\Phi$  are written in (4.30), (4.31) and (4.32).

The coefficients  $\beta_{k, (\mu_m + 1)} \ell^z \mu_m$  are zero unless  $\mu_k = \mu_m$ . In the other cases this parameter will be set to zero and will not be identified.





$$\begin{array}{c}
 \begin{array}{ccccccc}
 & + & & p & & & + \\
 \beta_{1,11} & \beta_{1,12} & \beta_{1,13} & \cdot & \cdot & \cdot & \beta_{1,1p} \\
 \beta_{1,21} & \beta_{1,22} & \beta_{1,23} & \cdot & \cdot & \cdot & \beta_{1,2p} \\
 \vdots & \vdots & \vdots & & & & \vdots \\
 \vdots & \vdots & \vdots & & & & \vdots \\
 \beta_{1,\mu_m 1} & \beta_{1,\mu_m 2} & \beta_{1,\mu_m 3} & \cdot & \cdot & \cdot & \beta_{1,\mu_m p} \\
 [ \beta_{1,(\mu_m+1)1} & \beta_{1,(\mu_m+1)2} & \beta_{1,(\mu_m+1)3} & \cdot & \cdot & \cdot & \beta_{1,(\mu_m+1)p} ] \\
 \hline
 \beta_{2,11} & \beta_{2,12} & \beta_{2,13} & \cdot & \cdot & \cdot & \beta_{2,1p} \\
 \beta_{2,21} & \beta_{2,22} & \beta_{2,23} & \cdot & \cdot & \cdot & \beta_{2,2p} \\
 \vdots & \vdots & \vdots & & & & \vdots \\
 \vdots & \vdots & \vdots & & & & \vdots \\
 \beta_{2,\mu_m 1} & \beta_{2,\mu_m 2} & \beta_{2,\mu_m 3} & \cdot & \cdot & \cdot & \beta_{2,\mu_m p} \\
 [ \beta_{2,(\mu_m+1)1} & \beta_{2,(\mu_m+1)2} & \beta_{2,(\mu_m+1)3} & \cdot & \cdot & \cdot & \beta_{2,(\mu_m+1)p} ] \\
 \hline
 \vdots & & & & & & \\
 \vdots & & & & & & \\
 \beta_{i,11} & \beta_{i,12} & \beta_{i,13} & \cdot & \cdot & \cdot & \beta_{i,1p} \\
 \beta_{i,21} & \beta_{i,22} & \beta_{i,23} & \cdot & \cdot & \cdot & \beta_{i,2p} \\
 \vdots & \vdots & \vdots & & & & \vdots \\
 \vdots & \vdots & \vdots & & & & \vdots \\
 \beta_{i,\mu_m 1} & \beta_{i,\mu_m 2} & \beta_{i,\mu_m 3} & \cdot & \cdot & \cdot & \beta_{i,\mu_m p} \\
 [ \beta_{i,(\mu_m+1)1} & \beta_{i,(\mu_m+1)2} & \beta_{i,(\mu_m+1)3} & \cdot & \cdot & \cdot & \beta_{i,(\mu_m+1)p} ] \\
 \hline
 \vdots & & & & & & \\
 \vdots & & & & & & \\
 \beta_{q,11} & \beta_{q,12} & \beta_{q,13} & \cdot & \cdot & \cdot & \beta_{q,1p} \\
 \beta_{q,21} & \beta_{q,22} & \beta_{q,23} & \cdot & \cdot & \cdot & \beta_{q,2p} \\
 \vdots & \vdots & \vdots & & & & \vdots \\
 \vdots & \vdots & \vdots & & & & \vdots \\
 \beta_{q,\mu_m 1} & \beta_{q,\mu_m 2} & \beta_{q,\mu_m 3} & \cdot & \cdot & \cdot & \beta_{q,\mu_m p} \\
 [ \beta_{2,(\mu_m+1)1} & \beta_{2,(\mu_m+1)2} & \beta_{2,(\mu_m+1)3} & \cdot & \cdot & \cdot & \beta_{2,(\mu_m+1)p} ] \\
 \hline
 \end{array} \\
 \end{array}
 \begin{array}{l}
 + \\
 \mu_m \\
 [\mu_m+1] \\
 + \\
 + \\
 \mu_m \\
 [\mu_m+1] \\
 + \\
 + \\
 \mu_m \\
 [\mu_m+1] \\
 + \\
 + \\
 \mu_m \\
 [\mu_m+1] \\
 +
 \end{array}
 \end{array}$$

4.31 Matrix B (pseudo-canonical form)

$$\Phi = \begin{array}{c}
 \begin{array}{cccccc}
 & + & & & & p \\
 d_{11} & d_{12} & d_{13} & \dots & \dots & d_{1p} \\
 b_{1,11} & b_{1,12} & b_{1,13} & \dots & \dots & b_{1,1p} \\
 b_{1,21} & b_{1,22} & b_{1,23} & \dots & \dots & b_{1,2p} \\
 \vdots & \vdots & \vdots & & & \vdots \\
 \vdots & \vdots & \vdots & & & \vdots \\
 b_{1,\mu_1 1} & b_{1,\mu_1 2} & b_{1,\mu_1 3} & \dots & \dots & b_{1,\mu_1 p} \\
 \hline
 d_{21} & d_{22} & d_{23} & \dots & \dots & d_{2p} \\
 b_{2,11} & b_{2,12} & b_{2,13} & \dots & \dots & b_{2,1p} \\
 b_{2,21} & b_{2,22} & b_{2,23} & \dots & \dots & b_{2,2p} \\
 \vdots & \vdots & \vdots & & & \vdots \\
 \vdots & \vdots & \vdots & & & \vdots \\
 b_{2,\mu_2 1} & b_{2,\mu_2 2} & b_{2,\mu_2 3} & \dots & \dots & b_{2,\mu_2 p} \\
 \hline
 \vdots & & & & & \\
 \hline
 d_{i1} & d_{i2} & d_{i3} & \dots & \dots & d_{ip} \\
 b_{i,11} & b_{i,12} & b_{i,13} & \dots & \dots & b_{i,1p} \\
 b_{i,21} & b_{i,22} & b_{i,23} & \dots & \dots & b_{i,2p} \\
 \vdots & \vdots & \vdots & & & \vdots \\
 \vdots & \vdots & \vdots & & & \vdots \\
 b_{i,\mu_i 1} & b_{i,\mu_i 2} & b_{i,\mu_i 3} & \dots & \dots & b_{i,\mu_i p} \\
 \hline
 \vdots & & & & & \\
 \hline
 d_{q1} & d_{q2} & d_{q3} & \dots & \dots & d_{qp} \\
 b_{q,11} & b_{q,12} & b_{q,13} & \dots & \dots & b_{q,1p} \\
 b_{q,21} & b_{q,22} & b_{q,23} & \dots & \dots & b_{q,2p} \\
 \vdots & \vdots & \vdots & & & \vdots \\
 \vdots & \vdots & \vdots & & & \vdots \\
 b_{q,\mu_q 1} & b_{q,\mu_q 2} & b_{q,\mu_q 3} & \dots & \dots & b_{q,\mu_q p}
 \end{array}
 \end{array}
 \begin{array}{l}
 + \\
 \mu_1 + 1 \\
 + \\
 + \\
 \mu_2 + \\
 + \\
 \vdots \\
 + \\
 \mu_i + 1 \\
 + \\
 \vdots \\
 + \\
 \mu_q + 1 \\
 +
 \end{array}$$

4.32 Matrix  $\Phi$

In the obtained input/output pair  $(P(z), Q(z))$  the relations among the degrees of the entries of  $P(z)$  and  $Q(z)$  are given by:

$$\begin{aligned} \deg \{ p_{ii}(z) \} &> \deg \{ p_{ji}(z) \} \quad i \neq j \\ \deg \{ q_{ij}(z) \} &= \rho_i < \mu_m \end{aligned} \quad (4.33)$$

where  $\rho_i$  denotes the degree of the  $i$ -th row of  $P(z)$ .

It is important to note that the row degree,  $\rho_i$ , of a row of  $P(z)$  can exceed the degree  $\mu_i = \deg \{ p_{ii}(z) \}$  of the monic polynomial  $p_{ii}(z)$ . The number of significant entries in  $A$  is equal of the number of significant coefficients in the entries of  $P(z)$ . The number of significant coefficients in the entries of  $Q(z)$  can exceed the number of entries of  $B$  plus the number of entries of  $D$ . The input/output pair  $(P(z), Q(z))$  is thus, in general, described by a number of parameters larger than the number of significant entries of  $\{A, B, C, D\}$ .

Since the row degree can be larger than the degree of the monic polynomial the obtained input/output multistructural model is, apparently, anticipatory. This can be seen if we consider the  $i$ -th equation of (4.18):

$$\sum_{j=1}^q p_{ij}(z) y_j(k) = \sum_{j=1}^p q_{ij} u_j(k) \quad (4.34)$$

so that:

$$p_{ii}(z) y_i(k) = - \sum_{\substack{j=1 \\ j \neq i}}^q p_{ij}(z) y_j(k) + \sum_{j=1}^p q_{ij}(z) u_j(k)$$

and

$$y_i(k) = z^{-\mu_i} \left[ - \sum_{\substack{j=1 \\ j \neq i}}^q p_{ij}(z) y_j(k) - p'_{ii}(z) y_i(k) + \sum_{j=1}^p q_{ij}(z) u_j(k) \right] \quad (4.35)$$

$$\text{with } p_{ii}'(z) = p_{ii}(z) - z^{\mu_i}$$

For each power of  $z$  within the  $p_{ij}(z)$  and  $q_{ij}(z)$  polynomials greater than  $\mu_i$  an anticipatory output or input will appear in the expression of  $y_i(k)$ . Anticipatory samples from output  $j$  will appear on output  $i$  if the difference  $(\mu_j - \mu_i)$  is more or equal than 2. Anticipatory samples from inputs will appear on output  $i$  as soon as  $(\mu_m - \mu_i)$  is more than or equal to 2.

Physically the I/O model, which is strictly equivalent to the causal state space model, is not anticipatory. The apparently excessive degrees in the entries of  $Q(z)$  are requested to balance the anticipatory information flowing from the off-diagonal polynomials with degree greater than  $\mu_i$ . In the state space model this information can be buffered into the state space vector so that no anticipatory effects appear in this representation.

#### 4.3 From canonical form to I/O model

As it has been remarked in section 3.4 the main difference between the canonical and the pseudo-canonical form resides in the supplementary structural zero's that can appear in the system matrix  $A$  of the canonical form (see 4.42). However, the equation (4.14) is still valid:

$$(zI - A) V(z) \underline{y}(k) = [(zI - A) X Z(z) + B] \underline{u}(k) \quad (4.36)$$

The same calculations as in the previous paragraph can be applied with the difference that matrix  $A$  has now some parameters equal to zero. These calculations lead to the following equivalent I/O model:

$$P(z) \underline{y}(k) = Q(z) \underline{u}(k) \quad (4.37)$$

$P(z)$  and  $Q(z)$  defined as in (4.15) with :

$$p_{ii}(z) = z^{v_i} - \alpha_{ii,v_i} z^{v_i-1} - \dots - \alpha_{ii,2} z - \alpha_{ii,1} \quad (4.38)$$

$$p_{ij}(z) = -\alpha_{ij,v_{ij}} z^{v_{ij}-1} - \alpha_{ij,v_{ij}-1} z^{v_{ij}-2} - \dots - \alpha_{ij,2} z - \alpha_{ij,1} \quad (4.39)$$

$i \neq j$

$$\begin{aligned} \text{with : } v_{ij} &= v_i && \text{for } i=j \\ v_{ij} &= \min(v_i+1, v_j) && \text{for } i>j \\ v_{ij} &= \min(v_i, v_j) && \text{for } i<j \end{aligned}$$

The entries of the  $q_{ij}(z)$  polynomials are:

$$q_{ij}(z) = \beta_{i,(v_i+1)j} z^{v_i} + \beta_{i,v_{ij}j} z^{v_i-1} + \dots + \beta_{i,2j} z + \beta_{i,1j} \quad (4.39)$$

From (4.38) and (4.39) it follows that the degrees of the polynomials of  $P(z)$  and  $Q(z)$  satisfy the conditions:

$$\begin{aligned} \deg\{p_{ii}(z)\} &> \deg\{p_{ij}(z)\} && \text{if } i>j \\ \deg\{p_{ii}(z)\} &> \deg\{p_{ij}(z)\} && \text{if } i<j \\ \deg\{p_{ii}(z)\} &> \deg\{p_{ji}(z)\} && \text{if } i \neq j \\ \deg\{p_{ii}(z)\} &> \deg\{q_{ij}(z)\} && \end{aligned} \quad (4.40)$$

An important consequence of (4.40) is that the I/O model equivalent to the canonical form will satisfy the causality principle ( each output is a linear combination of previous and simultaneous inputs and outputs). This was not always the case for the I/O model equivalent to the pseudo-canonical form as it has been showed in the previous section.

The same transformation rules as for the pseudo-canonical form are valid. The  $\alpha$ -parameters can directly be obtained from the  $p_{ij}$  polynomials while the  $\beta$ -parameters can be obtained by solving the equation:

$$\bar{B}_c = M_c \Phi \quad (4.41)$$

Matrix  $\Phi$  is exactly the same as already introduced in (4.32) while the matrices  $M_c$  and  $\bar{B}_c$  (see resp. (4.43) and (4.44)) differ slightly from  $M$  and  $B$ .

|  |            |            |            |     |                   |            |            |            |       |                   |     |            |            |            |     |                   |     |            |            |            |     |                |       |            |            |            |     |                |   |   |
|--|------------|------------|------------|-----|-------------------|------------|------------|------------|-------|-------------------|-----|------------|------------|------------|-----|-------------------|-----|------------|------------|------------|-----|----------------|-------|------------|------------|------------|-----|----------------|---|---|
|  | +          | $v_1$      |            |     |                   | +          |            | +          | $v_2$ |                   |     |            | +          |            | +   | $v_1$             |     |            |            | +          |     | +              | $v_q$ |            |            |            | +   |                |   |   |
|  | 0          | 1          | 0          | ... | 0                 | 0          | 0          | 0          | ...   | 0                 | ... | 0          | 0          | 0          | ... | 0                 | ... | 0          | 0          | 0          | ... | 0              | 0     | 0          | 0          | ...        | 0   |                |   |   |
|  | 0          | 0          | 1          | ... | 0                 | 0          | 0          | 0          | ...   | 0                 | ... | 0          | 0          | 0          | ... | 0                 | ... | 0          | 0          | 0          | ... | 0              | 0     | 0          | 0          | ...        | 0   |                |   |   |
|  | ⋮          | ⋮          | ⋮          | ⋮   | ⋮                 | ⋮          | ⋮          | ⋮          | ⋮     | ⋮                 | ⋮   | ⋮          | ⋮          | ⋮          | ⋮   | ⋮                 | ⋮   | ⋮          | ⋮          | ⋮          | ⋮   | ⋮              | ⋮     | ⋮          | ⋮          | ⋮          | ⋮   | ⋮              | ⋮ | ⋮ |
|  | 0          | 0          | 0          | ... | 1                 | 0          | 0          | 0          | ...   | 0                 | ... | 0          | 0          | 0          | ... | 0                 | ... | 0          | 0          | 0          | ... | 0              | 0     | 0          | 0          | ...        | 0   |                |   |   |
|  | $a_{11,1}$ | $a_{11,2}$ | $a_{11,3}$ | ... | $a_{11,v_1}$      | $a_{12,1}$ | $a_{12,2}$ | $a_{12,3}$ | ...   | $a_{12,v_2}^0$    | ... | $a_{11,1}$ | $a_{11,2}$ | $a_{11,3}$ | ... | $a_{11,v_1}^0$    | ... | $a_{1q,1}$ | $a_{1q,2}$ | $a_{1q,3}$ | ... | $a_{1q,v_q}^0$ | ...   | $a_{1q,1}$ | $a_{1q,2}$ | $a_{1q,3}$ | ... | $a_{1q,v_q}^0$ |   |   |
|  | 0          | 0          | 0          | ... | 0                 | 0          | 1          | 0          | ...   | 0                 | ... | 0          | 0          | 0          | ... | 0                 | ... | 0          | 0          | 0          | ... | 0              | 0     | 0          | 0          | ...        | 0   |                |   |   |
|  | 0          | 0          | 0          | ... | 0                 | 0          | 0          | 1          | ...   | 0                 | ... | 0          | 0          | 0          | ... | 0                 | ... | 0          | 0          | 0          | ... | 0              | 0     | 0          | 0          | ...        | 0   |                |   |   |
|  | ⋮          | ⋮          | ⋮          | ⋮   | ⋮                 | ⋮          | ⋮          | ⋮          | ⋮     | ⋮                 | ⋮   | ⋮          | ⋮          | ⋮          | ⋮   | ⋮                 | ⋮   | ⋮          | ⋮          | ⋮          | ⋮   | ⋮              | ⋮     | ⋮          | ⋮          | ⋮          | ⋮   | ⋮              | ⋮ | ⋮ |
|  | 0          | 0          | 0          | ... | 0                 | 0          | 0          | 0          | ...   | 1                 | ... | 0          | 0          | 0          | ... | 0                 | ... | 0          | 0          | 0          | ... | 0              | 0     | 0          | 0          | ...        | 0   |                |   |   |
|  | $a_{21,1}$ | $a_{21,2}$ | $a_{21,3}$ | ... | $a_{21,v_1}^0$    | $a_{22,1}$ | $a_{22,2}$ | $a_{22,3}$ | ...   | $a_{22,v_2}^0$    | ... | $a_{21,1}$ | $a_{21,2}$ | $a_{21,3}$ | ... | $a_{21,v_1}^0$    | ... | $a_{2q,1}$ | $a_{2q,2}$ | $a_{2q,3}$ | ... | $a_{2q,v_q}^0$ | ...   | $a_{2q,1}$ | $a_{2q,2}$ | $a_{2q,3}$ | ... | $a_{2q,v_q}^0$ |   |   |
|  | ⋮          | ⋮          | ⋮          | ⋮   | ⋮                 | ⋮          | ⋮          | ⋮          | ⋮     | ⋮                 | ⋮   | ⋮          | ⋮          | ⋮          | ⋮   | ⋮                 | ⋮   | ⋮          | ⋮          | ⋮          | ⋮   | ⋮              | ⋮     | ⋮          | ⋮          | ⋮          | ⋮   | ⋮              | ⋮ | ⋮ |
|  | 0          | 0          | 0          | ... | 0                 | 0          | 0          | 0          | ...   | 0                 | ... | 0          | 1          | 0          | ... | 0                 | ... | 0          | 0          | 0          | ... | 0              | 0     | 0          | 0          | ...        | 0   |                |   |   |
|  | 0          | 0          | 0          | ... | 0                 | 0          | 0          | 0          | ...   | 0                 | ... | 0          | 0          | 1          | ... | 0                 | ... | 0          | 0          | 0          | ... | 0              | 0     | 0          | 0          | ...        | 0   |                |   |   |
|  | ⋮          | ⋮          | ⋮          | ⋮   | ⋮                 | ⋮          | ⋮          | ⋮          | ⋮     | ⋮                 | ⋮   | ⋮          | ⋮          | ⋮          | ⋮   | ⋮                 | ⋮   | ⋮          | ⋮          | ⋮          | ⋮   | ⋮              | ⋮     | ⋮          | ⋮          | ⋮          | ⋮   | ⋮              | ⋮ | ⋮ |
|  | 0          | 0          | 0          | ... | 0                 | 0          | 0          | 0          | ...   | 0                 | ... | 0          | 0          | 0          | ... | 1                 | ... | 0          | 0          | 0          | ... | 0              | 0     | 0          | 0          | ...        | 0   |                |   |   |
|  | $a_{11,1}$ | $a_{11,2}$ | $a_{11,3}$ | ... | $a_{11,v_1}^0$    | $a_{12,1}$ | $a_{12,2}$ | $a_{12,3}$ | ...   | $a_{12,v_2}^0$    | ... | $a_{11,1}$ | $a_{11,2}$ | $a_{11,3}$ | ... | $a_{11,v_1}$      | ... | $a_{1q,1}$ | $a_{1q,2}$ | $a_{1q,3}$ | ... | $a_{1q,v_q}^0$ | ...   | $a_{1q,1}$ | $a_{1q,2}$ | $a_{1q,3}$ | ... | $a_{1q,v_q}^0$ |   |   |
|  | ⋮          | ⋮          | ⋮          | ⋮   | ⋮                 | ⋮          | ⋮          | ⋮          | ⋮     | ⋮                 | ⋮   | ⋮          | ⋮          | ⋮          | ⋮   | ⋮                 | ⋮   | ⋮          | ⋮          | ⋮          | ⋮   | ⋮              | ⋮     | ⋮          | ⋮          | ⋮          | ⋮   | ⋮              | ⋮ | ⋮ |
|  | 0          | 0          | 0          | ... | 0                 | 0          | 0          | 0          | ...   | 0                 | ... | 0          | 0          | 0          | ... | 0                 | ... | 0          | 1          | 0          | ... | 0              | 0     | 0          | 0          | ...        | 0   |                |   |   |
|  | 0          | 0          | 0          | ... | 0                 | 0          | 0          | 0          | ...   | 0                 | ... | 0          | 0          | 0          | ... | 0                 | ... | 0          | 0          | 1          | ... | 0              | 0     | 0          | 0          | ...        | 0   |                |   |   |
|  | ⋮          | ⋮          | ⋮          | ⋮   | ⋮                 | ⋮          | ⋮          | ⋮          | ⋮     | ⋮                 | ⋮   | ⋮          | ⋮          | ⋮          | ⋮   | ⋮                 | ⋮   | ⋮          | ⋮          | ⋮          | ⋮   | ⋮              | ⋮     | ⋮          | ⋮          | ⋮          | ⋮   | ⋮              | ⋮ | ⋮ |
|  | 0          | 0          | 0          | ... | 0                 | 0          | 0          | 0          | ...   | 0                 | ... | 0          | 0          | 0          | ... | 0                 | ... | 0          | 0          | 0          | ... | 1              | 0     | 0          | 0          | ...        | 0   |                |   |   |
|  | $a_{q1,1}$ | $a_{q1,2}$ | $a_{q1,3}$ | ... | $a_{q1,v_{q1}}^0$ | $a_{q2,1}$ | $a_{q2,2}$ | $a_{q2,3}$ | ...   | $a_{q2,v_{q2}}^0$ | ... | $a_{q1,1}$ | $a_{q1,2}$ | $a_{q1,3}$ | ... | $a_{q1,v_{q1}}^0$ | ... | $a_{qq,1}$ | $a_{qq,2}$ | $a_{qq,3}$ | ... | $a_{qq,v_q}$   | ...   | $a_{qq,1}$ | $a_{qq,2}$ | $a_{qq,3}$ | ... | $a_{qq,v_q}$   |   |   |

4.42 System matrix A (canonical form)



$$M_c = \begin{bmatrix} \begin{array}{c|c|c} \begin{array}{cccc} + & & & + \\ -a_{11,1} & -a_{11,2} & -a_{11,3} & \dots & -a_{11,v_1} & 0 \\ -a_{11,2} & -a_{11,3} & & & & 0 \\ -a_{11,3} & & & & & 0 \\ \vdots & & & & & \vdots \\ -a_{11,v_1} & -a_{11,v_1} & 1 & & & 0 \\ 0 & 0 & 0 & & & 0 \end{array} & \begin{array}{cccc} + & & & + \\ -a_{12,1} & -a_{12,2} & -a_{12,3} & \dots & -a_{12,v_{12}} & 0 \\ -a_{12,2} & -a_{12,3} & & & & 0 \\ -a_{12,3} & & & & & 0 \\ \vdots & & & & & \vdots \\ -a_{12,v_{12}} & -a_{12,v_{12}} & 1 & & & 0 \\ 0 & 0 & 0 & & & 0 \end{array} & \begin{array}{cccc} + & & & + \\ -a_{1q,1} & -a_{1q,2} & -a_{1q,3} & \dots & -a_{1q,v_{1q}} & 0 \\ -a_{1q,2} & -a_{1q,3} & & & & 0 \\ -a_{1q,3} & & & & & 0 \\ \vdots & & & & & \vdots \\ -a_{1q,v_{1q}} & -a_{1q,v_{1q}} & 1 & & & 0 \\ 0 & 0 & 0 & & & 0 \end{array} \\ \hline \begin{array}{cccc} + & & & + \\ -a_{21,1} & -a_{21,2} & -a_{21,3} & \dots & -a_{21,v_{21}} & 0 \\ -a_{21,2} & -a_{21,3} & & & & 0 \\ -a_{21,3} & & & & & 0 \\ \vdots & & & & & \vdots \\ -a_{21,v_{21}} & -a_{21,v_{21}} & 1 & & & 0 \\ 0 & 0 & 0 & & & 0 \end{array} & \begin{array}{cccc} + & & & + \\ -a_{22,1} & -a_{22,2} & -a_{22,3} & \dots & -a_{22,v_2} & 1 \\ -a_{22,2} & -a_{22,3} & & & & 0 \\ -a_{22,3} & & & & & 0 \\ \vdots & & & & & \vdots \\ -a_{22,v_2} & -a_{22,v_2} & 1 & & & 0 \\ 0 & 0 & 0 & & & 0 \end{array} & \begin{array}{cccc} + & & & + \\ -a_{2q,1} & -a_{2q,2} & -a_{2q,3} & \dots & -a_{2q,v_{2q}} & 0 \\ -a_{2q,2} & -a_{2q,3} & & & & 0 \\ -a_{2q,3} & & & & & 0 \\ \vdots & & & & & \vdots \\ -a_{2q,v_{2q}} & -a_{2q,v_{2q}} & 1 & & & 0 \\ 0 & 0 & 0 & & & 0 \end{array} \\ \hline \begin{array}{cccc} + & & & + \\ -a_{q1,1} & -a_{q1,2} & -a_{q1,3} & \dots & -a_{q1,v_{q1}} & 0 \\ -a_{q1,2} & -a_{q1,3} & & & & 0 \\ -a_{q1,3} & & & & & 0 \\ \vdots & & & & & \vdots \\ -a_{q1,v_{q1}} & -a_{q1,v_{q1}} & 1 & & & 0 \\ 0 & 0 & 0 & & & 0 \end{array} & \begin{array}{cccc} + & & & + \\ -a_{q2,1} & -a_{q2,2} & -a_{q2,3} & \dots & -a_{q2,v_{q2}} & 0 \\ -a_{q2,2} & -a_{q2,3} & & & & 0 \\ -a_{q2,3} & & & & & 0 \\ \vdots & & & & & \vdots \\ -a_{q2,v_{q2}} & -a_{q2,v_{q2}} & 1 & & & 0 \\ 0 & 0 & 0 & & & 0 \end{array} & \begin{array}{cccc} + & & & + \\ -a_{q,1} & -a_{q,2} & -a_{q,3} & \dots & -a_{q,v_q} & 1 \\ -a_{q,2} & -a_{q,3} & & & & 0 \\ -a_{q,3} & & & & & 0 \\ \vdots & & & & & \vdots \\ -a_{q,v_q} & -a_{q,v_q} & 1 & & & 0 \\ 0 & 0 & 0 & & & 0 \end{array} \end{array} \end{bmatrix}$$

4.43 Transformation matrix  $M_c$  (canonical form)

$$\begin{array}{c}
 \bar{B}_c = \left[ \begin{array}{cccccc}
 & + & & p & & + \\
 \beta_{1,11} & \beta_{1,12} & \beta_{1,13} & \cdot & \cdot & \cdot & \beta_{1,1p} \\
 \beta_{1,21} & \beta_{1,22} & \beta_{1,23} & \cdot & \cdot & \cdot & \beta_{1,2p} \\
 \vdots & \vdots & \vdots & & & & \vdots \\
 \vdots & \vdots & \vdots & & & & \vdots \\
 \beta_{1,v_1 1} & \beta_{1,v_1 2} & \beta_{1,v_1 3} & \cdot & \cdot & \cdot & \beta_{1,v_1 p} \\
 [ \beta_{1,(v_1+1)1} & \beta_{1,(v_1+1)2} & \beta_{1,(v_1+1)3} & \cdot & \cdot & \cdot & \beta_{1,(v_1+1)p} ] \\
 \hline
 \beta_{2,11} & \beta_{2,12} & \beta_{2,13} & \cdot & \cdot & \cdot & \beta_{2,1p} \\
 \beta_{2,21} & \beta_{2,22} & \beta_{2,23} & \cdot & \cdot & \cdot & \beta_{2,2p} \\
 \vdots & \vdots & \vdots & & & & \vdots \\
 \vdots & \vdots & \vdots & & & & \vdots \\
 \beta_{2,v_2 1} & \beta_{2,v_2 2} & \beta_{2,v_2 3} & \cdot & \cdot & \cdot & \beta_{2,v_2 p} \\
 [ \beta_{2,(v_2+1)1} & \beta_{2,(v_2+1)2} & \beta_{2,(v_2+1)3} & \cdot & \cdot & \cdot & \beta_{2,(v_2+1)p} ] \\
 \hline
 & & & & & & : \\
 \hline
 \beta_{i,11} & \beta_{i,12} & \beta_{i,13} & \cdot & \cdot & \cdot & \beta_{i,1p} \\
 \beta_{i,21} & \beta_{i,22} & \beta_{i,23} & \cdot & \cdot & \cdot & \beta_{i,2p} \\
 \vdots & \vdots & \vdots & & & & \vdots \\
 \vdots & \vdots & \vdots & & & & \vdots \\
 \beta_{i,v_i 1} & \beta_{i,v_i 2} & \beta_{i,v_i 3} & \cdot & \cdot & \cdot & \beta_{i,v_i p} \\
 [ \beta_{i,(v_i+1)1} & \beta_{i,(v_i+1)2} & \beta_{i,(v_i+1)3} & \cdot & \cdot & \cdot & \beta_{i,(v_i+1)p} ] \\
 \hline
 & & & & & & : \\
 \hline
 \beta_{q,11} & \beta_{q,12} & \beta_{q,13} & \cdot & \cdot & \cdot & \beta_{q,1p} \\
 \beta_{q,21} & \beta_{q,22} & \beta_{q,23} & \cdot & \cdot & \cdot & \beta_{q,2p} \\
 \vdots & \vdots & \vdots & & & & \vdots \\
 \vdots & \vdots & \vdots & & & & \vdots \\
 \beta_{q,v_q 1} & \beta_{q,v_q 2} & \beta_{q,v_q 3} & \cdot & \cdot & \cdot & \beta_{q,v_q p} \\
 [ \beta_{2,(v_q+1)1} & \beta_{2,(v_q+1)2} & \beta_{2,(v_q+1)3} & \cdot & \cdot & \cdot & \beta_{2,(v_q+1)p} ]
 \end{array} \right] \begin{array}{c}
 + \\
 v_1 \\
 [v_1+1] \\
 + \\
 v_2 \\
 [v_2+1] \\
 + \\
 : \\
 + \\
 v_i \\
 [v_i+1] \\
 + \\
 : \\
 + \\
 v_q \\
 [v_q+1] \\
 +
 \end{array}
 \end{array}$$

4.44 Matrix  $\bar{B}_c$  (canonical form)

## 5.0 Structural identification. Method of Guidorzi

### 5.1 Introduction

The MIMO identification problem will be split into  $q$  (=number of outputs) independent MISO identification problems. But before that we can identify the parameters of the system we must know the dimension or degree of complexity of each of the MISO sub-systems. In other words, we have to identify the structure of the system.

In this chapter we will treat several methods to accomplish the structural identification of a linear multivariable system (see Guidorzi (1975), (1981), (1982), Meertens (1983) and Renes (1983) ).

The structural identification is only required in the case that the system is represented in the canonical form. In the pseudo-canonical form all structures (if representable) can describe the process, so that in this case only the order of the system is necessary for the parametric identification.

### 5.2 Structural identification in the noise-free case

The goal of the structural identification is to determine the Kronecker invariants  $v_1, v_2, \dots, v_q$  as they have been defined in paragraph 3.2 . These numbers coincide with the degree of the  $q$  MISO sub-systems in which the system is split.

The dependence property (3.5) can also be written in terms of input and output vectors. Each Kronecker invariant  $v_i$  can be defined as being the smallest integer for which at any time  $t$  holds :

$$y_i(t+v_i) = \sum_{j=1}^q \sum_{k=1}^{v_{ij}} \alpha_{ij,k} y_j(t+k-1) + \sum_{j=1}^p \sum_{k=1}^{v_i+1} \beta_{ik,j} u_j(t+k-1) \quad (5.1)$$

$$\text{with : } \quad v_{ij} = \begin{cases} v_i & \text{for } i=j \\ \min(v_i+1, v_j) & \text{for } i>j \\ \min(v_i, v_j) & \text{for } i<j \end{cases}$$

This property will be used to perform the structural identification of the system. But first let's introduce the following notations.

$$y^i(t+k)^T = [ y_i(t+k) \quad y_i(t+k+1) \quad \dots \quad y_i(t+k-L+1) ] \quad (5.2)$$

$$u^i(t+k)^T = [ u_i(t+k) \quad u_i(t+k+1) \quad \dots \quad u_i(t+k-L+1) ] \quad (5.3)$$

where L is the number of samples being used.

Consider the following vector sequence:

$$\begin{aligned} u^1(t) \dots u^p(t) \quad y^1(t) \dots y^q(t) \quad u^1(t+1) \dots u^p(t+1) \dots \\ y^1(t+1) \dots y^q(t+1) \quad u^1(t+2) \dots u^p(t+2) \dots \end{aligned} \quad (5.4)$$

and select the vectors linearly independent from preceding ones.

Let  $y^i(t+X)$  be the first vector of the i-th output linearly dependent on the preceding ones. From (5.1) it holds that :

$$y^i(t+X) = \sum_{j=1}^q \sum_{k=1}^{v_{ij}} \alpha_{ij,k} y^j(t+k-1) + \sum_{j=1}^p \sum_{k=1}^{v_i+1} \beta_{ik,j} u^j(t+k-1) \quad (5.5)$$

So that  $v_i = X$  can be concluded.

The number of samples L must be large enough to write relation (5.1).

This relation holds for every input sequence and initial state. The only restriction is that the input components  $u^i(t+s)$ ,  $i=1,q$  and  $s=1, v_m$  must be linearly independent. The input signal must be "rich" enough, so that no dependence will arise between input vectors of (5.4).

The following algorithm describes a practical way to obtain the structure of a system (Guidorzi (1975), (1981)).

Algorithm for structural identification.

Denote  $L_k(y_i)$  and  $L_k(u_i)$  as follows:

$$L_k(y_i) = [ y^i(t) \quad y^i(t+1) \quad \dots \quad y^i(t+k-1) ] \quad (5.6)$$

$$L_k(u_i) = [ u^i(t) \quad u^i(t+1) \quad \dots \quad u^i(t+k-1) ] \quad (5.7)$$

with vectors  $y^i(t)$  and  $u^i(t)$  as introduced at (5.2) and (5.3).

Let

$$R(\delta_1, \delta_2, \dots, \delta_{p+q}) = [ L_{\delta_1}(u_1) \quad \dots \quad L_{\delta_p}(u_p) \quad L_{\delta_{p+1}}(y_1) \quad \dots \quad L_{\delta_{p+q}}(y_q) ] \quad (5.8)$$

Define the matrix S as follows:

$$S(\delta_1, \delta_2, \dots, \delta_{p+q}) = R^T(\delta_1, \delta_2, \dots, \delta_{p+q}) R(\delta_1, \delta_2, \dots, \delta_{p+q}) \quad (5.9)$$

S is a square matrix whose dimension is given by  $(\delta_1 + \delta_2 + \dots + \delta_{p+q})$ . Construct then the sequence of increasing dimension matrices

$$\begin{aligned} & S(2, 1, 1, \dots, 1), S(2, 2, 1, \dots, 1), \dots, S(2, 2, 2, \dots, 2) \\ & S(3, 2, 2, \dots, 2), S(3, 3, 2, \dots, 2), \dots, S(3, 3, 3, \dots, 3) \quad (5.10) \\ & S(4, 3, 3, \dots, 3), S(4, 4, 3, \dots, 3), \dots, \dots \text{ etc} \end{aligned}$$

and select from (5.10) the non singular ones. When a singular matrix is found, one of the structural indices  $v_i$  can be determined.

Let  $S(\delta_1, \delta_2, \dots, \delta_i, \dots, \delta_{p+q})$  be a singular matrix in (5.10) and let  $\delta_i$

be the index increased by one with respect to the previous non singular matrix in the sequence. Then it can be concluded that  $v_i = \delta_i - 1$ . The procedure continues from matrix  $S(\delta_1, \delta_2, \dots, v_i, \dots, \delta_{p+q})$ , the  $i$ -th index not being increased anymore. The structural identification ends (all remaining matrices of (5.10) are singular) when all  $q$  Kronecker invariants has been found (see fig. 5.1).

If two adjacent matrices in sequence (5.10) are considered, all the elements of the first are present in the subsequent one that can thus be obtained by computing only a limited number of terms.

The rank of  $R$  is investigated by studying the singularity of matrix  $S = R^T R$ . It can be proved that  $S$  is singular if and only if  $R$  contains dependent vectors. One must realise that this could introduce numerical problems (the condition number of the problem is squared). The great advantage is that  $S$  occupies much less computer memory than matrix  $R$  does. While  $R$  and  $S$  have the same amount of columns ( $=\delta_1 + \delta_2 + \dots + \delta_{p+q}$ ), the amount of rows of  $R$  (=no. of samples) will be in general much larger than the amount of rows of  $S$  ( $=\delta_1 + \delta_2 + \dots + \delta_{p+q}$ ).

### 5.3 Structural identification in the stochastic case

In the deterministic case the structural identification consists of testing the singularity of matrix  $S$ . This can be done in a very simple way by checking whether the determinant of  $S$  becomes zero or not. In the stochastic case, and thus in all practical cases, the matrix may never become singular. Therefore some tests has been developed to find whether the structure of a MISO subsystem has been reached during an extension of matrix  $S$ .

Three methods, two that test the singularity of matrix  $S$  and one that estimates the reconstruction error for a given structure, will be introduced in this paragraph; those methods are:

- the determinant method (see Bollen 1980).
- a test based on the singular value decomposition of the matrix  $S$  (see Renes (1983)).
- the "Range Error test" which gives a prediction of the

reconstruction error (see Guidorzi (1982) and Meertens (1983)).

The determinant method is quite evident. A strong decrease of the determinant of matrix  $S$  indicates a "quasi" singularity of it. By computing successively the determinants of  $\tilde{S}$  of the following sequence of noise corrupted matrices,

$$\begin{array}{ccccccc} \tilde{S}(2,1,1,\dots,1) & \tilde{S}(2,2,1,\dots,1) & \dots & \tilde{S}(2,2,2,\dots,2) & & & \\ \tilde{S}(3,2,2,\dots,2) & \tilde{S}(3,3,2,\dots,2) & \dots & \tilde{S}(3,3,3,\dots,3) & & & (5.11) \\ \tilde{S}(4,3,3,\dots,3) & \dots & \dots & \dots & \dots & \dots & \text{etc} \end{array}$$

it can be possible to obtain the structure of the system. However it seems that for low S/N this method doesn't always lead to good results (see Bollen (1980)). The determinant has properties that make the structural identification sensitive to multiplication scaling. At each iteration (see fig. 5.1) the size of matrix  $\tilde{S}$  is increased by one. The determinant obeys the property:

$$\det(\beta S_n) = \beta^n \det(S_n) \quad (5.12)$$

where:

$S_n$  is a  $n \times n$  matrix.  
 $\beta$  is a real coefficient.

The size of  $S$  has an influence on the determinant. Therefore scale factors in the data will affect the determinant test in a way that it will cause downward (if  $\beta < 1$ ) trends in  $\det(S_n)$  due to scaling to be attributed incorrectly satisfying a dependency relation due to structure. In a same way, scale factors can also hide (if  $\beta > 1$ ) dependency relations due to structure.

The two other above mentioned methods are more complicated and will therefore be discussed in the next paragraphs.

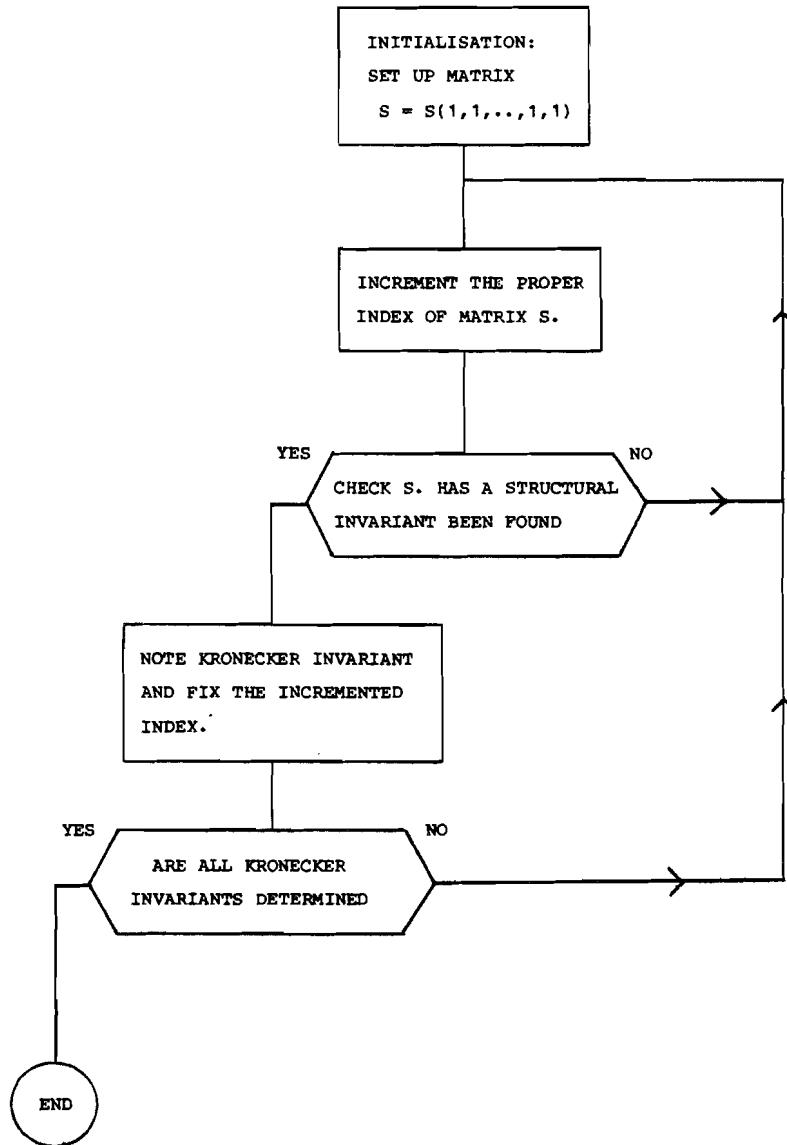


Fig. 5.1 Flow-chart of the structural identification algorithm .



### 5.3.1 The Singular Value test

The singular value test is a method that indicates "how far" a matrix  $\tilde{E}$  is from an exact singular matrix  $E$ . When the distance  $\text{dist}(E-\tilde{E}) < \epsilon$ ,  $\epsilon$  being a threshold value, then we will assume that the matrix  $\tilde{E}$  is "singular". Obviously the threshold value must be related to the noise (process noise, noise due to rounding errors,...) contained in the elements of  $S$ .

Let  $E$  be a real  $n \times n$  matrix. From the Singular Value Decomposition (see Klema and Laub (1980)) we know that  $E$  can be decomposed as follows:

$$E = U \Sigma V^T \quad (5.13)$$

with  $E : n \times n$  matrix  
 $U : n \times n$  unitary matrix  
 $V : n \times n$  unitary matrix

$$\Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n), \quad \sigma_1 > \sigma_2 > \dots > \sigma_r > \sigma_{r+1} = \dots = \sigma_n = 0$$

The  $\sigma_i$ 's,  $i=1, n$  are called the singular values of  $E$  and  $r$  is the rank of  $E$ . The singular values are the non-negative roots of the eigenvalues or the symmetric matrix  $EE^T$ . This makes them naturally insensitive to small perturbations (see Purviance (1980)).

The most important propertie of the singular value is the link between them and the spectral norm. Define  $\lambda_i(E)$ ,  $i=1, n$ , be the eigenvalues of  $E$  ordered so that  $|\lambda_1(E)| > \dots > |\lambda_n(E)|$ . Usefull properties are :

$$\sigma_i(E) = (\lambda_i(EE^T))^{1/2} \quad (5.14)$$

$$\sigma_1(E) = \sup_{\underline{x}} \frac{|E \underline{x}|}{|\underline{x}|} = || E ||_2$$

where  $\underline{x}$  is a  $n \times 1$  vector,  $\left| \cdot \right|$  denotes the Euclidian norm and  $\left\| \cdot \right\|_2$  indicates the spectral norm (see Veltkamp and Geurts (1980)). We also have:

$$0 < \sigma_n(E) < \left| \lambda_n(E) \right| < \left| \lambda_1(E) \right| < \sigma_1(E) \quad (5.15)$$

Note that if  $E$  is singular,  $\lambda_n(E) = 0$  and consequently  $\sigma_n(E) = 0$ . Using this properties, Golub and Wilkinson (1976) proved the key result needed to implement the structure identification by means of the singular values. They showed that:

$$\sigma_n(E) = \left\| F \right\|_2 \quad (5.16)$$

where  $F$  is the matrix of minimum norm such that  $E + F$  is singular. Thus  $\sigma_n(E)$  is the minimum "distance", in the spectral norm sense, that  $E$  is from being singular. When  $E$  is highly non-singular,  $\sigma_n(E)$  is large; when  $E$  is nearly singular, then  $\sigma_n \approx 0$ . Thus the smallest singular value of  $E$  is a good indicator of the "nearness" of  $E$  to being singular.

The singular values have properties similar to those of the determinant. A supplementary desirable property is:

$$\sigma(\beta S_n) = \beta(S_n) \quad (5.17)$$

where:

$S_n$  is a  $n \times n$  matrix.  
 $\beta$  is a real coefficient.

The singular values are independent of the size of  $S_n$ . This property will make the singular value test insensitive for data scaling.

Each matrix  $\tilde{S}$  of (5.11) can be decomposed into:

$$\tilde{S} = S + N \quad (5.18)$$

where  $S$  is the data matrix that would be used if exact covariance information were available and  $N$  is an error matrix which accounts for the noise and for finite sample size (with uncorrelated and zero mean disturbances we have  $\text{plim}_{L \rightarrow \infty} \tilde{S} = S$ ).

From the knowledge of  $\tilde{S}$ , it has to be decided whether the matrix  $S$  is (or has a great probability to be) singular. The influence of  $N$  on the singular values can be studied by using the following properties (see Rao (1973)):

$$\sigma_i(S+N) < \sigma_j(S) + \alpha_k(N) \quad j+k < i+1$$

$$i=1, n \quad (5.19)$$

$$\sigma_i(S+N) > \sigma_j(S) + \alpha_k(N) \quad j+k > n+i$$

$$i=1, n$$

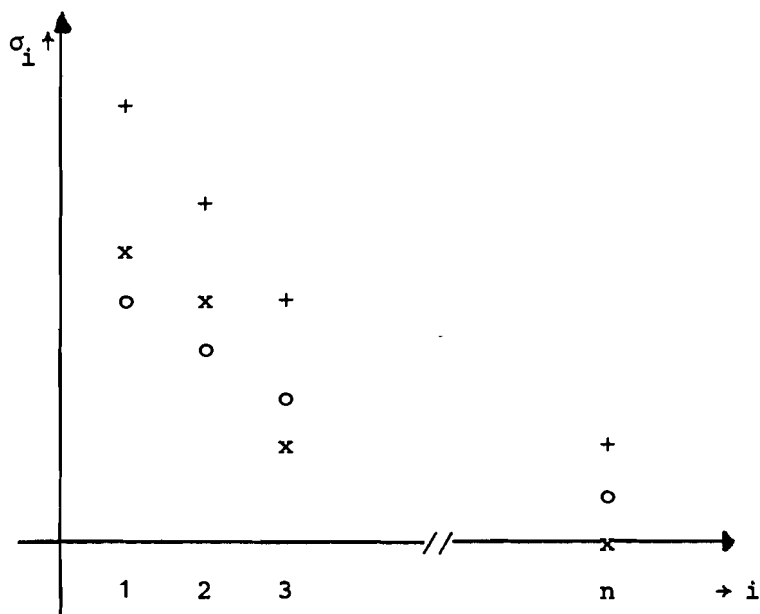
where  $S$  and  $N$  are positive semidefinite  $n \times n$  matrices. In the particular case of the least singular value we have:

$$\sigma_n(S+N) < \sigma_n(S) + \alpha_1(N)$$

$$(5.20)$$

$$\sigma_n(S+N) > \sigma_n(S) + \alpha_n(N)$$

This means that the error  $N$  will result in an upwards shift of the singular values from at least  $\alpha_n(N)$  and at most  $\alpha_1(N)$  (see fig. 5.2):



where:  $x$ :  $\sigma_i(S)$ ,  $o$ :  $\sigma_i(N)$ ,  $+$ :  $\sigma_i(\tilde{S}=S+N)$

Fig. 5.2 Influence of noise on the singular values of S (singular)

In the case that S is singular ( $\sigma_n(S)=0$ ), then (5.20) becomes:

$$\sigma_n(S+N) = \sigma_n(\tilde{S}) < \sigma_1(N) = \left\| \left\| N \right\| \right\|_2 \quad (5.21)$$

$$\sigma_n(S+N) = \sigma_n(\tilde{S}) > \sigma_n(N)$$

This gives a necessary, but not sufficient condition to  $\tilde{S}$  so that S is singular.

The problem resides in the fact that the noise covariance matrix is unknown. It is quite difficult to define the distribution of the least singular value as a function of the disturbances and the number of samples being used. An attempt has been made in Purviance (1980) to define a "singularity threshold" for  $\sigma_n(\tilde{S})$  that would guarantee with a very high probability the singularity of S. Purviance derives this threshold from the estimation of matrix N in the following way.

The elements of  $\tilde{S}$  are sample covariances and crosscovariances of the form:

$$\tilde{s}_{ij} = \frac{1}{L-k+1} \sum_{f=k}^L x(t+f) y(t-k+f) \quad (5.22)$$

where  $x$  and  $y$  are dummy variables representing an input or an output depending upon  $i$  and  $j$ . Using the normal assumptions on  $x$  and  $y$  it can be shown (see Purviance (1980)) that the random variable  $n_{ij} = \tilde{s}_{ij} - s_{ij}$  is asymptotically normal with zero mean and with a variance that can be approximated by::

$$\sigma_{ij} \approx \text{var}(n_{ij}) \approx \frac{1}{L-k} \sum_{-\infty}^{+\infty} P_{xx}(f) P_{yy}(f) \quad (5.23)$$

Since the covariances  $P_{xx}$  and  $P_{yy}$  can be replaced by their sample estimate, we assume that  $n_{ij}$  are known.

Purviance proposes  $t_\epsilon$  as "singularity threshold" for  $\sigma_n(\tilde{S})$ :

$$t_\epsilon = \left\| \left\| G \right\| \right\|_2 \quad (5.24)$$

where:  $G = \{ \gamma \sigma_{ij} \}$  for  $\gamma$  fixed.

When

$$\sigma_n(\tilde{S}) < t_\epsilon \quad (5.25)$$

then the probability of  $S$  being singular is high. In other words, when (5.25) is fulfilled we can expect that the "distance"  $F$  from  $\tilde{S}$  to the nearest singular matrix is only due to the error matrix  $N$ . With practical experiments, Purviance found out that a good value for  $\gamma$  was 0.05.

### 5.3.2 The range error test.

The range error test, method proposed by Guidorzi (see Guidorzi, Losito

and Muratori (1982)), estimates the Per Cent Reconstruction Error (PCRE) which shall be called the Predicted PCRE (PPCRE). During the execution of the structural identification the PPCRE gives an idea about the accuracy of the model (if estimated with a least squares method) for a given structure without having to perform the entire parametric identification. For a given output  $i$  the PPCRE will stabilise itself after the true order  $v_i$  of the corresponding MISO subsystem has been reached.

The range error test is based on the following principle:

Let  $\Phi$  be a regular matrix which has to be extended with a vector  $\underline{w}$ , partly not belonging to the range of  $\Phi$ .  $\underline{w}$  can be decomposed into a projection into the range of  $\Phi$ ,  $\underline{w}_\Phi$  and a component  $\underline{e}$  orthogonal to the range of  $\Phi$  part  $\underline{e}$ .

$$\underline{w} = \underline{w}_\Phi + \underline{e} = \Phi \underline{\tau} + \underline{e} \quad (5.26)$$

Consider the extended matrix  $\psi$  :

$$\begin{aligned} \psi &= (\Phi \mid \underline{w})^T (\Phi \mid \underline{w}) = (\Phi \mid \underline{w}_\Phi + \underline{e})^T (\Phi \mid \underline{w}_\Phi + \underline{e}) \\ \psi &= \begin{bmatrix} \Phi^T \Phi & \Phi^T \underline{w}_\Phi + \Phi^T \underline{e} \\ \underline{w}_\Phi^T \Phi + \underline{e}^T \Phi & \underline{w}_\Phi^T \underline{w}_\Phi + \underline{e}^T \underline{e} \end{bmatrix} \end{aligned} \quad (5.27)$$

Using the following property

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A) \det(D - CA^{-1}B) \quad (5.28)$$

and remarking that  $\underline{e}^T \Phi = 0$ , we can write the determinant from  $\psi$  as follows:

$$\det(\psi) = \det(\Phi^T \Phi) \det(\underline{w}_\Phi^T \underline{w}_\Phi + \underline{e}^T \underline{e} - \underline{w}_\Phi^T \Phi (\Phi^T \Phi)^{-1} \Phi^T \underline{w}_\Phi) \quad (5.29)$$

we can rewrite (5.19) as follows:

$$\begin{aligned}
\det(\psi) &= \det(\Phi^T \Phi) \det(\underline{w}_{\Phi-\Phi}^T + \underline{e}^T \underline{e} - \underline{w}_{\Phi}^T \Phi(\Phi^T \Phi)^{-1} \Phi^T \Phi \underline{e}) \\
&= \det(\Phi^T \Phi) \det(\underline{e}^T \underline{e}) = \underline{e}^T \underline{e} \det(\Phi^T \Phi) \quad (5.30)
\end{aligned}$$

Therefore the Euclidian norm of  $\underline{e}$  is equal to:

$$\| \underline{e} \| = (\det(\psi) / \det(\Phi^T \Phi))^{1/2} \quad (5.31)$$

If we substitute :  
 $\Phi$  by  $S$   
 $\psi$  by extended  $S$   
 $\underline{w}$  by input/output extension vector

it will be evident that in the deterministic case vector  $\underline{y}_1(k+s)$  for  $s > v_1$  will belong to the range of  $S$ . In that case the norm of  $\underline{e}$  will be zero. In the stochastic case the noise will cause a non zero error vector  $\underline{e}$ . For  $s < v_1$  this error vector is emphasized by a bad fit of the model because a wrong structure was used. If the noise parameters are time invariant further extension of matrix  $S$  will have as result a stabilisation of the norm of vector  $\underline{e}$ . This stabilisation will occur once the true structure of the MISO sub-system has been reached. By looking to the variations of the norm of this error vector it will be possible to determine the structure of the system.

The quantity we will use in this report will be the PPCRE introduced by Guidorzi and defined by:

$$\Pi = 100 \text{ norm}(\underline{e}) / \sigma_1^2 (L-1)^{1/2} \quad (5.32)$$

where  $\sigma_1^2$  : variance of extended output.

$L$  : number of samples used in  $\underline{y}^1$ .

The structural identification algorithm in the stochastic case runs as follows:

Let's consider the sequence of noise corrupted matrices (5.11) and compute the associated PPCRE's:

$$\begin{aligned}
& \Pi(2, 1, 1, \dots, 1) \quad \Pi(2, 2, 1, \dots, 1) \quad \dots \quad \Pi(2, 2, 2, \dots, 2) \\
& \Pi(3, 2, 2, \dots, 2) \quad \Pi(3, 3, 2, \dots, 2) \quad \dots \quad \Pi(3, 3, 3, \dots, 3) \\
& \Pi(4, 3, 3, \dots, 3) \quad \dots \quad \dots \quad \dots \quad \text{etc}
\end{aligned}
\tag{5.33}$$

When the PPCRE  $\Pi(\delta_1, \delta_2, \dots, \delta_i, \dots, \delta_{p+q})$  associated to the matrix  $S(\delta_1, \delta_2, \dots, \delta_i, \dots, \delta_{p+q})$ , where the increased argument with respect to the preceding non singular matrix is  $\delta_i$ , does not decrease significantly with respect to  $\Pi(\delta_1-1, \delta_2-1, \dots, \delta_i-1, \dots, \delta_{p+q}-1)$  then we can conclude that  $v_i = \lambda_i - 2$ . The test continues from  $\Pi(v_{i1}, v_{i2}, \dots, v_i, \dots, v_{i(p+q)})$  without increasing anymore the  $i$ -th argument of the matrices  $\tilde{S}$  and ends when all indices are found.

The prediction of the reconstruction error on the  $i$ -th output given by this method coincides exactly with the reconstruction error obtained with a model having the considered order for the  $i$ -th subsystem when:

- the model parameters have been estimated with a least square method. (see chapter 6).
- the model is used for one step ahead prevision (i.e. the actual output is constructed, by means of the estimated model, using the past inputs and outputs measured on the process and not the outputs obtained in the previous simulation steps by means of the model).

In fact, this method is a disguised determinants ratio method with a certain multiplication factor to obtain an estimate of the reconstruction error. Therefore it will have all advantages and inconvenients of this method.

### 5.3.3 Coarse estimation of the D matrix

It can happen that the  $\underline{D}_u$  term of the output vector in the state space representation is much larger than the  $\underline{C}_x$  term, then:



$$\underline{y}(k) = \underline{C}\underline{x}(k) + \underline{D}\underline{u}(k) \approx \underline{D}\underline{u}(k) \quad (5.34)$$

If we are identifying the structure of the system this can be quite embarrassing since we are only interested in the  $\underline{C}\underline{x}$  term ( the dimension of  $D$  is given by the number of inputs and outputs). A way to reduce the influence of the  $\underline{D}\underline{u}$  term is by correcting the output data with  $\hat{\underline{D}}\underline{u}$ . A coarse estimate  $\hat{D}$  of  $D$  can be obtained by:

$$\hat{d}_{ij} = \frac{P_{jj}(0)}{P_{ij}(0)} \quad (5.35)$$

$i=1,p$   
 $j=1,q$

where:  $\hat{d}_{ij}$  : entry  $i,j$  of matrix  $\hat{D}$ .

$P_{jj}(0)$  : autocorrelation of output  $j$  for time shift zero.

$P_{ij}(0)$  : correlation between between input  $i$  and residual  $j$  for time shift zero.

The estimation of  $D$  will be better according as  $\underline{D}\underline{u}$  becomes larger with regard to  $\underline{C}\underline{x}$ . By inspecting the statistics of the signals (mean and standard deviation), it can be possible to get an impression about the accuracy of  $\hat{D}$ . Depending of this accuracy (which indicates whether  $\underline{D}\underline{u} \gg \underline{C}\underline{x}$  or not), the user can decide to perform the  $\underline{D}\underline{u}$  correction of the output vectors by doing:

$$\text{for } k=1,L \quad \underline{y}_{\text{corr}}(k) = \underline{y}(k) - \hat{\underline{D}}\underline{u}(k) = \underline{C}\underline{x}(k) + (D-\hat{D})\underline{u}(k) \quad (5.36)$$

If  $\hat{D}$  is a reasonable estimate of  $D$ , the influence of  $(D-\hat{D})\underline{u}$  on the corrected outputs will not be as preponderate as the influence of  $\underline{D}\underline{u}$  on the original outputs, so that a better conditioning of the structural identification problem will be obtained. In fact, this is nothing else than elementary row operation of a matrix. The columns of  $R$  (5.8) corres-

ponding to an output are subtracted by a linear combination of columns of the same matrix corresponding to the inputs of the process. The coefficients of this linear combination are the entries of  $\hat{d}_{ij}$   $i=1,q$  and  $j=1,p$ .

#### 5.3.4 Influence of offsets on the structural identification

By adding constant and small (with regard to the elements of a vector) offset vectors to different dependent sample vectors, one can destroy their dependency. Large offsets (with regard to the elements of a vector) can "increase the dependency" between independent vectors.

The offsets can affect strongly the dependency between vectors, and therefore the structural identification (see also section 6.4 where the influence on the parametric identification is treated).

A simple way to avoid the embarrassing influence of the offsets is by extending the matrix R (5.8) with an L-dimensional vector filled with ones. During the structural identification, the offsets contained in all vectors of R will be "caught" by this vector. The extended matrix R becomes:

$$R_{\text{ext}}(\delta_1, \delta_2, \dots, \delta_{p+q}) = \left[ \begin{array}{ccc|c} L_{\delta_1}(u_1) & \dots & L_{\delta_{p+q}}(y_q) & \underline{1} \end{array} \right] \quad (5.37)$$

## 6.0 Parametric identification of multivariable systems.

### 6.1 Introduction

Once the structure (i.e. the order of all MISO sub-systems) is known or assumed to be known, we can proceed to identify the parameters of the  $q$  MISO subsystems. As it has been mentioned, the parametric identification will be performed using the Input/Output model. For subsystem  $i$  the equation that has to be solved for every sample  $k$  is:

$$y_i(k) = z^{-\mu_i} \left[ -\sum_{\substack{j=1 \\ i \neq j}}^q p_{ij}(z)y_j(k) - p'_{ii}(z)y_i(k) + \sum_{j=1}^p q_{ij}(z)u_j(k) \right]$$

(6.1)

where  $p'_{ii}(z) = p_{ii}(z) - z^{\mu_i}$

The order of the polynomials  $p_{ij}(z)$  and  $q_{ij}(z)$  will depend of the structure of the system and whether we will represent the system into a pseudo-canonical or a canonical form (see sections 3.2 and 3.3).

In a general form, (6.1) can be written as follows:

$$y_i(k) = \underline{\underline{Q}}_i^T(k) \underline{\underline{\theta}}_i \quad (6.2a)$$

where:

$$(6.2b) \quad \underline{\underline{Q}}_i^T(k) = \begin{bmatrix} y_1(k-\delta), y_1(k-\delta+1), \dots, y_1(k-\delta+\delta_1-1), \\ y_2(k-\delta), y_2(k-\delta+1), \dots, y_2(k-\delta+\delta_2-1), \\ \vdots \\ y_i(k-\delta), y_i(k-\delta+1), \dots, y_i(k-1), \\ \vdots \\ y_q(k-\delta), y_q(k-\delta+1), \dots, y_q(k-\delta+\delta_q-1), \end{bmatrix}$$

$$\begin{bmatrix} u_1(k-\delta), u_1(k-\delta+1), \dots, u_1(k-\delta+\delta'-1), \\ u_2(k-\delta), u_2(k-\delta+1), \dots, u_2(k-\delta+\delta'-1), \\ \vdots \\ u_p(k-\delta), u_p(k-\delta+1), \dots, u_p(k-\delta+\delta'-1) \end{bmatrix}$$

the parameter vector of subsystem  $i$  is:

$$(6.2c) \quad \underline{\theta}_i^T = \begin{bmatrix} -\alpha_{i1,1}', -\alpha_{i1,2}', \dots, -\alpha_{i1,(\delta_1-1)}', -\alpha_{i1,\delta_1}', \\ -\alpha_{i2,1}', -\alpha_{i2,2}', \dots, -\alpha_{i2,(\delta_2-1)}', -\alpha_{i2,\delta_2}', \\ \vdots \\ -\alpha_{ii,1}', -\alpha_{ii,2}', \dots, -\alpha_{ii,(\delta_i-1)}', -\alpha_{ii,\delta_i}', \\ \vdots \\ -\alpha_{iq,1}', -\alpha_{iq,2}', \dots, -\alpha_{iq,(\delta_q-1)}', -\alpha_{iq,\delta_q}', \\ +\beta_{i,11}', +\beta_{i,21}', \dots, +\beta_{i,(\delta'-1)1}', +\beta_{i,\delta'1}', \\ +\beta_{i,12}', +\beta_{i,22}', \dots, +\beta_{i,(\delta'-1)2}', +\beta_{i,\delta'2}', \\ \vdots \\ +\beta_{i,1p}', +\beta_{i,2p}', \dots, +\beta_{i,(\delta'-1)p}', +\beta_{i,\delta'p}' \end{bmatrix}$$

From (4.19), (4.27), (4.28), (4.38) and (4.39) we can derive the expressions of  $\delta, \delta', \delta_1, \dots, \delta_q$ :

Pseudo canonical form

- $\delta = \mu_i$
- if  $\mu_i < \mu_m$  then  $\delta' = \mu_m$
- if  $\mu_i = \mu_m$  then  $\delta' = \mu_m + 1$
- $\delta_j = \mu_j$ , for  $j=1, q$

Canonical form

- $\delta = v_i$
- $\delta' = v_i + 1$
- if  $i > j$  then  $\delta_j = \min(v_i + 1, v_j)$
- if  $i = j$  then  $\delta_j = v_i$
- if  $i < j$  then  $\delta_j = \min(v_i, v_j)$

(6.2d)

Assuming that the system is time-invariant and that equation (6.2) holds for every sample time  $k$ , the  $L$  (no. of samples) systems equations can be written as follows:

$$\underline{y}_i = \underline{Q}_i \underline{\theta}_i \quad (6.3)$$

where :

$$\underline{y}_i = \underline{y}_i(t) = [ y_i(t) \ y_i(t+1) \ .. \ y_i(t+L-1) ]^T$$

$$\underline{Q}_i = \underline{Q}_i(t) = [ \underline{Q}_i(t) \ \underline{Q}_i(t+1) \ .. \ \underline{Q}_i(t+L-1) ]^T$$

Remark: To simplify notations the time index  $(t)$  will be omitted in  $\underline{y}$  and  $\underline{Q}$ .  $t$  is the first sample that will be used.

In the deterministic case a number of equations (6.2) equal to the number of parameters to identify would be enough to solve "exactly" the parametric identification problem. In the stochastic case, when additive noise is present on the output, the system equation (6.3) becomes:

$$\tilde{\underline{y}}_i = \tilde{\underline{Q}}_i \underline{\theta}_{it} + \underline{e}_i \quad (6.4)$$

where:

$\tilde{\underline{y}}_i$  : disturbed output vector  $i$ . Defined as  $\underline{y}_i$  (6.3)

$\tilde{\underline{Q}}_i$  : defined as  $\underline{Q}_i$  (6.2b), but now containing disturbed samples.

$\underline{\theta}_{it}$  : true parameter vector  $i$  (see (6.2c))

$\underline{e}_i = [ e_i(t) \ e_i(t+1) \ .. \ e_i(t+L-1) ]$  : equation error  $i$ .

The output vector can also be seen as being the sum of an undisturbed output term and an output error term  $\underline{e}'_i$ :

$$\tilde{\underline{y}}_i = \underline{y}_i + \underline{e}'_i \quad (6.5)$$

where:

$$\underline{e}'_i = [ e'_i(t) \ e'_i(t+1) \ .. \ e'_i(t+L-1) ]$$
 : output error  $i$ .

For this kind of problems we know (see van den Boom (1982)) that except

for particular cases, the Least Squares method will lead to biased estimates. In this chapter we will propose two methods which lead to consistent estimates; those are the Instrumental Variable (IV) method (section 6.2) and the Compensated Least Squares (CLS) method (section 6.3). An extension in order to identify systems with offsets on input or/and output will be described in section 6.4. In section 6.5 some facilities to check the validity of the estimates will be discussed.

## 6.2 The Instrumental Variable (IV) method

Consider the equation of the  $i$ -th MISO subsystem (6.4) and suppose that we have a matrix  $Z_i$  (with same structure as  $Q_i$ ) that fulfils the following requirements:

$$\left. \begin{aligned} \text{plim}_{L \rightarrow \infty} \frac{1}{L} Z_i^T e_i &= \underline{0} \\ \text{plim}_{L \rightarrow \infty} \frac{1}{L} Z_i^T \tilde{Q}_i &= Q \text{ not singular.} \end{aligned} \right\} \quad (6.6)$$

These two requirements mean that  $Z_i$  is not correlated with the disturbances and that  $Z_i$  is correlated with the process signals. Multiplication of (6.4) by  $Z_i^T$  leads to:

$$Z_i^T \tilde{y}_i = Z_i^T \tilde{Q}_i \theta_{-it} + Z_i^T e_i \quad (6.7)$$

so that

$$\theta_{-it} = (Z_i^T \tilde{Q}_i)^{-1} Z_i^T \tilde{y}_i - (Z_i^T \tilde{Q}_i)^{-1} Z_i^T e_i$$

The IV-estimator of  $\theta_{-it}$  is:

$$\hat{\theta}_{-it, IV} = (Z_i^T \tilde{Q}_i)^{-1} Z_i^T \tilde{y}_i \quad (6.8)$$

The proof that this estimate is consistent (converges to the true para-

meter vector when  $L \rightarrow \infty$ ) follows from equations (6.7) and (6.8):

$$\begin{aligned} \hat{\theta}_{i_{IV}} &= (Z_i^T \tilde{Q}_i)^{-1} Z_i^T \tilde{Q}_i^T \theta_{it} - (Z_i^T \tilde{Q})^{-1} Z_i^T e_i \\ &= \theta_{it} + (Z_i^T \tilde{Q})^{-1} Z_i^T e_i \end{aligned} \quad (6.9)$$

From the properties of  $Z_i$  (6.6) we can derive:

$$\text{plim}_{L \rightarrow \infty} \hat{\theta}_{i_{IV}} = \theta_{it} + \text{plim}_{L \rightarrow \infty} \frac{(Z_i^T \tilde{Q}_i)^{-1}}{L} \text{plim}_{L \rightarrow \infty} \frac{Z_i^T e_i}{L}$$

which leads to:

$$\text{plim}_{L \rightarrow \infty} \hat{\theta}_{i_{IV}} = \theta_{it} \quad (6.10)$$

Now rests the question: how to obtain such an Instrumental Variable. The choice of the IV is central in the problem for the IV estimation, since both the consistency and the accuracy properties of the parameter estimates depend heavily on the IV scheme employed (see Soderstrom, Stoica (1983)). The computational complexity may also be a point to take into account. Some of the most popular IV's are:

1 - The bootstrap IV . This operator is build with the undisturbed input samples of the system and the output samples of a filter with the original input samples of the system on input. As filter, the bootstrap IV uses the model obtained during the previous iteration. It can be proved that the ideal IV operator (Hsia (1977)) is the one composed by undisturbed input and output samples. Because this information is not available, the bootstrap IV will approximate the undisturbed output samples as above is mentioned. This introduces an iterative scheme, since at each iteration the estimation of the model is expected to be better. This provides a better approximation of the undisturbed outputs so that an IV closer from the ideal IV can be build for the next iteration. When the iterative scheme converges (which will not always be the case) , this

will again improve the accuracy of the model.

2 - The IV composed by delayed inputs.  $Z$  is like matrix  $Q$ , but with delayed inputs substituting the noise corrupted outputs. To fulfil (6.6) the delay must be larger than the largest structural index of the system being identified.

3 - As 1, but using a fixed filter .

We will use the bootstrap IV as describe in 1. During the first iteration, no model being available, an IV operator with delayed inputs (see IV no. 2) or a least squares estimator will be used. The entire IV estimation algorithm is illustrated on fig 6.1.

The iterations are pursued until stabilisation of a certain parameter (e.g. eigenvalue, reconstruction error ..) has been reached. The advantage of the IV method is that no knowledge is required about the noise processes to obtain consistent estimators. In Soica, Soderstrom (1982) it is shown that the Instrumental Variable method has better convergence properties than the Compensated Least Squares method (see section 6.3).

Practical remark: As it was expected and happend during the practical tests of the IV method, the iterative algorithm may estimate an unstable model which makes the construction of the bootstrap IV operator impossible. This starts happening for  $S/N \approx 30$  dB on both outputs, an becomes more frequent as noise increases.

Probably the reason is that we are using a finite number of samples. The IV- method tries to minimise the correlation between inputs and residuals ( like LS minimises the equation error) and finds at a certain iteration an unstable model that fulfils this criterion.

This problem was solved by extending a practical remark in Ljung (1983) for recursive estimators to iterative estimators. The idea is, in the case that an unstable model is found, to project the system matrix  $A$  (responsible for the dynamics of the system) into the set of stable system matrices. This will provide a worse IV but allows the continuation of the iterative algorithm. An easy way to perform this "stability projection" is:

```
LABEL : IF ( A IS UNSTABLE ) THEN
```



```
FOR ALL  $\alpha_{ijk}$  FROM A
   $\alpha_{ijk} := 0.5 ( \alpha_{ijk} + \alpha_{ijk}(\text{previous iteration}) )$ 
CONTINUE
GOTO LABEL
ELSE
ENDIF
```

A stable system will always be found since the matrix of the previous iteration is a stable one. In practice, this "stabilisation projection" resulted in a significant increase of the number of successful ( converging estimates ) runs.

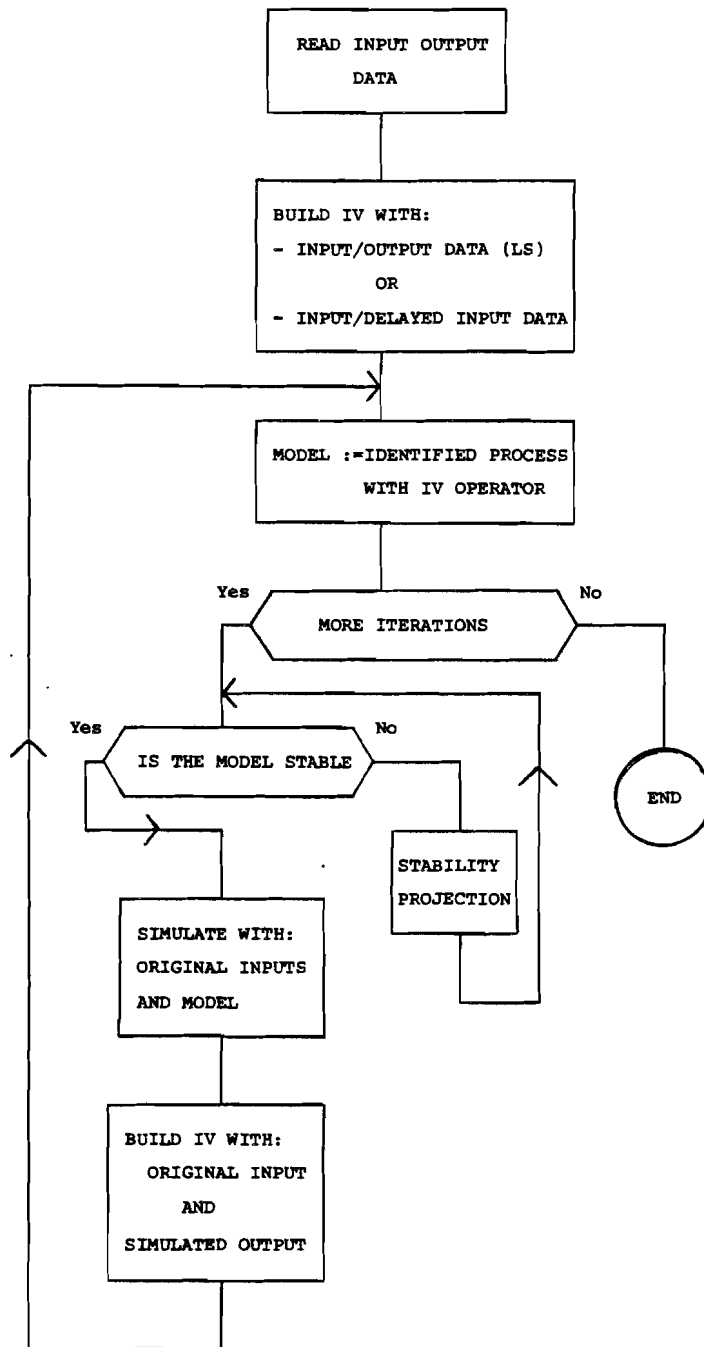


Fig. 6.1 Flow-chart of the IV estimation algorithm

### 6.3 The Compensated Least Squares (CLS) estimator

In the deterministic case the Least Squares (LS) method will provide us with unbiased estimates of the process. Those are given by:

$$\hat{\underline{\theta}}_{i_{LS}} = (\underline{\Omega}_i^T \underline{\Omega}_i)^{-1} \underline{\Omega}_i^T \underline{y}_i \quad (6.11)$$

From (5.8), (5.9), and (6.2) we can remark that:

$$R_i = R (\delta_1, \delta_2, \dots, \delta_q, \underbrace{\delta^r, \dots, \delta^r}_{p\text{-times}}) = \underline{\Omega}_i \quad (6.12)$$

$$S_i = R_i^T R_i$$

So that (6.11) can be written as:

$$\hat{\underline{\theta}}_{i_{LS}} = S_i^{-1} R_i^T \underline{y}_i \quad (6.13)$$

In the stochastic case it is known that, in general, the LS method will lead to biased estimates. If we can assume that the disturbances are white and uncorrelated processes, then it is possible to (asymptotically) eliminate the bias. We will use the term of Compensated Least Squares (CLS) method when the bias is more or less subtracted from the LS estimates of the parameters.

In our case (noise on the outputs of the proces), the noisy outputs can be denoted by:

$$\tilde{y}_i(k) = y_i(k) + d(y_i(k)) \quad (6.14)$$

where:

- $\tilde{y}_i(k)$  : disturbed output  $i$  at time  $k$ .
- $y_i(k)$  : noiseless output  $i$  at time  $k$ .
- $d(y_i(k))$ : noise on output  $i$  at time  $k$ .

The noise having zero mean and being an uncorrelated process, we have:

$$\begin{aligned} \text{plim}_{L \rightarrow \infty} \frac{1}{L} \tilde{S}_i &= \text{plim}_{L \rightarrow \infty} \frac{1}{L} S_i + N(\underline{d}_i) \\ \text{plim}_{L \rightarrow \infty} \frac{1}{L} \tilde{R}_i^T \tilde{\underline{y}}_i &= \text{plim}_{L \rightarrow \infty} \frac{1}{L} R_i^T \underline{y}_i \end{aligned} \quad (6.15)$$

where

- $\tilde{S}_i$  : defined as  $S_i$  (6.12), but now with disturbed signals.
- $N(\underline{d}_i)$  : covariance matrix of the noise vector  $\underline{d}_i$ .
- $\underline{d}_i$  : noise vector. Defined as  $\underline{\Omega}_i$  (6.2b), but containing only the disturbances (the part corresponding to the input is in this case filled with zero's).
- $\tilde{R}_i$  : defined as  $R_i$  (6.12), but now with disturbed signals.

If the statistics of the noises are known and the above mentioned requirements are fulfilled, we can approximate  $S_i$  and  $R_i^T \underline{y}_i$  by:

$$\begin{aligned} \hat{(R_i^T \underline{y}_i)} &= \tilde{R}_i \tilde{\underline{y}}_i \\ \hat{S}_i &= \tilde{S}_i - L N(\underline{d}_i) \end{aligned} \quad (6.16)$$

The Compensated Least Squares estimator can now be derived by substituting (6.16) in (6.13):

$$\hat{\underline{\theta}}_{i, \text{CLS}} = \hat{S}_i^{-1} (\hat{R}_i^T \underline{y}_i) \quad (6.17)$$

In the case that white, independent noise of equal variance  $\sigma^2$  is present on all outputs, the estimated parameter vector becomes:

$$\hat{\underline{\theta}}_{i, \text{CLS}} = (\tilde{S}_i - L \sigma^2 I')^{-1} \tilde{R}_i \tilde{y}_i \quad (6.18)$$

where:  $\sigma^2$  : variance of all output disturbances.  
 $I'$  : identity matrix with its last  $p \times \delta$  diagonal elements equal to zero (no input noise).

The covariance matrix of the estimated parameter vector is (see Guidorzi (1975), (1981) ):

$$\text{cov}(\hat{\underline{\theta}}_{i, \text{CLS}}) = (1 + \hat{\underline{\theta}}_{i, \text{CLS}}^T \hat{\underline{\theta}}_{i, \text{CLS}}) (\tilde{S}_i - L \sigma^2 I)^{-1} \sigma^2 \quad (6.19)$$

Guidorzi suggests (see Guidorzi (1975), (1981) ) to use the least eigenvalue of the matrix  $\tilde{S}$  as an estimate for  $L \sigma^2$ . This point is worked out in Meertens (1983).

The unrealistic assumptions made about the noise processes and the necessity of knowing the statistics of the noise make the CLS method less applicable than the IV method (see section 6.2).

#### 6.4 Offset estimation

In practical cases it can happen that the recorded data contains a constant offset. Also in cases that the dynamic part of the signal is much smaller than the DC component, it is desirable to correct the signals with their mean value in order to avoid numerical problems. This correction is nothing else than an "artificial offset" which is introduced by the user during the preprocessing of the signals.

These offsets must be taken into account during the parametric but also (see section 5.4) during the structural identification. The offsets can be concentrated on the outputs of the process (offsets on input can be transferred to the output by multiplying them by the DC amplification factor of the system). The output can be written as:

$$\underline{y}^+(k) = \underline{y}(k) + \underline{c} \quad (6.20)$$

where:  $\underline{y}^+(k)$  : output vector with offset. Defined as vector  $\underline{y}$  in equation (2.9).  
 $\underline{y}(k)$  : output vector without offset (see (2.9)).  
 $\underline{c} = [c_1, c_2, \dots, c_q]^T$  : offset vector.

By the same way we can split the matrix  $\Omega_i^+$  containing the I/O data with offsets:

$$\Omega_i^+ = \Omega_i + \Omega_i^C \quad (6.21)$$

where:

$\Omega_i^+$  : defined as  $\Omega_i$  in (6.2b) with samples containing an offset.

$\Omega_i$  : as defined in (6.2b) (without offsets).

$$\Omega_i^C = \begin{bmatrix} \underline{c}_1 & \underline{c}_1 \dots \underline{c}_1 & \underline{c}_2 & \underline{c}_2 \dots \underline{c}_2 & \dots & \underline{c}_q & \underline{c}_q \dots \underline{c}_q & \underline{0} & \underline{0} \dots \underline{0} \end{bmatrix}$$

$\delta_1$ -columns     $\delta_2$ -columns    ...     $\delta_q$ -columns     $p \times \delta$ -columns

$\delta_1, \delta_2, \dots, \delta_q, \delta$  are defined at (6.2d).  $q$  is the number of outputs.

$\underline{c}_i$  : vector containing  $L$  elements equal to  $c_i$ .

The offset on the output of MISO sub-system  $i$  can be introduced in the system equations and be considered as a new parameter in parameter vector  $\underline{\theta}_i$  in the following way:

$$\Omega_i^{++} = \left( \Omega_i^+ \mid \underline{1} \right) = \left( \Omega_i \mid \underline{0} \right) + \left( \Omega_i^C \mid \underline{1} \right) \quad (6.22)$$

$\Omega_i^{++}$  : extended matrix  $\Omega_i^+$  (see 6.21).

$\underline{0}$  :  $L$ -dimensional zero vector.

$\underline{1}$  :  $L$ -dimensional vector filled with ones.

The extended equation to solve (with IV, CLS or other method) is:

$$\underline{y}_i^+ = \Omega_i^{++} \underline{\theta}_i^+ \quad (6.23)$$

where:  $\underline{y}_i^+ = \underline{y}_i + \underline{c}_i$  : defined as in  $\underline{y}_i$  (6.3)  
 $\underline{\theta}_i^+ = (\underline{\theta}_i \mid \Delta_i)$  : extended parameter vector.  
 $\underline{\Delta} = [\Delta_1, \dots, \Delta_i, \dots, \Delta_q]$  : vector containing  
 "information" about the offsets.

$\Delta_i$  is the offset seen from the equation error while  $\underline{c}_i$  is the offset on the output. This is illustrated by fig. 6.2 with a SISO system represented in the ARMA representation which is related to the MFD by:

$$z^{-n} p(z) = 1 + A(z^{-1}) \quad (6.24)$$

$$z^{-n} q(z) = b_0 + B(z^{-1})$$

where  $n$  is the order of the system.

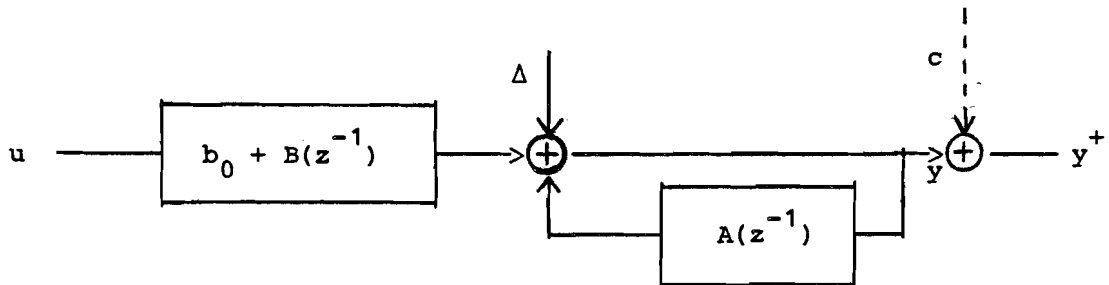


Fig. 6.2. I/O (ARMA) representation of SISO system.

Consequently:

$$c = \frac{\Delta}{1 + \sum_{i=1}^n a_i} \quad (6.25)$$

Because the I/O model might present non causal equations (see chapter 3), the simulation with original input data and the estimated model ( to build the bootstrap IV or to calculate the reconstruction error) will be performed with the state space model. We have to know how to introduce the offset estimated with an Input/Output model in a simulation in the state space. The state space model equivalent to (6.24) is showed in fig.6.3.

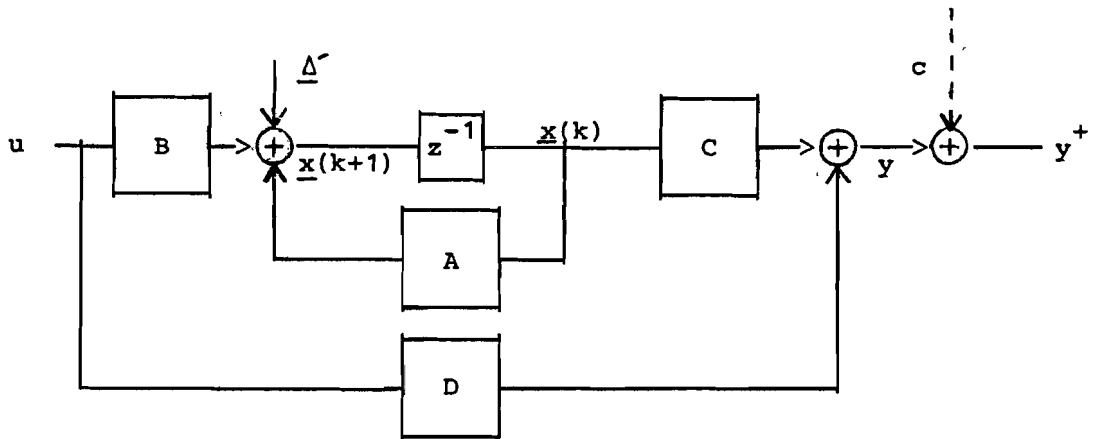


Fig. 6.3. State space representation of SISO system.

If we consider the following SISO second order state space model in canonical form:

$$\underline{x}(k+1) = \begin{bmatrix} 0 & 1 \\ \alpha_{11,1} & \alpha_{11,2} \end{bmatrix} \underline{x}(k) + \begin{bmatrix} b_{11} \\ b_{12} \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ \Delta^r \end{bmatrix}$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}(k) + \begin{bmatrix} d \end{bmatrix} u(k) \quad (6.26)$$

By transforming its equations to the I/O model we obtain:



$$\begin{bmatrix} z^2 - \alpha_{11,2} z - \alpha_{11,1} \end{bmatrix} y(k) = \begin{bmatrix} dz^2 + (b_{11} - \alpha_{11,2}^d)z - \alpha_{11,1}^d - \alpha_{11,2} b_{11} + \\ b_{12} \end{bmatrix} u(k) + \Delta^r \quad (6.27)$$

which is equal to (see (4.22) and 4.26)):

$$p_{11}(z) y(k) = q_{11}(z) u(k) + \Delta^r \quad (6.28)$$

This proves that for the SISO case we have :

$$\underline{\Delta}^r = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \Delta \end{bmatrix} \begin{matrix} \uparrow \\ n \\ \downarrow \end{matrix} \quad (6.29)$$

n : order of the system.

In the multivariable case, instead of using the relation between  $\underline{\Delta}$  and  $\underline{\Delta}^r$  (similar to (6.29)), we will derive from  $\underline{\Delta}$  the corresponding offset vector  $\underline{c}$  seen from the output. This is the quantity to add to the output vector in order to perform a correct simulation, and is independent of the type of model used (as long as they are equivalent).

The relation between  $\underline{\Delta}$  and  $\underline{c}$  can be obtained from (6.22) and (6.23):

$$\left[ \left( \underline{Q}_i \mid \underline{0} \right) + \left( \underline{Q}_i^c \mid \underline{1} \right) \right] \left( \underline{\theta}_i^T \mid \underline{\Delta}_i \right)^T = \underline{y}_i + \underline{c}_i \quad (6.30)$$

This equation also holds for the estimates  $\hat{\underline{\theta}}_i$ ,  $\hat{\underline{\Delta}}_i$  and  $\hat{\underline{c}}_i$  of  $\underline{\theta}_i$ ,  $\underline{\Delta}_i$  and  $\underline{c}_i$ . By recognizing the i-th system equation (6.3) in (6.30), we obtain:

$$\underline{Q}_i^c \hat{\underline{\theta}}_i + \hat{\underline{\Delta}}_i \underline{1} = \hat{\underline{c}}_i \quad (6.31)$$

Which consists in L identical equations :

$$\sum_{\ell=1}^q \left( \sum_{j=1}^{\delta_\ell} \hat{\alpha}_{i\ell,j} \hat{c}_j \right) + \underline{\Delta}_i = \hat{\underline{c}}_i \quad (6.32)$$

where:  $\delta_\ell$   $\ell=1, q$  are defined at (6.2d)

$\hat{\alpha}_{i\ell,j}$  estimates of the entries of system matrix A.

This relation stands for all q MISO sub-systems. We can obtain the offset vector  $\underline{c}$  by solving:

$$\chi \underline{\hat{c}} = \underline{\hat{\Delta}} \quad (6.33)$$

where:

$$\chi = \begin{bmatrix} \chi_{11} & \cdot & \cdot & \chi_{1q} \\ \vdots & & & \vdots \\ \chi_{q1} & & & \chi_{qq} \end{bmatrix}$$

$$\text{with : } \chi_{ii} = 1 - \frac{\delta_i}{\sum_{\ell=1}^i \hat{\alpha}_{ii,\ell}}$$

$$\chi_{ij} = - \frac{\delta_j}{\sum_{\ell=1}^j \hat{\alpha}_{ij,\ell}} \quad i \neq j$$

### 6.5 Some validity tests

Once the structural and parametric identification has been performed, rests the validation phase. In this phase it will be determined whether a model should be accepted or not to describe the behaviour of a process. Important remark: the user must be aware that the LS method (which minimises the equation error) and the IV method (which minimises the correlation between residuals and inputs) as they are presented in this report will not always minimise the "distance" between original and model outputs. In general we may expect that this "distance" will be small as long as the model describes the process in an accurate way. However there exists a particular class of models for which an equation error or cross-correlation minimisation leads to very bad reconstruction errors. Some research is being done at the department of Measurement and Control of the THE in order to define this class of models.

The following tests will give the user an idea about the simulation quality of the estimated model:

- 1 - Comparison on a plot of the original outputs and the simulated outputs of the model (with the original inputs on input).

- 2 - Calculation of the reconstruction error. This test is very close related to the precedent one because, for each output, it concentrates in one number the "relative distance" between original and model output. The reconstruction error is defined by:

$$RE(i)_{i=1,q} = \frac{\sum_{k=F}^L (\tilde{y}_i(k) - \hat{y}_i(k))^2}{\sum_{k=F}^L \tilde{y}_i(k)^2} = \frac{\sum_{k=F}^L \hat{e}_i^r(k)^2}{\sum_{k=F}^L \tilde{y}_i(k)^2} \quad (6.34)$$

where: -  $\tilde{y}_i(k)$ : measured output  $i$  at time  $k$ .  
 -  $\hat{y}_i(k)$ : model output  $i$  at time  $k$ .  
 -  $\hat{e}_i^r(k)$ : estimated output error (residual)  $i$  at time  $k$ .  
 -  $F$  is the first sample used to calculate the reconstruction error (and the crosscorrelation, see 6.36). It corresponds to an estimate of the length of the impulse response of the system and is defined by:

$$(\text{largest eigenvalue of system})^F = 10^{-4}$$

Assuming that the disturbances are not correlated with the outputs and, that the model provides a good reconstruction error ( $\hat{e}_i^r \approx e_i^r$ ), (6.34) can be written as:

$$\frac{1}{RE(i)} = \frac{\sum_{k=F}^L (y_i(k)^2 + \hat{e}_i^r(k)^2)}{\sum_{k=F}^L \hat{e}_i^r(k)^2} = (\hat{S/N})_i + 1 \quad (6.35)$$

where : -  $(S/N)_i$  is the estimated signal to noise ratio on output  $i$ .  
 -  $F$  defined as in (6.34).

- 3 - Crossvalidation. This is done by confronting the results from one set of data with the results from another independent set of data.

- 4 - Calculation of the crosscorrelation between the residuals (estimated output errors) and the inputs. When a process is identified with success, it also results in a good estimation of the disturbances on the outputs (in our case). These disturbances were supposed to be uncorrelated with the process signals. A deterioration of the estimated model will cause a deterioration of the estimates of the residuals. Then, the residuals will not only contain a coarse estimation of the noises, but also a term corresponding to the input data which has been filtered by the model. The crosscorrelation between inputs and residuals will increase according as the estimation becomes worse.

The crosscorrelation between input  $i$  and residual  $j$ , for time shift  $\tau$ , is defined by:

$$\text{crosscor}(i,j,\tau) = \frac{1}{L'-F} \sum_{k=F}^{L'} \hat{e}_{js}^{\tau}(k+\tau) u_{is}(k) \quad (6.36)$$

where :  $L'$ : adapted number of samples according to the maximum time shift  $\tau$  being used.  
 $F$  : defined as in (6.34).

Subscript  $s$  indicates that signals are standardised. As an example, the standardised signal from  $x(k)$  is given by:

$$x_s = \frac{x(k) - \text{mean}(x)}{\text{standard deviation}(x)} \quad (6.37)$$

A plot of the correlations for several time shifts  $\tau$  will give an idea about the validity of the estimated model.

## 7.0 Results

### 7.1 Choice of a test system

It is difficult to prove the validity of a method with the help of a few examples. Instead of performing many runs with different test systems which often leads to an enormous and hard to digest amount of data, we will build a system  $\{F, G, H, K\}$  in which several trouble sources arise. The following requirements were demanded to our test system:

- a - order 5
- b - 3 inputs
- c - 2 outputs
- d - 3 "slow" eigenvalues
- e - 2 "fast" eigenvalues
- f - eigenvalues should be close to each other (as well for the fast as for the slow eigenvalues)
- g - at least one pseudo-canonical form in which the system is not representable
- h - approximately equal energy of all modes in impulse response
- i - more or less balanced representation: the variances of the states are approximately of the same magnitude. This is important for the numerical stability of the simulation

In order to fulfil requirements a, d, e and f the chosen system matrix F was:

$$F = \text{diag} [ 0.08 , 0.9 , 0.1 , 0.92 , 0.95 ] \quad (7.1)$$

The chosen structure of the system was  $v_1 = 2$  ,  $v_2 = 3$  . This implies that the the system is represented in the basis

$$\underline{h}_1 \quad \underline{h}_2 \quad (F^T)\underline{h}_1 \quad (F^T)\underline{h}_2 \quad (F^T)^2\underline{h}_2 \quad (7.2)$$

and that

$$(F^T)^2 \underline{h}_1 = \alpha \underline{h}_1 + \beta \underline{h}_2 + \gamma (F^T) \underline{h}_1 + \delta (F^T) \underline{h}_2 \quad (7.3)$$

Property (7.3) of  $(F^T)^2 \underline{h}_1$  will provide the system with the desired structure 2,3 (as long as vectors (7.2) are linearly independent). A simple way to satisfy (7.3) is by choosing 3 of the 5 elements of vector  $\underline{h}_1$  equal to zero.  $F$  being diagonal, it is obvious that in this case

$$(F^T)^2 \underline{h}_1 = \alpha \underline{h}_1 + \gamma (F^T) \underline{h}_1 \quad (7.4)$$

the coefficients  $\beta$  and  $\delta$  being equal to zero. The only constraints on vector  $\underline{h}_2$  is that it has to be independent of  $\underline{h}_1$ . This confers on the matrix  $H$  the following structure:

$$H = \begin{bmatrix} x & x & 0 & 0 & 0 \\ x & x & x & x & x \end{bmatrix} \quad (7.5)$$

A direct consequence is that the pseudo-canonical representations with structure  $(\mu_1=3, \mu_2=2)$  and  $(\mu_1=4, \mu_2=1)$  are not representable (requirement g). The entries of the non-zero elements of  $H$  and of the elements of  $G$  will be fixed according to requirement  $h$  and  $i$ . The energy in each mode is defined as being the total energy in the impulse responses when only the relevant mode is excited. If we denote

$$F_i = \text{diag} [ 0, 0, \dots, \lambda_i, 0, \dots, 0 ]$$

with  $\lambda_i$ : eigenvalue corresponding to mode  $i$ .

The energy in the impulse response in mode  $i$  is:

$$EN(i) = \sum_{l=1}^p \sum_{m=1}^q \sum_{k=1}^{\infty} (H F_i^{k-1} G)_{ml}^2 \quad (7.6)$$

where  $ml$  indicates the entry  $m, l$  of the matrix.

$$EN(i) = \sum_{\ell=1}^p \sum_{k=1}^{\infty} \underline{f}_{-\ell}^T G^T F_i^{k-1} H^T H F_i^{k-1} G \underline{f}_{-\ell}$$

$$\text{where } \underline{f}_{-\ell}^T = \begin{bmatrix} \leftarrow & & & & & & \rightarrow \\ 0 & 0 & \dots & 1 & \dots & 0 & 0 \\ & & & \uparrow & & & \\ & & & \ell & & & \end{bmatrix}$$

$$EN(i) = \sum_{\ell=1}^p \sum_{k=1}^{\infty} \lambda_i^{2k-2} \underline{h}_{-i}^T \underline{h}_{-i} g_{i\ell}^2$$

where  $\underline{h}_{-i}$  is the  $i$ -th column of  $H$  and  $g_{i\ell}$  the element  $i, \ell$  of  $G$ .

$$EN(i) = \frac{\underline{h}_{-i}^T \underline{h}_{-i} \underline{g}_i^T \underline{g}_i}{1 - \lambda_i^2} \quad (7.7)$$

where  $\underline{g}_i^T$  is the  $i$ -th row of  $G$ .

The entries of  $H$  and  $G$  has been adjusted in such a way that:

$$EN(i) = 200 \begin{matrix} + \\ - \end{matrix} 10$$

$i=1, n$

where  $n$  is the order of the system ( $=5$ ).

Furthermore all entries are in the range of 0.5 till 9.0 which, together with matrix  $F$ , guarantees a rather balanced representation (see requirement  $i$ ).

Requirements  $a, b, \dots, i$  result in the following system configuration:

$$F = \begin{bmatrix} 0.08 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.9 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.1 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.92 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.95 \end{bmatrix} \quad (7.8 \text{ a})$$

$$G = \begin{bmatrix} 1.0 & 0.0 & -1.0 \\ 1.0 & 1.7 & 0.0 \\ 0.0 & 1.0 & -1.5 \\ 7.0 & 0.0 & 8.0 \\ 0.0 & 7.0 & 5.0 \end{bmatrix} \quad (7.8 \text{ b})$$

$$H = \begin{bmatrix} 4.0 & 3.0 & 0.0 & 0.0 & 0.0 \\ 9.0 & 1.0 & 8.0 & -0.5 & 0.5 \end{bmatrix} \quad (7.8 \text{ c})$$

$$K = \begin{bmatrix} 0.1 & 0.3 & 0.5 \\ 0.2 & 0.7 & 0.4 \end{bmatrix} \quad (7.8 \text{ d})$$

The choice of the matrix K was free. No special constraints were imposed to this matrix.

The corresponding canonical representation can be obtained by performing the similarity transformation defined by (see par. 3.2)

$$T = [ \underline{h}_1 \quad (F^T)\underline{h}_1 \quad \underline{h}_2 \quad (F^T)\underline{h}_2 \quad (F^T)^2\underline{h}_2 ]^T \quad (7.9)$$

$$T = \begin{bmatrix} 4.0 & 3.0 & 0.0 & 0.0 & 0.0 \\ 3.2 & 2.7 & 0.0 & 0.0 & 0.0 \\ 9.0 & 1.0 & 8.0 & -0.5 & 0.5 \\ 0.72 & 0.9 & 0.8 & -0.46 & 0.475 \\ 0.058 & 0.81 & 0.08 & -0.423 & 0.451 \end{bmatrix} \quad (7.10 \text{ a})$$

$$T^{-1} = \begin{bmatrix} 0.274 & -0.304 & 0.0 & 0.0 & 0.0 \\ -0.032 & 0.406 & 0.0 & 0.0 & 0.0 \\ -0.324 & 0.359 & 0.157 & -0.335 & 0.179 \\ -3.600 & 5.203 & 7.723 & -85.37 & 86.30 \\ -3.295 & 4.125 & 7.216 & -80.00 & 78.43 \end{bmatrix} \quad (7.10 \text{ b})$$



$$\begin{array}{l}
 A = TFT^{-1} = \\
 \text{struct. } 2,3
 \end{array}
 \begin{bmatrix}
 0. & 1. & 0. & 0. & 0. \\
 -0.072 & 0.980 & 0.000 & 0.000 & 0.000 \\
 0. & 0. & 0. & 1. & 0. \\
 0. & 0. & 0. & 0. & 1. \\
 -0.036 & 0.040 & 0.087 & -1.061 & 1.970
 \end{bmatrix}
 \quad (7.11 \text{ a})$$

Notation: - 0. (resp. 1.) indicates a structural zero (resp. 1.).  
 - 0.000 (resp. 1.) indicates the rounded value of an  $\alpha$  parameter which does not have to be equal to zero (resp. to one).

$$\begin{array}{l}
 B = TG = \\
 \text{struct. } 2,3
 \end{array}
 \begin{bmatrix}
 7.000 & 5.100 & -4.000 \\
 3.020 & 4.590 & -0.320 \\
 6.500 & 13.20 & -22.50 \\
 -1.600 & 5.655 & -3.225 \\
 -2.095 & 4.616 & -1.301
 \end{bmatrix}
 \quad (7.11 \text{ b})$$

$$\begin{array}{l}
 C = HT^{-1} = \\
 \text{struct. } 2,3
 \end{array}
 \begin{bmatrix}
 1. & 0. & 0. & 0. & 0. \\
 0. & 0. & 1. & 0. & 0.
 \end{bmatrix}
 \quad (7.11 \text{ c})$$

Matrix D is equal to matrix K.

By performing a similar transformation the system can be represented in the pseudo-canonical form with structure 1,4. This leads to the following representation.

$$\begin{array}{l}
 A = TAT^{-1} = \\
 \text{struct. } 1,4
 \end{array}
 \begin{bmatrix}
 0.893 & -2.162 & 26.243 & -48.73 & 24.73 \\
 0. & 0. & 1. & 0. & 0. \\
 0. & 0. & 0. & 1. & 0. \\
 0. & 0. & 0. & 0. & 1. \\
 0.0002 & -0.008 & 0.179 & -1.232 & 2.057
 \end{bmatrix}
 \quad (7.12 \text{ a})$$

$$\begin{array}{l}
 \text{B = TG =} \\
 \text{struct. 1,4}
 \end{array}
 \begin{bmatrix}
 7.000 & 5.100 & -4.000 \\
 6.500 & 13.20 & -22.500 \\
 -1.600 & 5.655 & 3.225 \\
 -2.094 & 4.616 & -1.307 \\
 -1.992 & 4.248 & -0.988
 \end{bmatrix}
 \quad (7.12 \text{ b})$$

$$\begin{array}{l}
 \text{C = HT}^{-1} = \\
 \text{struct. 1,4}
 \end{array}
 \begin{bmatrix}
 1. & 0. & 0. & 0. & 0. \\
 0. & 1. & 0. & 0. & 0.
 \end{bmatrix}
 \quad (7.12 \text{ c})$$

Matrix D is equal to matrix K.

These representations will also be obtained in the deterministic case when the system is identified into a pseudo- and canonical form with structure (2,3) and into a pseudo-canonical form of structure (1,4) (see further results).

The similarity transformation matrices that transform to structures (3,2) and (4,1) are singular because they contain both the dependent vector  $(F^T)^2 \underline{h}_1$ . The system is not representable in these two structures ( as demanded by requirement g).

## 7.2 Overview of the implemented methods and facilities.

Because of lack of time not all the methods and extensions presented in this report were implemented. Nevertheless, I wanted to treat them in order to give a more general overview to the reader and to be able to propose some recommendations (see section 7.4) for future research. The implemented methods were:

### - Structural identification.

The available software was not very programmers friendly, so I decided to rewrite this part starting from zero. The determinant and the least singular value method were implemented. Because of earlier remarks (similarity with determinant method), no attention was paid to the Range

Error method.

- Parametric identification.

Inspired by and making use of a part of the programs developed by Jean Pfennings (see Pfennings (1983)), the IV method has been implemented with the following extensions:

- preprocessing part of the signals (mean correction, scaling).
- estimation of D matrix (proper systems).
- estimation of systems in canonical and pseudo-canonical form.
- allowing systems with offsets on inputs or/and outputs.
- improvement of the validity tests.
- stability projection of (unstable) models.

The Compensated Least Squares method has not been implemented. Because of the better properties of the IV method, we gave this last method a higher priority. However, a Least Squares step can be chosen as first iteration of the IV method. The reader interested in the implementation of the Compensated Least Squares method can refer to Meertens (1983).

### 7.3 Presentation of the results

All the experiments that will be presented in this section have been performed using data files containing 1000 samples and generated with the simulation program SYSSIMUL. The model used was the one presented in section 7.1. The input signals were white sequences (zero mean and standard deviation equal to 1) and the noises were white and colored noise. The filters used to color the noises were defined by the following state space representation:

$$A_F = \begin{bmatrix} 0.7 & 0.0 \\ 0.0 & 0.8 \end{bmatrix} \quad B_F = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \quad (7.13)$$

$$C_F = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \quad D_F = \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}$$

(7.13) represents two first order filters without any interaction. During all the experiments equal S/N will be applied on both outputs. The following abbreviations has been used in some plots:

S/N : signal to noise ratio (in dB)  
 PCAN14 : pseudo-canonical form, structure  $\mu_1=1, \mu_2=4$   
 CAN23 : canonical form, structure  $\nu_1=2, \nu_2=3$   
 ITER : iterations  
 INF : infinity (used to indicate that no noise is present)  
 LS : Least Squares (thus only one iteration)

#### Structural identification

On fig. 7.1.1, 2, 3 and 4 we see the results obtained with the Least Singular value method for S/N = 80 dB, 60 dB, 40 dB and 20 dB. For S/N= 80 dB, 40 dB, 20 dB and 0 dB, the results using the determinant method are plotted on fig. 7.1.5, 6, 7 and 8. During all these experiments we used white noise sequences to disturb the outputs of the system. The threshold value used to detect dependency was obtained in an empiric way during previous runs. The true structure (2,3) was found by the Least Singular Value method and by the determinant method for S/N > 40 dB. For the LSV method, a threshold value derived from the noise intensity and the number of samples would give better results (see recommendations, section 7.4).

The increasing noise (from 80 dB to 0 dB) results in a lowering of the estimated structure. This fact was confirmed by the parametric identification, since for S/N < 40 dB the reconstruction error for structure (2,3) was almost equal to the reconstruction error for structure (2,2). The slowest eigenvalue ( $\lambda = 0.95$ ) disappeared when the system was iden-

tified with structure (2,2).

### Parametric identification

In fig. 7.2.1 to fig 7.2.4 the reconstruction errors are plotted as a function of the S/N on both outputs ( 100 dB, 80 dB, ..., 20 dB, 0 dB). This has been done with white and colored noise for the pseudo-canonical form, structure (1,4) and for the pseudo-canonical form, structure (2,3). The reconstruction errors were obtained by taking the average value of 20 runs with I/O data generated with 20 different start values. The standard deviation was smaller than 1/10 of the reconstruction error itself. This indicates the good reproducibility of the method.

For high S/N we can remark that an increase of the noise from 20 dB (noise energy multiplied by 100), corresponds exactly, as it is stated by equation (6.35), to a decrease in a factor 100 of the reconstruction error (two divisions of the plots). This indicates that the noise, and therefore also the process, is identified in a good way. This is not any more the case for low S/N, since the estimations become worse.

When structure (2,3) is used (fig 7.2.3 and 4), the reconstruction error 2 is worse than the one on output 1. This may be due to the fact that only two modes are present on output 1 (see test system 7.8 a and c) and that the slowest mode ( corresponding to  $\lambda=0.95$ ) is only present in output 2 which could introduce numerical problems.

We can also remark that the estimations obtained with I/O data corrupted by colored noise are slightly better than those obtained with I/O data corrupted by white noise. The most probable reason is that the equation error is possibly 'more white' in case of coloured output noise. Seen from the equation error, the white noise on output appears as colored noise ( because of the filtering effect of the autoregressive part of the system). In the same way, a coloring of the disturbances may result in a 'more white' equation error which will lead to better estimates and therefore a smaller reconstruction error.

In these figures we can also recognise the structure (2,3) of the system. A decrease (from 2 to 1) of the order of MISO subsystem 1 results in a strong increase of the reconstruction error, while an increase of the

order of MISO subsystem 2 (from 3 to 4) doesn't lead to a reduction of the reconstruction error.

On fig. 7.3.1 to fig. 7.3.13 we have some plots of the crosscorrelation between the inputs and the residuals for different structures ( (pseudo-) canonical 1,4 and 2,3), different number of iterations ( 1 (LS), 2, 15, and 20 iterations) and different S/N (inf., 40 dB, 20 dB, and 0 dB). All the noises corresponding to these plots are white. A reliability interval is defined (see dotted lines on plots) by  $[-3\sigma, +3\sigma]$  where  $\sigma$  is the standard deviation of the crosscorrelation:

$$\sigma = \frac{\sqrt{(\Psi_{rr}(0) \Psi_{ii}(0))}}{\sqrt{L}}$$

where:  $\Psi_{rr}(0)$ : autocorrelation of residual  $r$  ( $\tau=0$ )  
 $\Psi_{ii}(0)$ : autocorrelation of input  $i$  ( $\tau=0$ )  
 $L$  : number of samples

Because we are dealing with standard signals (see section 6.5) we have:

$$\sigma = 1 / \sqrt{L} = 0.0316$$

so that the reliability interval is  $\approx [-0.1, 0.1]$ . Crosscorrelations inside of this interval can be considered as being zero (no correlation). Inputs and residuals are completely correlated when the correlation equals 1 or -1.

In the case that no noise is applied, the residuals only contains disturbances corresponding to computer rounding and numerical errors. This can be recognised in the little stochastic character of the crosscorrelations on fig. 7.3.1 and 2.

Wrong structures (canonical (1,4)) can be recognised by their large crosscorrelation (see fig. 7.3.7, 10 and 13). The fact that the least squares method results in bad (biased) estimates can be seen by comparing fig. 7.3.6 (LS), 8 (LS) and 11 (run with "enough" - until stabilisation

of reconstruction error - iterations). Because of the little noise, the bias can hardly be noticed for  $S/N = 40$  dB (fig. 7.3.3 and 4).

It can also be remarked that (afortunately) the large correlations only appears for  $\tau > 0$  (delayed inputs). This confirms the causality principle.

The original and model output can be compared on fig. 7.4.1 to 7.4.8 for  $S/N = \text{INF}$ , 40 dB, 20dB and 0 dB (all with white noise). For correct structures and enough iterations no difference between original and model output can be appreciated for  $S/N > 20$  dB. Wrong structures lead to bad reconstructions (see fig. 7.4.1, 2, 3 and 6). Correct structures and enough iterations results in good fits (see fig. 7.4.4 and 7). A slight improvement as a result of enough iterations, instead of a single Least Squares step, can be appreciated by comparing fig. 7.4.4 and fig 7.4.5.

#### Experiments with practical data

The practical data that has been used was recorded at the glas production plant of Windschoten where quartz tubes are manufactured. This product is used in halogene lamps ( street illumination, auto industry, ...).

The approximate dimensions of the tube are: diameter  $\approx 1$ cm

wall thickness:  $\approx 0.08$  cm

The units on the plots do not correspond to physical units.

The basic components of the final product, ( silicium and others) are melted in a tank furnace. Because of gravity effects, the liquified glas leaves the tank through an orifice in its bottom. Through a small tube located in the middle of this orifice a special gas is blown in order to provide the outcoming, solidifying glas a cylindric form. The solid glas tube is pulled (or pushed) so that its drawing speed and therefore some of its physical properties can be controlled.

This process can be seen as a MIMO system with 2 inputs and two outputs, which are:

Input 1 : gas flow

Input 2 : drawing speed  
 Output 1 : wall thickness of quartz tube  
 Output 2 : diameter of quartz tube

Other factors that might influence this process ( temperature of glas and ambient, humidity ...) are considered as disturbances.

The data used was read from file SIGAN7.DAT (in PICOS format) which was previously preprocessed (filtered and drift correction) by members of the PICOS group. The first 800 samples of this file (from a total number of 802) were used during the identification experiments.

On fig. 7.5.1. a) and b) we see the plots corresponding to the structural identification ( resp. Least Singular Value and Determinant method). From both plots we can (rougly) derive the order of the MISO subsystem corresponding to output 2;  $\nu_2 = 2$ , since for  $\delta_2 > 3$  no important downward shift can be remarked. Since no downward shift can be appreciated for  $\delta_1 > 2$ , the order of subsystem could be  $\nu_1 = 1$ . However parameter identifications with different structures indicated that (2,2) lead to a better representaion (smaller reconstruction errors).

On fig. 7.5.2 a) and b) we see the original and simulated output in the case that the system is identified in the pseudo-canonical form with structure (2,2). The model was obtained after 20 iterations.

The fit on output 1 is bad. The shape of the original signal can hardly be recognised in the model simulation. If we consider the output 2, we can remark that the simulation fits quite well with the original signal.

On fig. 7.5.3. we have the crosscorrelations between inputs and residuals. No important correlations can be noticed. The correlations between inputs and residual 1 is smaller than the correlation between inputs and residual 2. This is in contradiction with the fact that the model has a better fit on output 2.

Identification using higher structures ( (2,3), (3,2) and (3,3) ) did not lead to a significant decrease of the reconstruction errors. The obtained model was:



$$A = \begin{bmatrix} 0. & 1. & 0. & 0. \\ 0.107 & 0.600 & 0.065 & -0.008 \\ 0. & 0. & 0. & 1. \\ -0.003 & -0.139 & -0.320 & 1.171 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.037 & -0.054 \\ -0.032 & -0.062 \\ -0.010 & -0.227 \\ 0.024 & -0.351 \end{bmatrix}$$

$$C = \begin{bmatrix} 1. & 0. & 0. & 0. \\ 0. & 0. & 1. & 0. \end{bmatrix}$$

$$D = \begin{bmatrix} -0.039 & -0.104 \\ -0.012 & -0.064 \end{bmatrix}$$

The eigenvalues of A are: -0.146  
 0.768 + i 0.138  
 0.768 - i 0.138  
 0.382

No definitive explanation has been found to explain the bad fit on output 1. However, some possible reasons are:

- to much noise on output 1.
- incorrect preprocessing (filtering, drift correction, delay estimation)
- incorrect choise of sample frequency

#### 7.4 Recommendations for further research

As a result of my experiences with the identification methods presented in this report I would like to give some recommendations for future investigation:

- Structural identification.

- study of the distribution of the least singular value of matrix  $\tilde{S}$  as a fonction of the different S/N and the number of samples . More specifically: how to define a threshold value that will indicate (with a high probability ) the singularity of matrix S from the knowledge of matrix  $\tilde{S}$  (see section 5.3.1 and Purviance (1980)).

- investigate a remark in Stoica (1984): the disturbances will lead to an over-estimation of the structure of the system. Our first results indicate the opposite.

- study and implementation of an IV-based structural identification method to decrease the influence of noise ( see Young (1980).
  
- Parametric identification.
  - study and implementation of the optimal IV method presented in Stoica, Soderstrom (1983).
  - investigate a way of choosing, according to a certain criterion, the best out of different possible pseudo-canonical parametrisations (see van Overbeek (1982) and Gevers and Wertz(1982)).

## 8.0 Conclusion

During this project I have been working on several identification methods that estimate the structure and the parameters of linear, time-discrete, multivariable systems.

Although the system used to test the behaviour of these methods contained several trouble sources, the experiments performed with the structural and parametric identification algorithms implemented during this work gave reasonable results.

For  $S/N > 40$  dB the Determinant method and the Least Singular Value method identified the true structure (2,3) of the system. For lower  $S/N$  both structural identification methods indicated a structure (2,2). Nevertheless, the parametric identification confirms that for  $S/N < 40$  dB, structure (2,3) and structure (2,2) result in almost the same reconstruction errors. The Least Singular Value method has some advantages that makes it quite attractive for future research.

Although the IV method (parametric identification) is not based in the minimisation of the reconstruction error, the simulation quality of the models obtained with this method was good. Good fits were obtained for all  $S/N$  (from 100 dB to 0 dB). The crosscorrelations and the reconstruction errors seem to be usefull validation tools in order to decide wheter a model should be accepted or not.

The experiments performed with practical data gave different results. The reconstruction error and the quality of the simulation on one of the two outputs was good. This was not the case for the other output where a bad fit and high reconstruction error was obtained. No significant correlation between inputs and residuals was remarked.

LIST OF SYMBOLS

|                             |  |
|-----------------------------|--|
| A                           | system matrix in (pseudo-)canonical form   |
| $A(z^{-1})$                 | autoregressive part of ARMA model  |
| $\alpha_{ij,k}$             | entry of matrix A (= - coefficients of polynomial $p_{ij}(z)$ )                    |
| B                           | distribution matrix in (pseudo-)canonical form                                     |
| $b_0 + B(z^{-1})$           | moving average part of ARMA model  |
| $\beta_{i,ik}$              | coefficients of polynomial $q_{ij}(z)$   |
| $\bar{B}$                   | matrix which contains the estimated MA parameters in a special way (see chapter 4) |
| C                           | output matrix in (pseudo-)canonical form   |
| D                           | Input/Output matrix in (pseudo-)canonical form                                     |
| $\Gamma$                    | special constructed matrix (see chapter 4)   |
| $\underline{e}_i(k)$        | equation error vector  |
| $\underline{e}'_i(k)$       | output error vector  |
| $\hat{\underline{e}}'_i(k)$ | estimated output error vector  |
| F                           | system matrix  |
| $\Phi$                      | matrix containing the entries of matrices B and D                                  |
| G                           | distribution matrix  |
| H                           | output matrix  |
| K                           | Input/Output matrix  |
| k                           | sample time  |

|                     |  |
|---------------------|--|
| $\theta_{-i}$       | parameter vector of the i-th subsystem   |
| $\hat{\theta}_{-i}$ | estimated parameter vector of the i-th subsystem   |
| $\theta_{-it}$      | true parameter vector of the i-th subsystem  |
| L                   | number of samples being used for the estimation  |
| $L_k(\dots)$        | matrix constructed with I/O sample vectors   |
| M                   | transformation matrix that contains the estimated AR parameters in a special way (see chapter 4) |
| $\mu_i$             | Kronecker number   |
| n                   | dimension of the system  |
| $\nu_i$             | Kronecker invariant  |
| $P(z)$              | left matrix fraction   |
| $p_{ij}(z)$         | polynomial of left matrix fraction $P(z)$  |
| p                   | number of inputs   |
| $Q(z)$              | right matrix fraction  |
| $q_{ij}(z)$         | polynomial of right matrix fraction $Q(z)$   |
| q                   | number of outputs  |
| $R(\dots)$          | matrix constructed with I/O sample vectors   |
| $RE(i)$             | reconstruction error on output i   |
| $S(\dots)$          | information matrix   |
| $\tilde{S}(\dots)$  | noise disturbed information matrix   |
| S/N                 | signal to noise ratio  |

|                              |   |
|------------------------------|---|
| $\sigma_i$                   | i-th singular value of a matrix                       |
| $\lambda_i$                  | i-th eigenvalue of matrix                             |
| t                            | time  |
| T                            | transformation matrix                                 |
| <u>u</u>                     | input vector  |
| V(z)                         | special constructed matrix (see chapter 4)            |
| X                            | special constructed matrix (see chapter 4)            |
| <u>x</u>                     | state vector  |
| <u>y</u>                     | output vector   |
| $\tilde{\underline{y}}_i(k)$ | noise disturbed output vector                         |
| $\hat{\underline{y}}_i(k)$   | estimated output vector                               |
| $z^{-1}$                     | unit delay operator                                   |
| Z(z)                         | special constructed matrix (see chapter 4)            |
| $Z_i$                        | instrumental variable for subsystem i                 |
| $\Omega_i$                   | matrix containing input-output samples                |
| $\tilde{\Omega}_i$           | matrix containing input and disturbing output samples |

STRUCTURAL IDENTIFICATION  
LEAST SINGULAR VALUE METHOD.

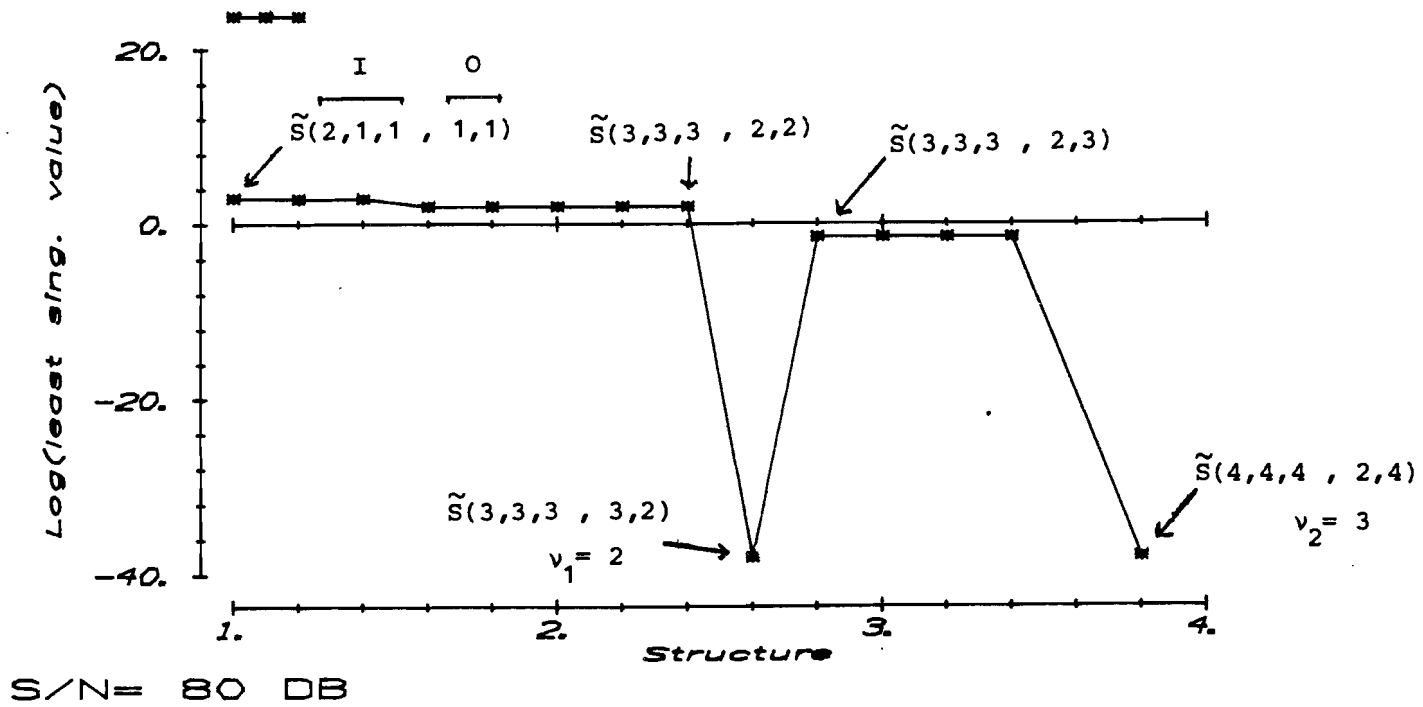


Fig. 7.1.1 Structural identification. LSV method (S/N=80 dB)

STRUCTURAL IDENTIFICATION  
LEAST SINGULAR VALUE METHOD.

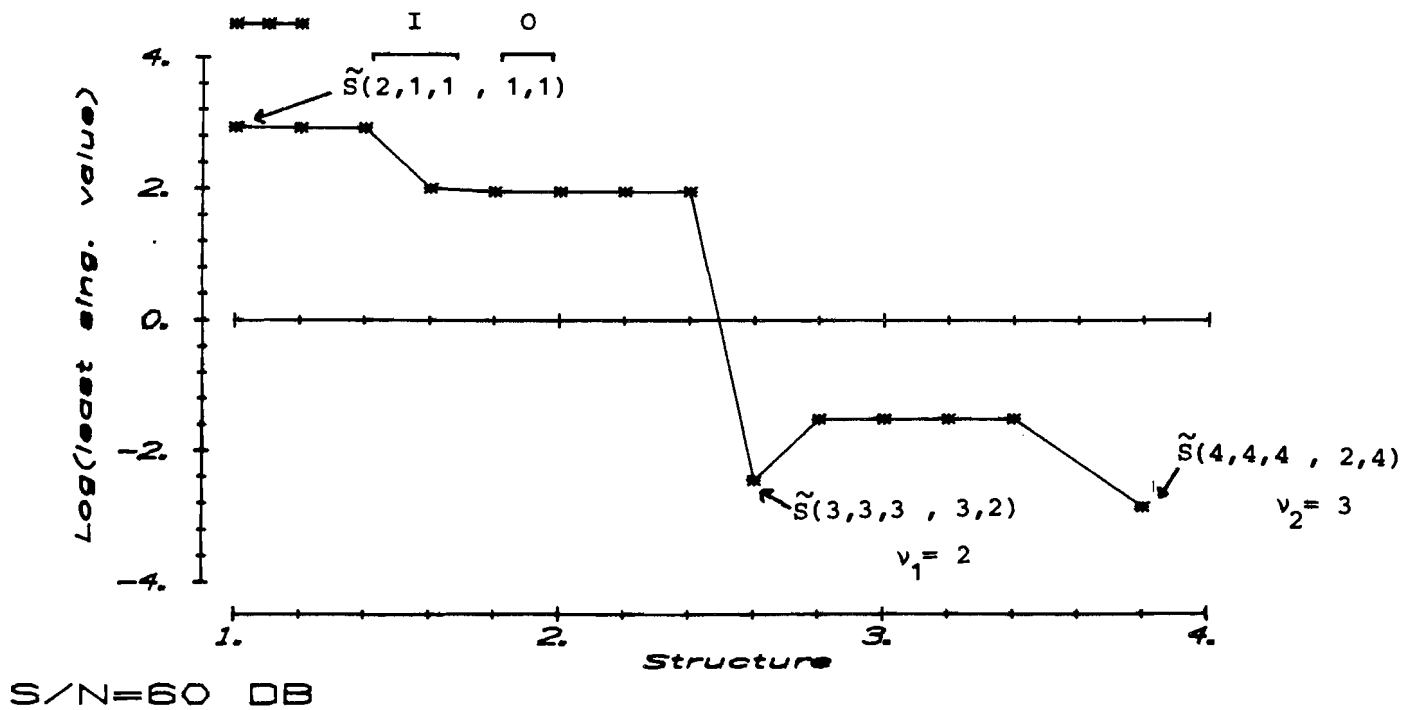
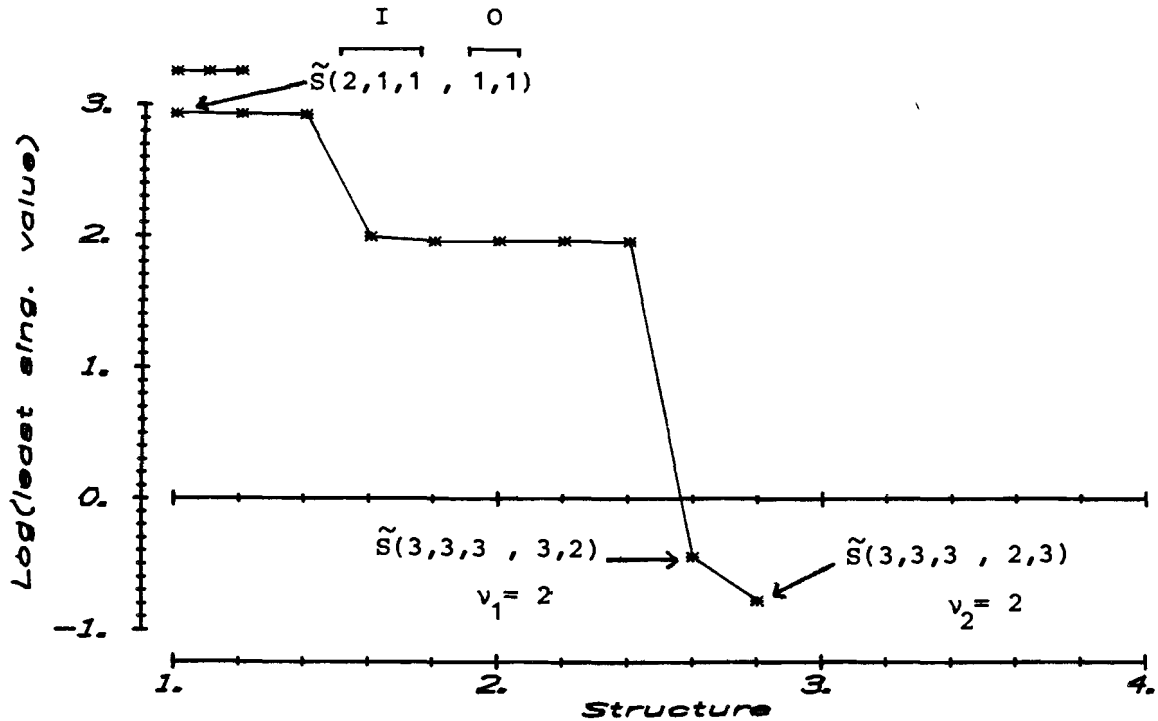


Fig. 7.1.2 Structural identification. LSV method (S/N=60 dB)

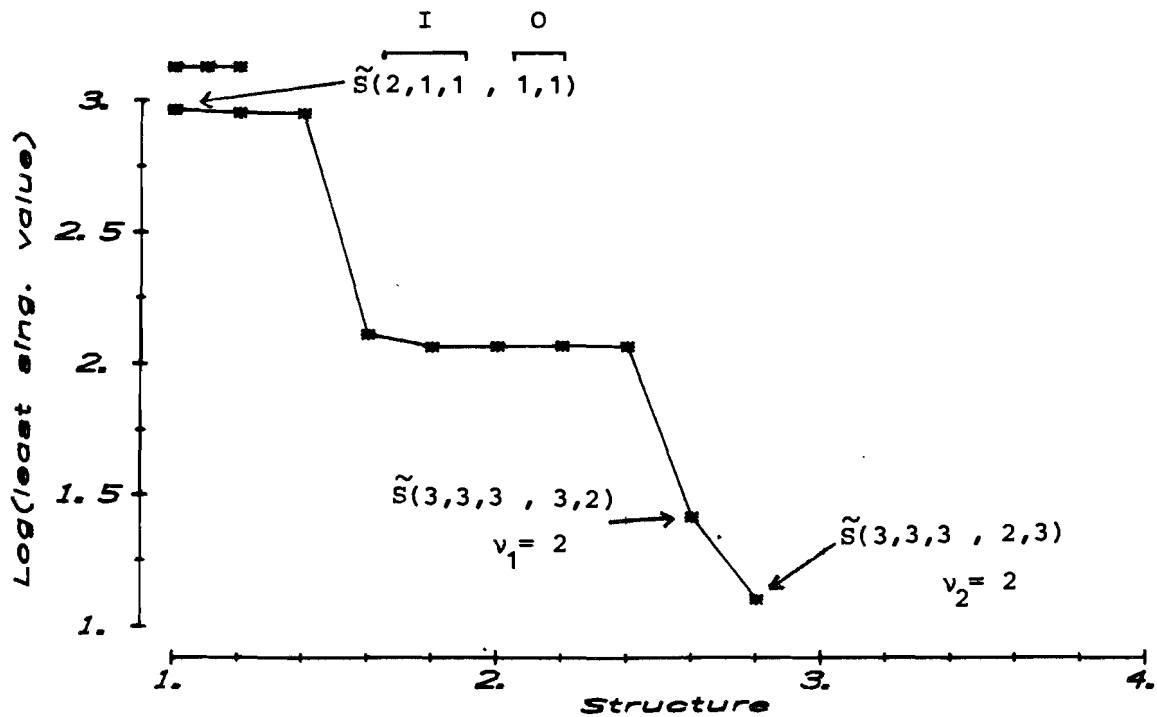
STRUCTURAL IDENTIFICATION  
LEAST SINGULAR VALUE METHOD.



S/N = 40 DB

Fig. 7.1.3 Structural identification. LSV method (S/N=40 dB)

STRUCTURAL IDENTIFICATION  
LEAST SINGULAR VALUE METHOD.



S/N=20 DB

Fig. 7.1.4 Structural identification. LSV method (S/N=20 dB)



STRUCTURAL IDENTIFICATION  
DETERMINANT METHOD

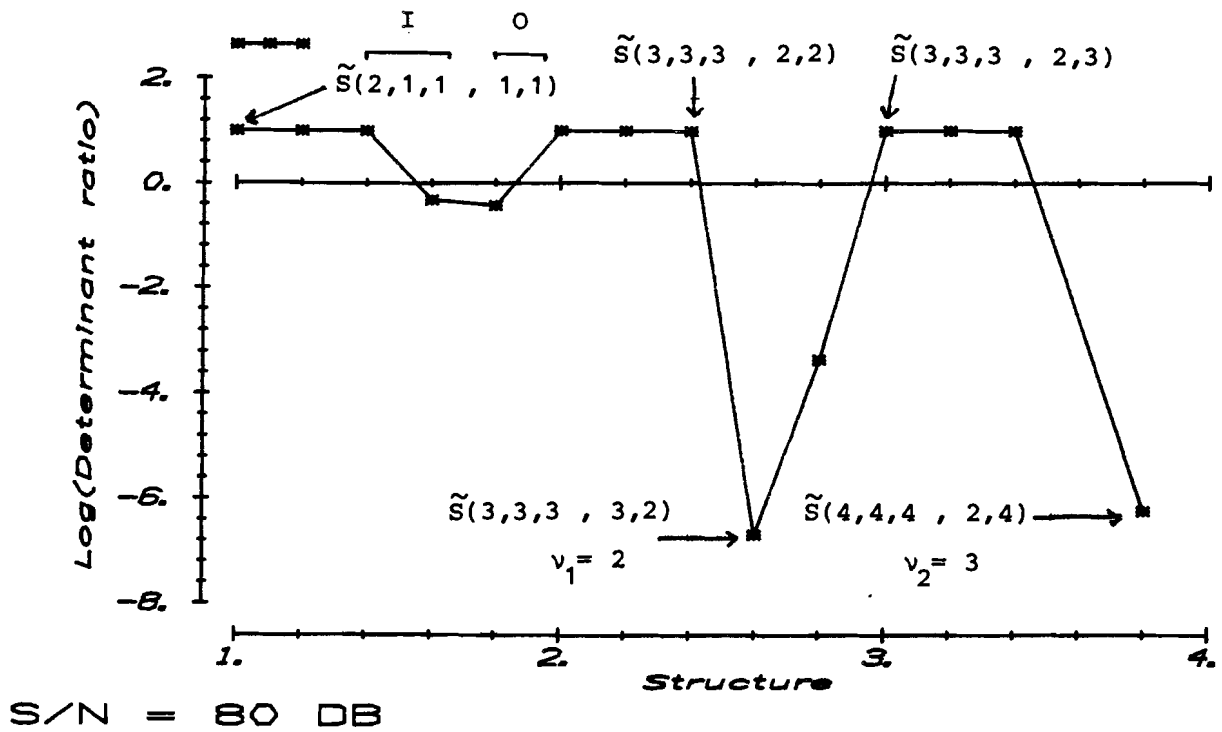


Fig. 7.1.5 Structural identification. Det. method (S/N=80 dB)

STRUCTURAL IDENTIFICATION  
DETERMINANT METHOD

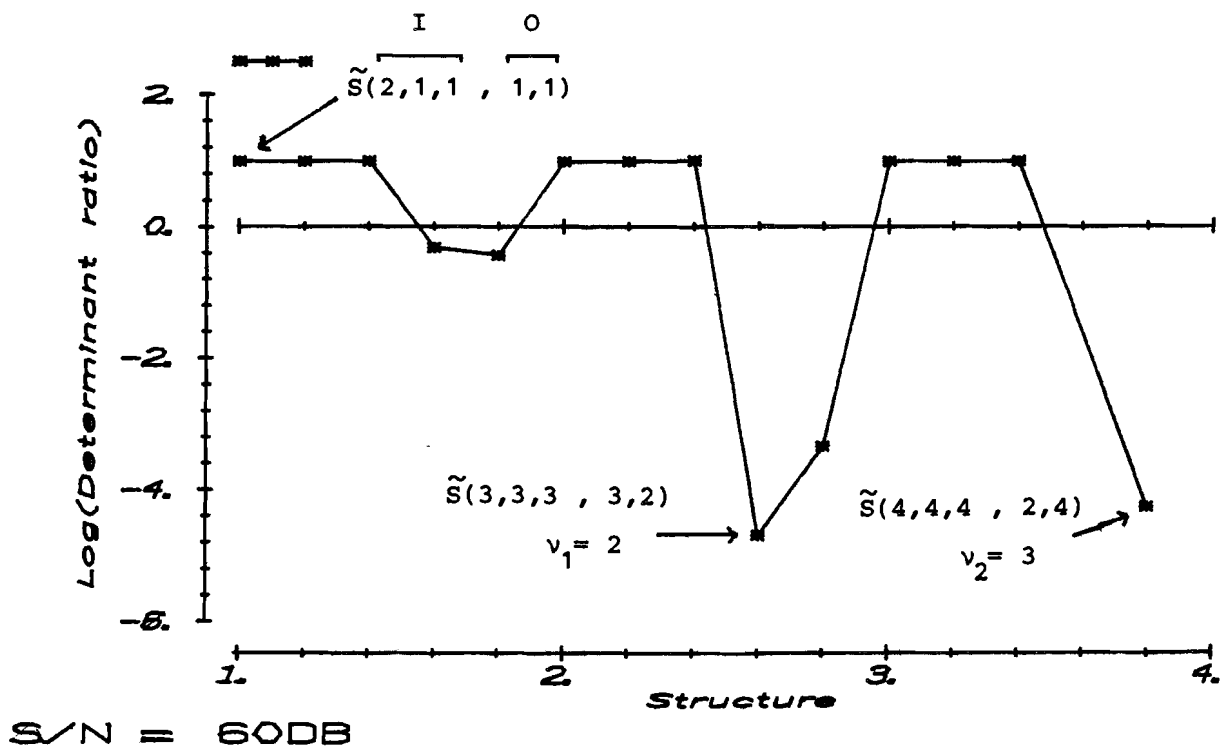
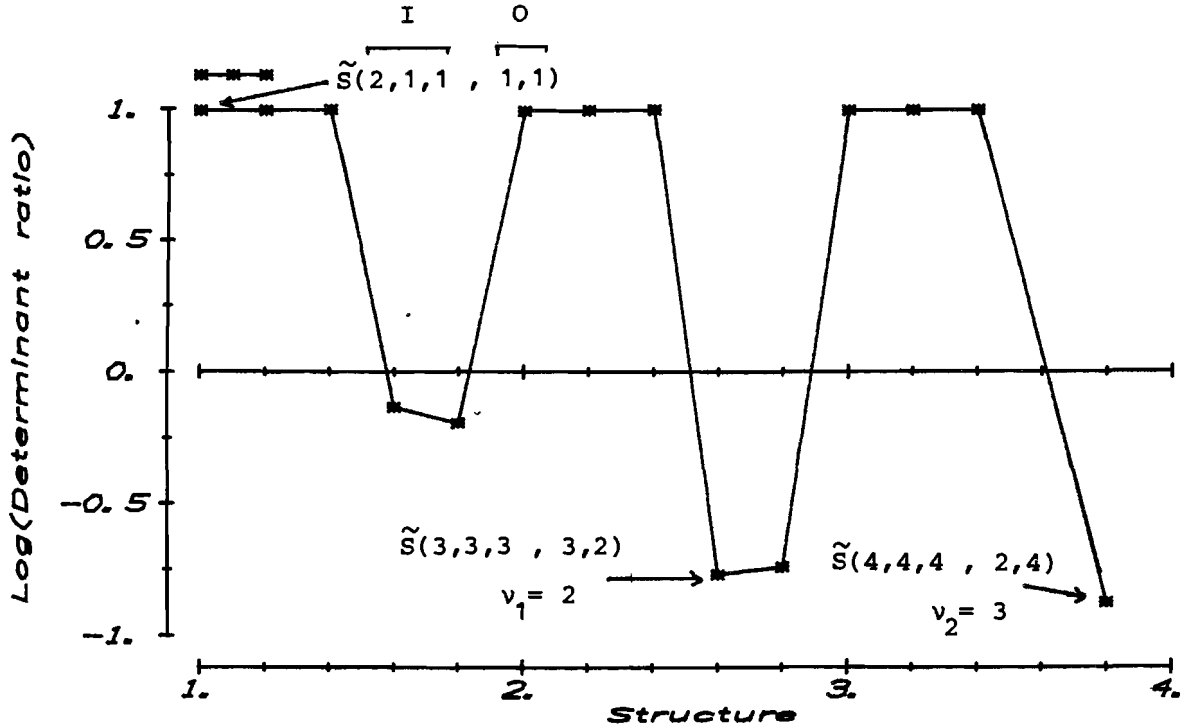


Fig. 7.1.6 Structural identification. Det. method (S/N=60 dB)

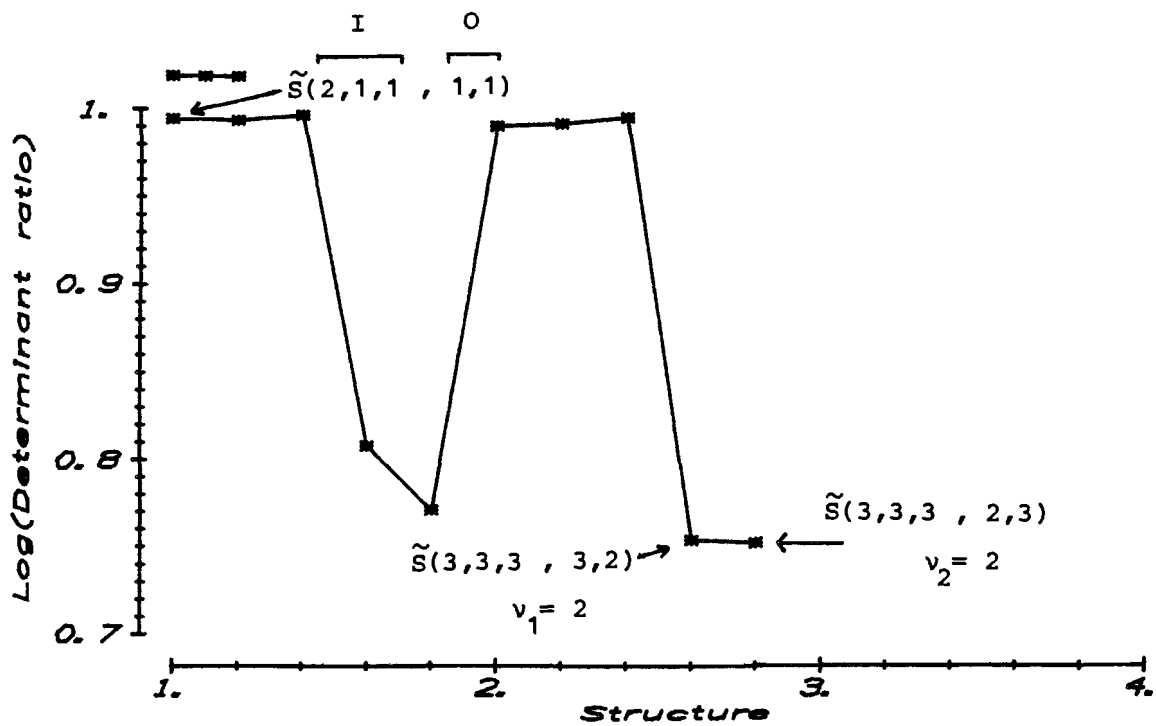
STRUCTURAL IDENTIFICATION  
DETERMINANT METHOD



S/N = 20 DB

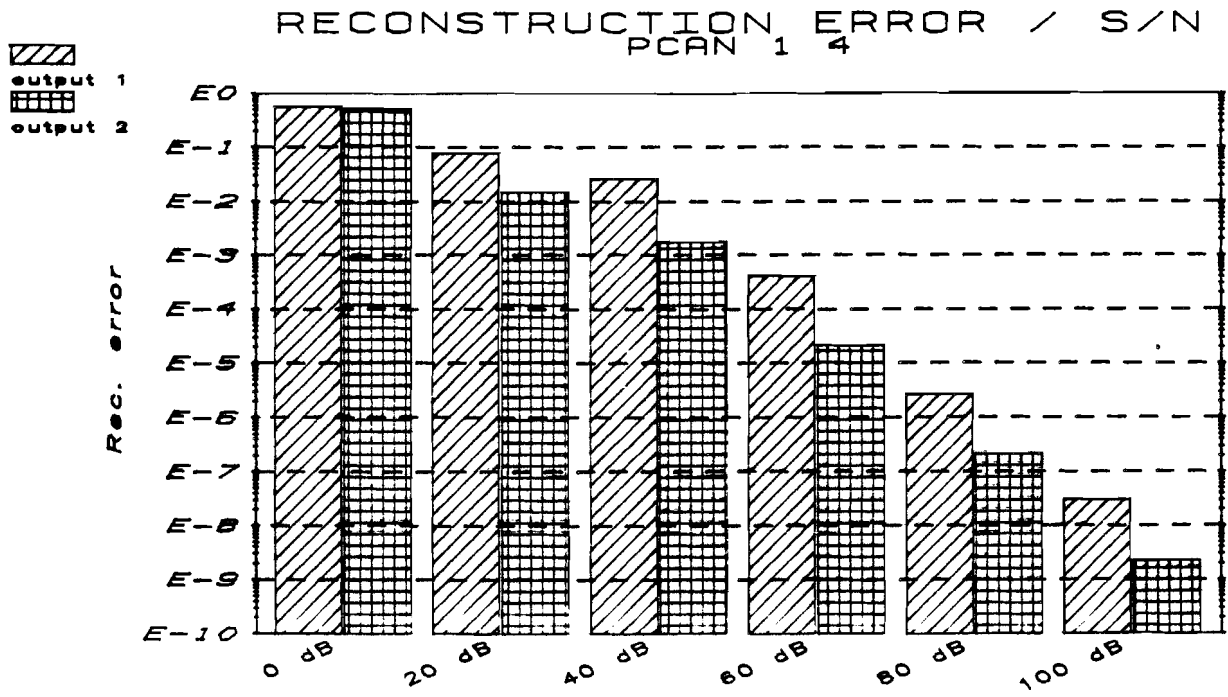
Fig. 7.1.7 Structural identification. Det. method (S/N=20 dB)

STRUCTURAL IDENTIFICATION  
DETERMINANT METHOD



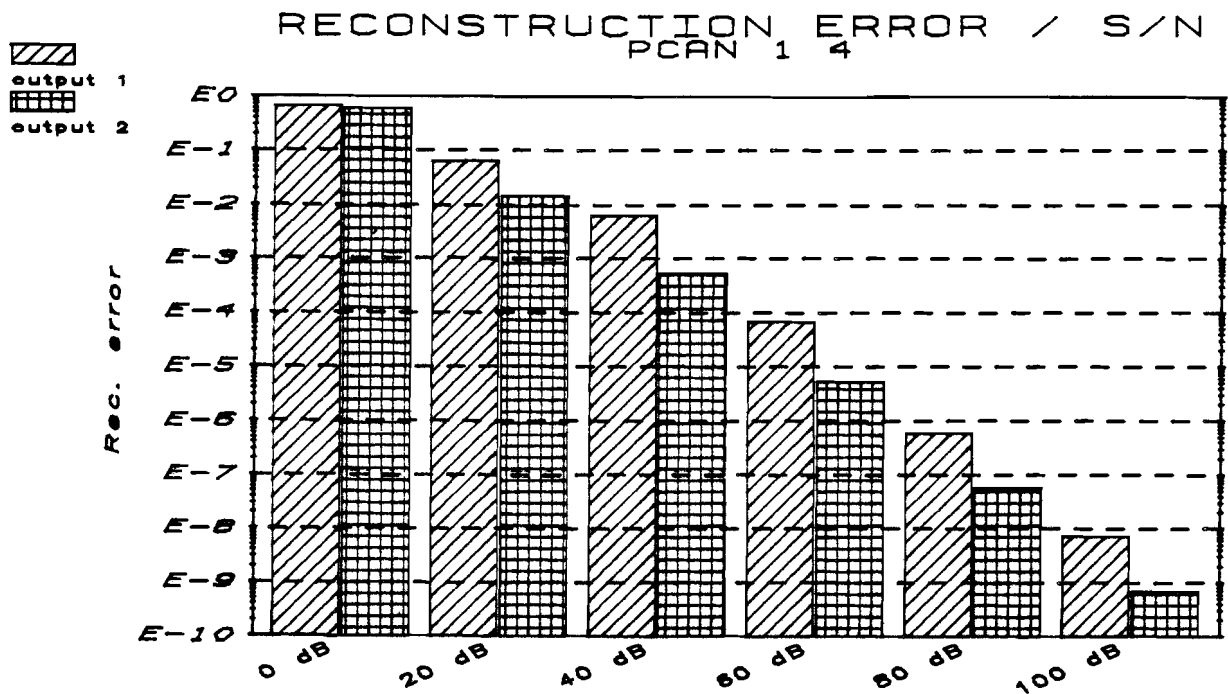
S/N = 0 DB

Fig. 7.1.8 Structural identification. Det. method (S/N= 0 dB)

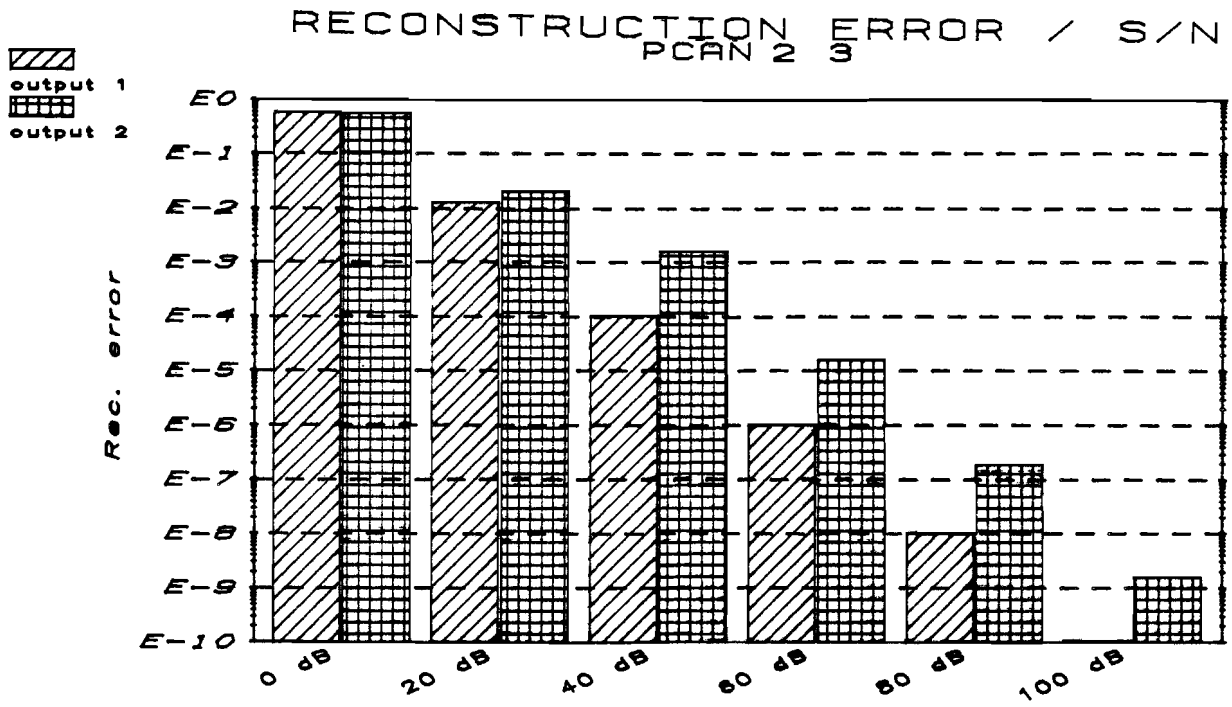


WHITE NOISE

Fig. 7.2.1 & 2 Reconstruction error as a function of the S/N.  
(pseudo-canonical form, structure 1 4)

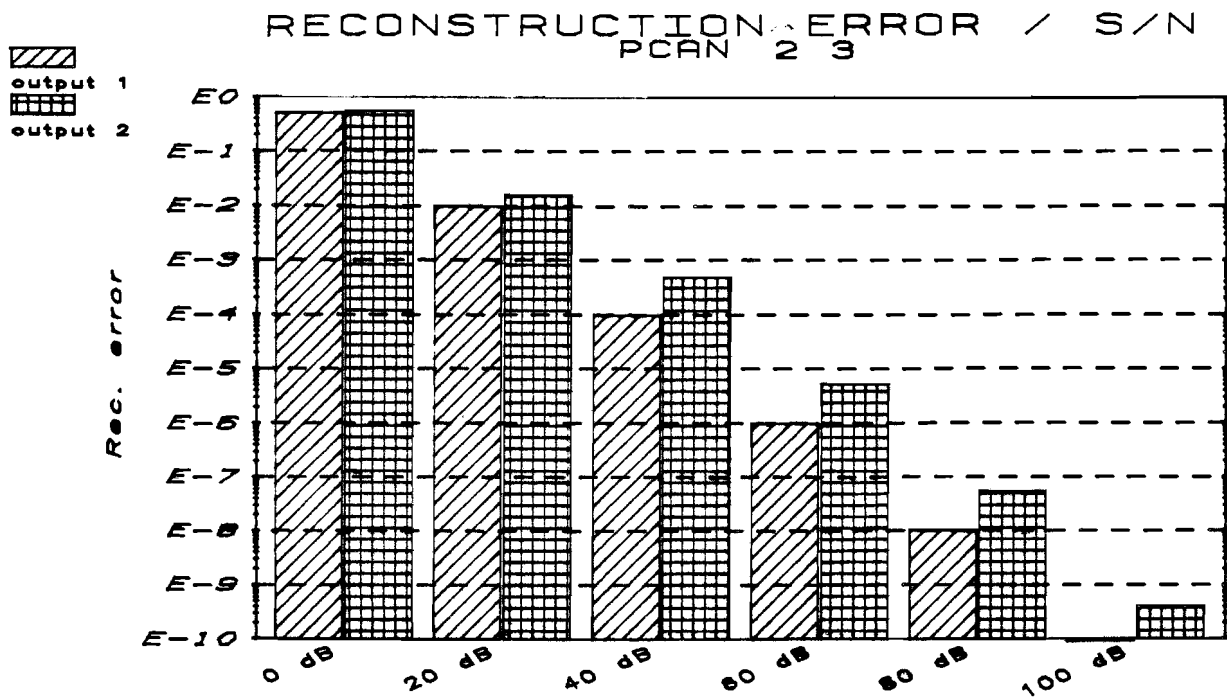


COLORLED NOISE



WHITE NOISE

Fig. 7.2.3 & 4 Reconstruction error as a function of the S/N.  
(pseudo-canonical form, structure 2 3)



COLORED NOISE

INPUTS/RESIDUALS CROSSCORRELATION.  
S/N= INF. PCAN 2 3. 2 ITER.

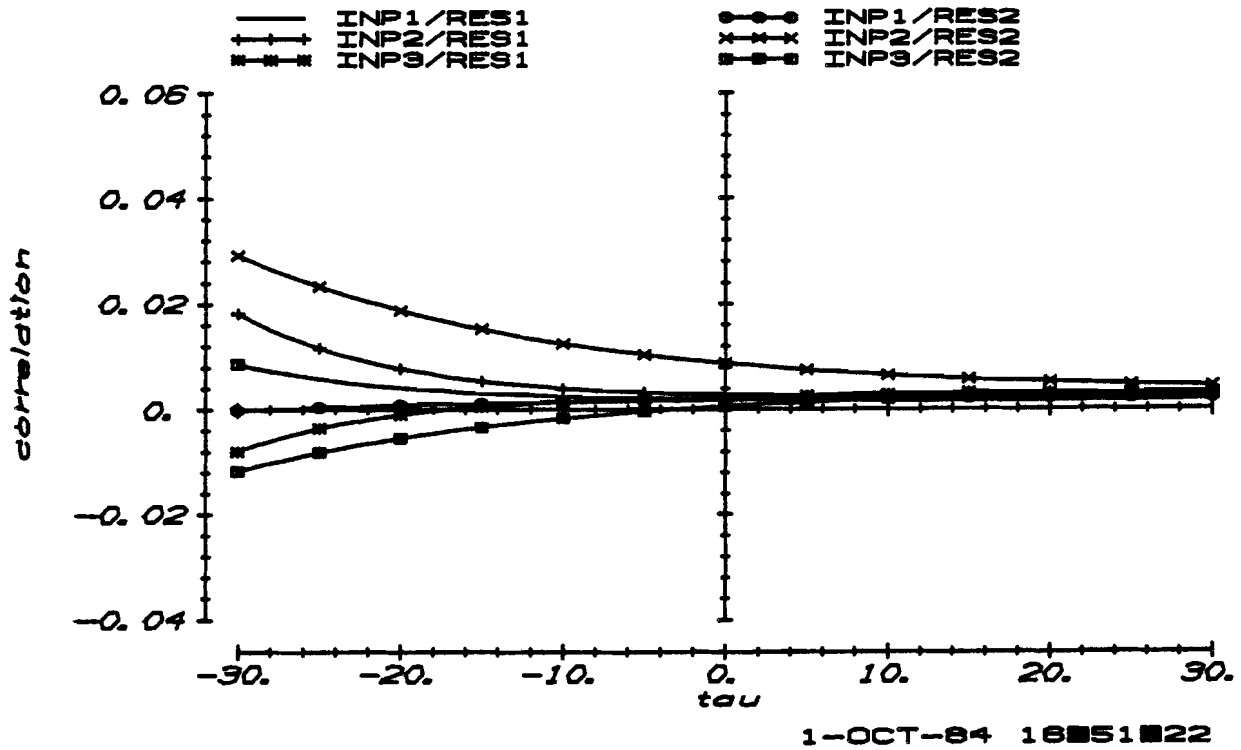


Fig. 7.3.1 Crosscorrelations (Rec. errors are 0.77D-21 & 0.39D-12)

INPUTS/RESIDUALS CROSSCORRELATIONS.  
S/N= INF.. CAN 1 4. LS.

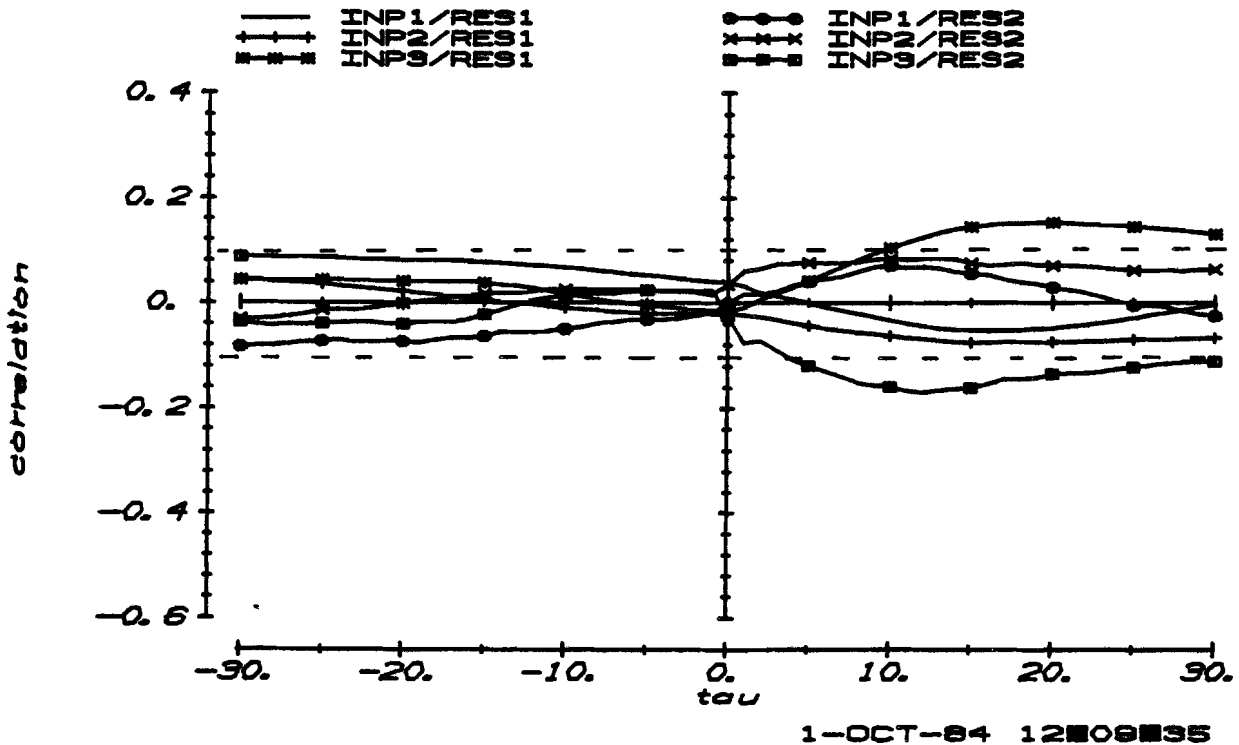


Fig. 7.3.2 Crosscorrelations (Rec. errors are 0.1D4 & 0.87D3)

INPUTS/RESIDUALS CROSSCORRELATIONS.  
S/N=40DB. CAN 2 3. 15 ITER.

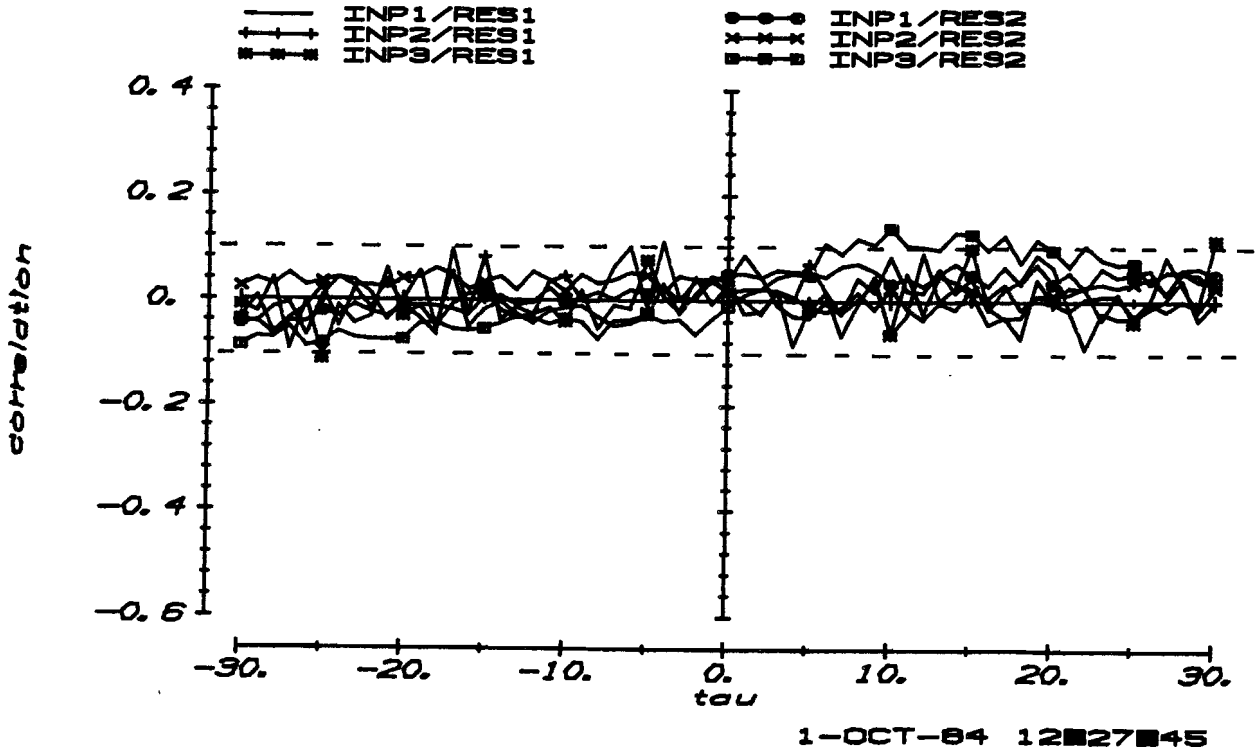


Fig. 7.3.3 Crosscorrelations (Rec. errors are  $0.11D-3$  &  $0.97D-3$ )

INPUTS/RESIDUALS CROSSCORRELATIONS.  
S/N=40DB. PCAN 2 3. LS.

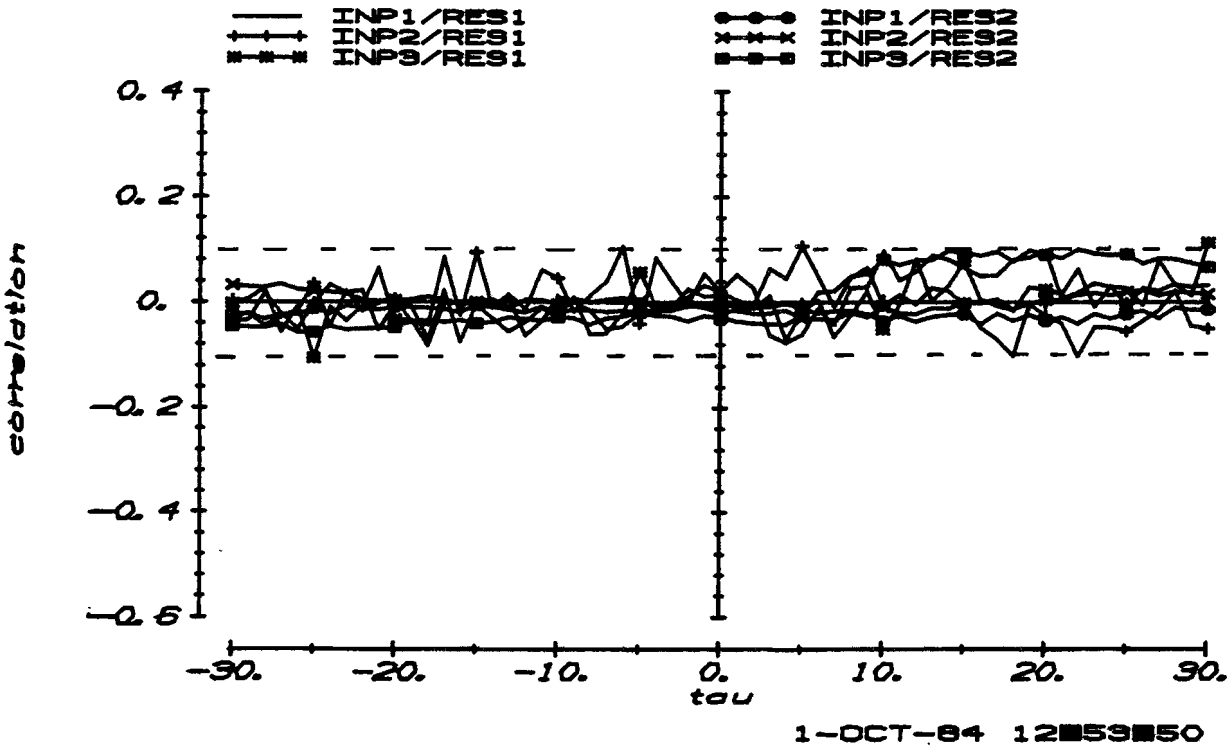


Fig. 7.3.4 Crosscorrelations (Rec. errors are  $0.13D-3$  &  $0.53D-2$ )

INPUTS/RESIDUALS CROSSCORRELATIONS.  
S/N=40DB. PCAN. 14. LS.

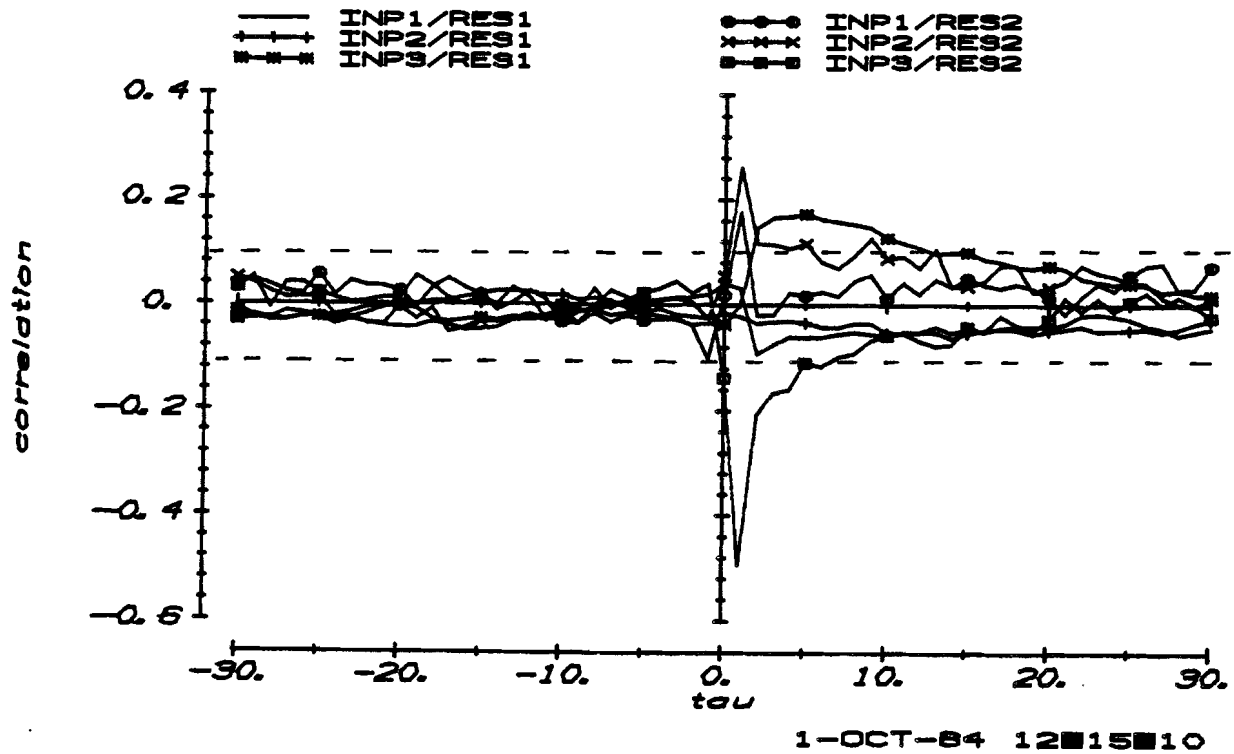


Fig. 7.3.5 Crosscorrelations (Rec. errors are 0.43D1 & 0.61D1)

INPUTS/RESIDUALS CROSSCORRELATIONS.  
S/N=40DB. PCAN. 14. 15 ITER.

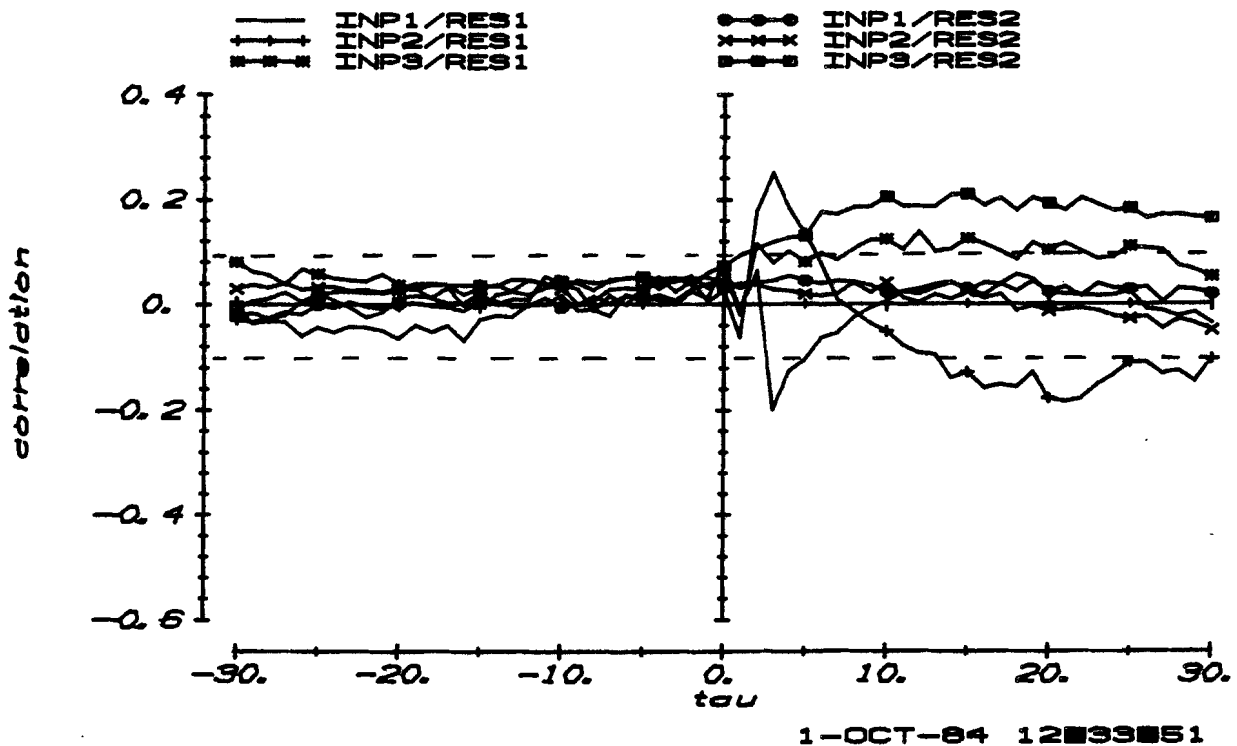


Fig. 7.3.6 Crosscorrelations (Rec. errors are 0.91D-2 & 0.31D-2)

INPUTS/RESIDUALS CROSSCORRELATIONS.  
S/N=40DB. CAN 1 4. LS.

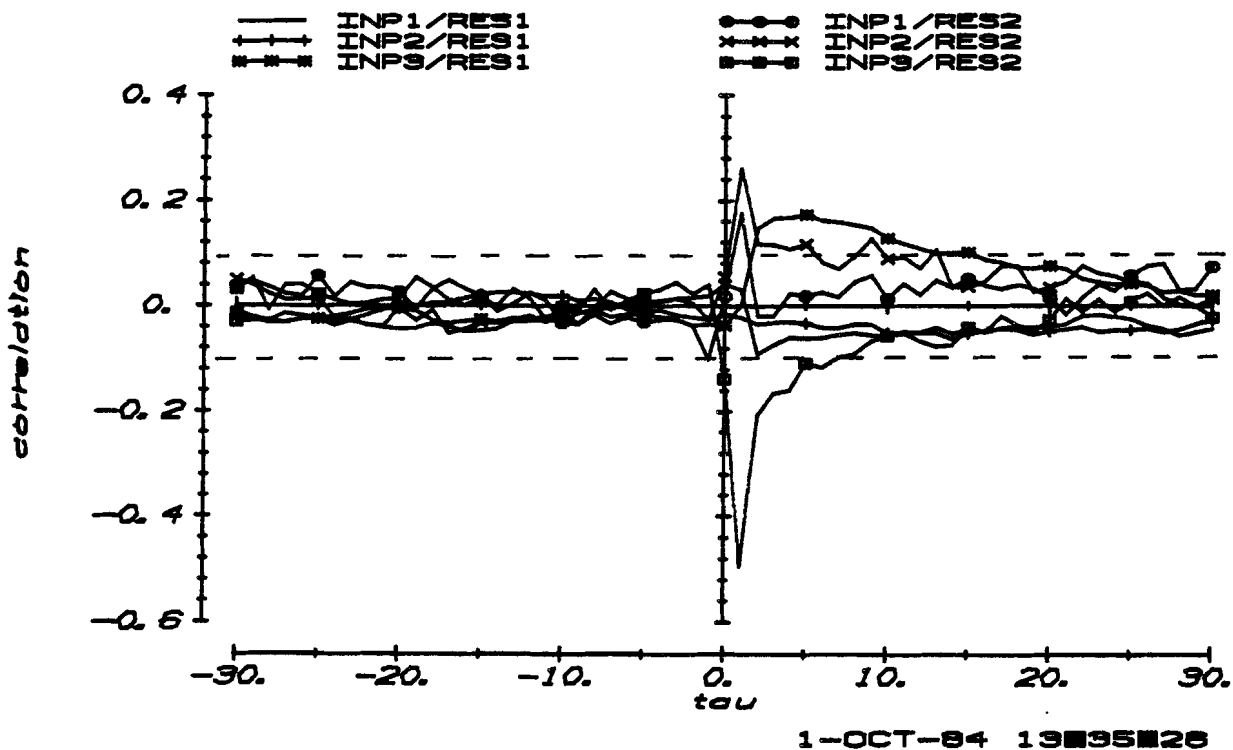


Fig. 7.3.7 Crosscorrelations (Rec. errors are 0.43D1 & 0.61D1)

INPUTS/RESIDUALS CROSSCORRELATIONS.  
S/N=200DB. PCAN 2 3. 20 ITER.

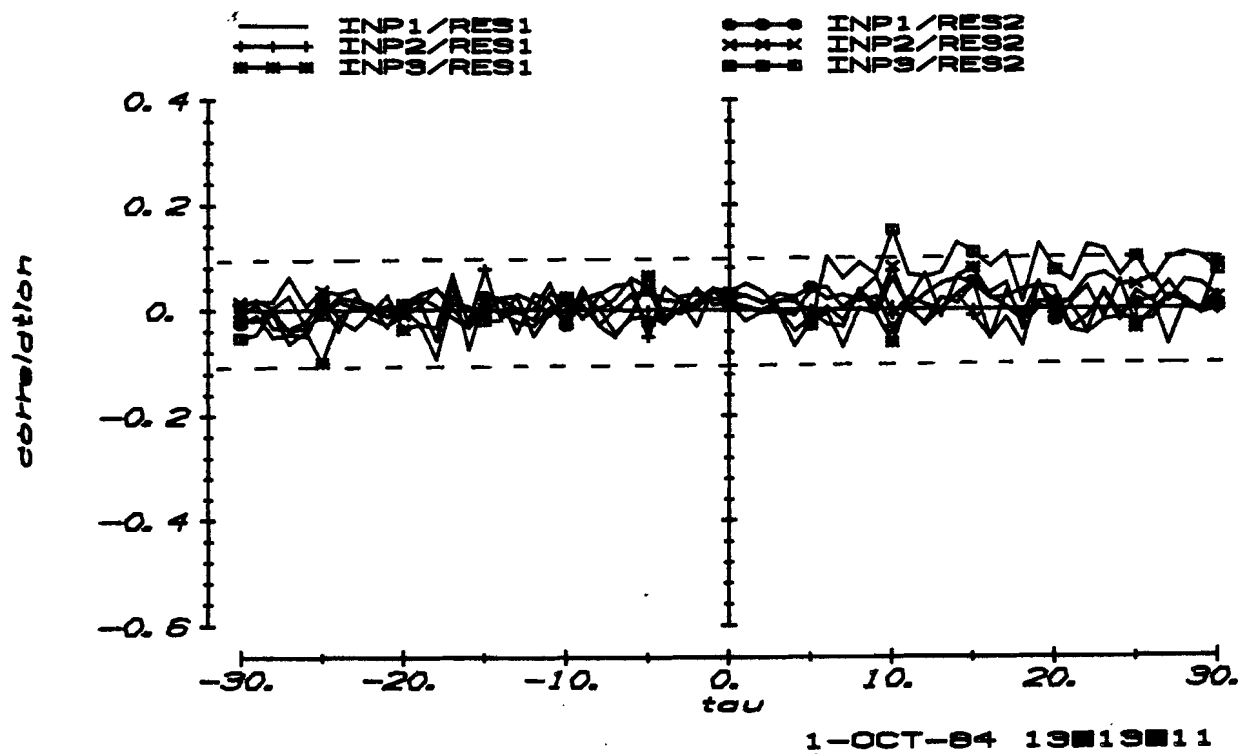


Fig. 7.3.8 Crosscorrelations (Rec. errors are 0.11D-1 & 0.16D-1)



INPUTS/RESIDUALS CROSSCORRELATIONS.  
S/N=20DB. PCAN. 2 3. LS.

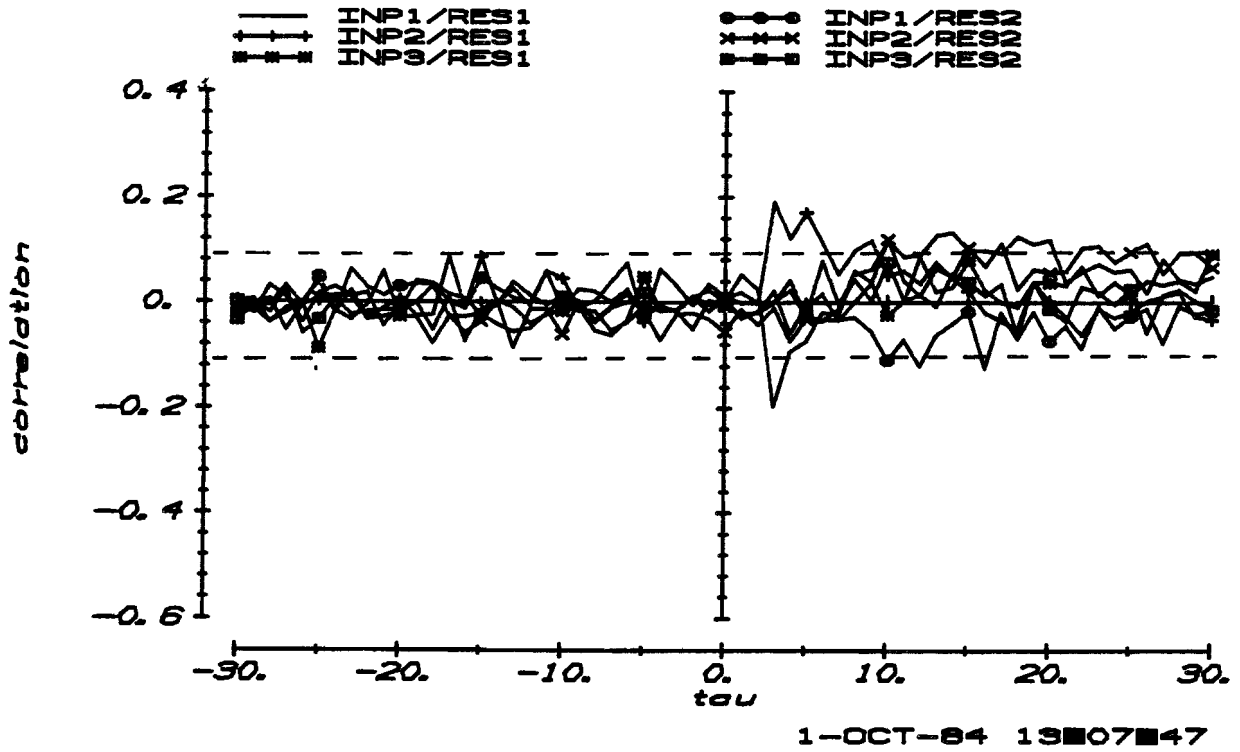


Fig. 7.3.9 Crosscorrelations (Rec. errors are  $0.18D^{-1}$  &  $0.20D^{-1}$ )

INPUTS/RESIDUALS CROSSCORRELATIONS.  
S/N=20DB. CAN. 1 4. LS.

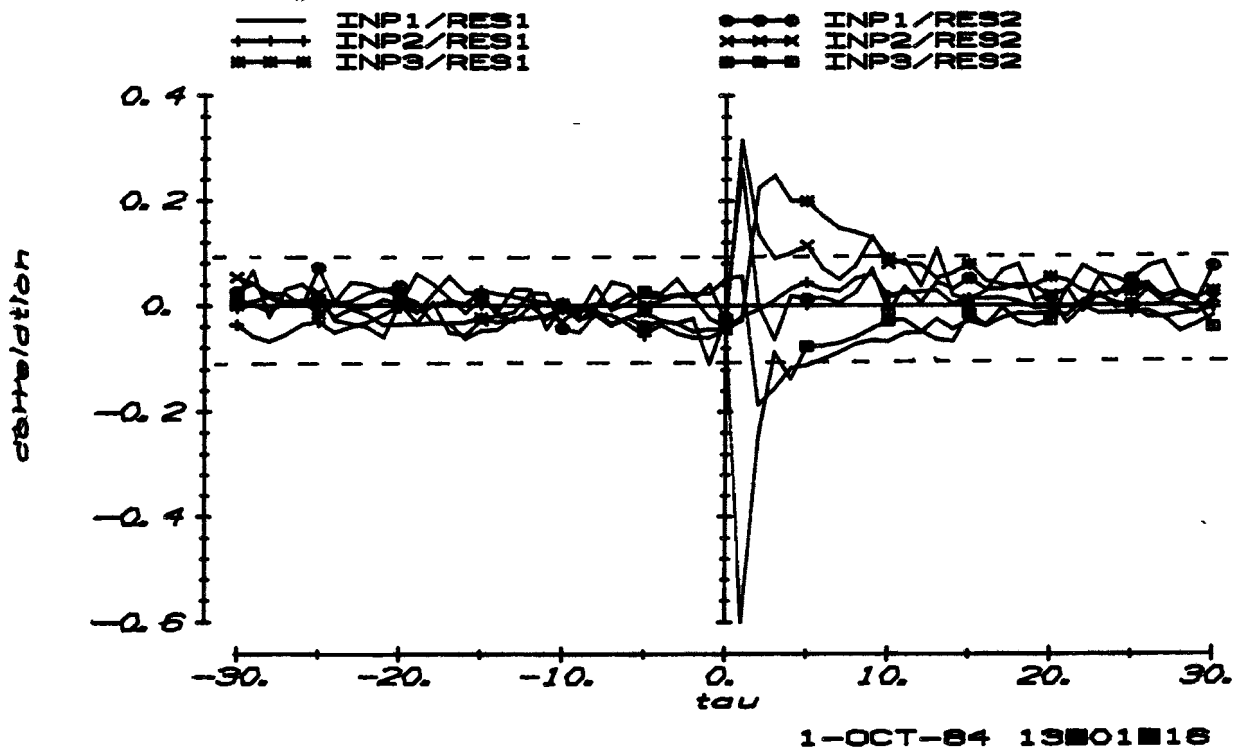


Fig. 7.3.10 Crosscorrelations (Rec. errors are  $0.92$  &  $0.21D^{-1}$ )

INPUTS/RESIDUALS CROSSCORRELATIONS.  
S/N=ODB. CAN. 2 3. 20 ITER.

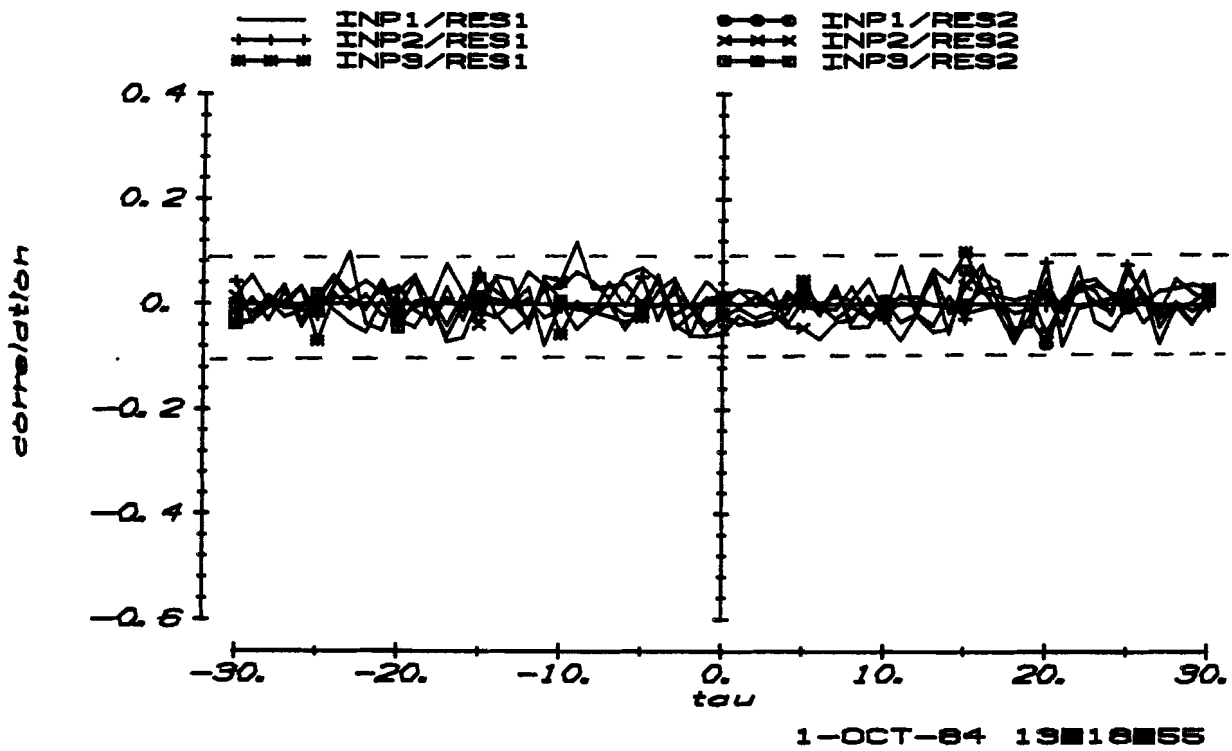


Fig. 7.3.11 Crosscorrelations (Rec. errors are 0.60 & 0.57)

INPUTS/RESIDUALS CROSSCORRELATIONS.  
S/N=ODB. CAN. 2 3. LS.

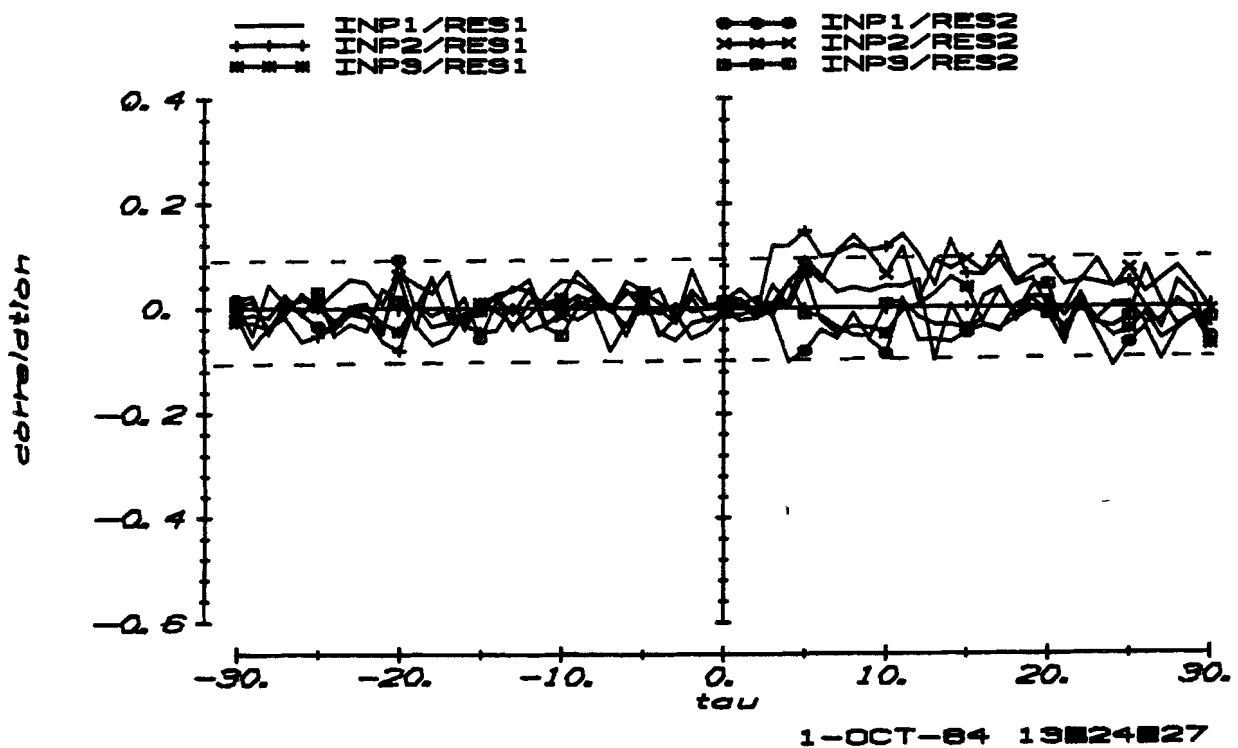


Fig. 7.3.12 Crosscorrelations (Rec. errors are 0.66 & 0.54)

INPUTS/RESIDUALS CROSSCORRELATIONS.  
S/N=008. CAN. 1 4. LS.

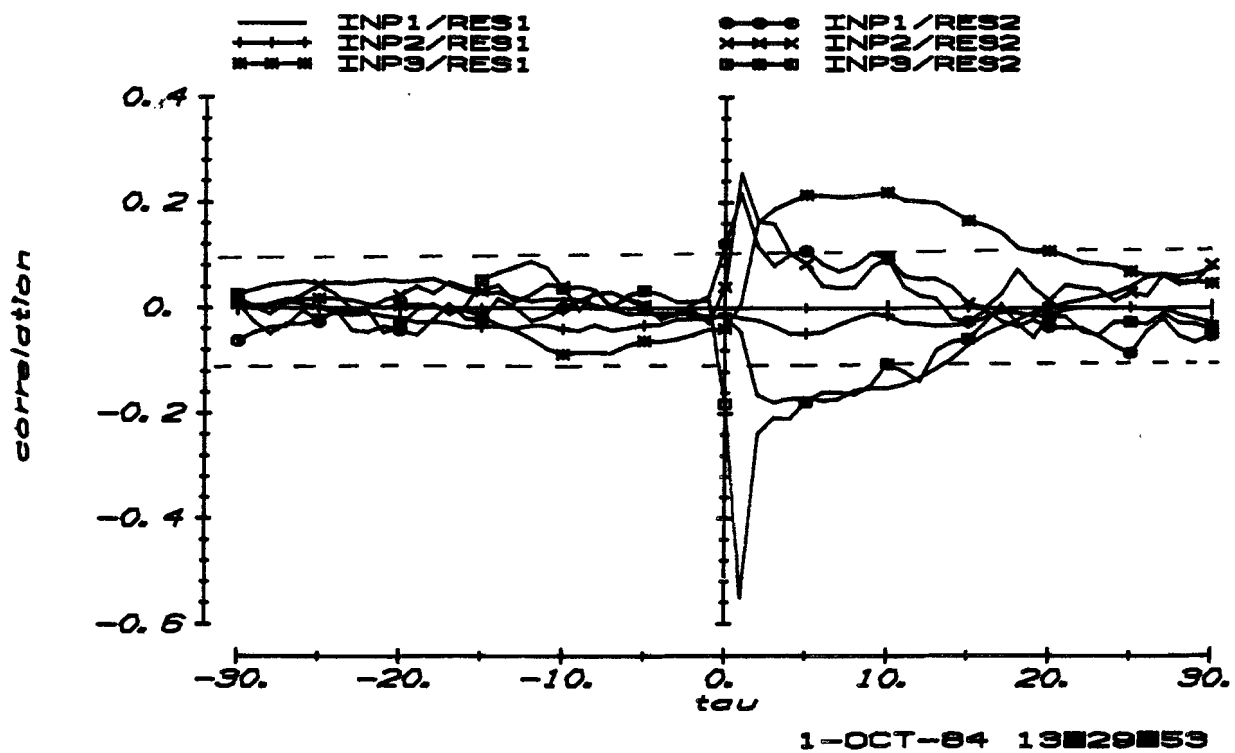


Fig. 7.3.13 Crosscorrelations (Rec. errors are 0.37D1 & 0.55D1)

ORIGINAL AND MODEL OUTPUT 1.  
S/N=INF. CAN. 1 4. LS.

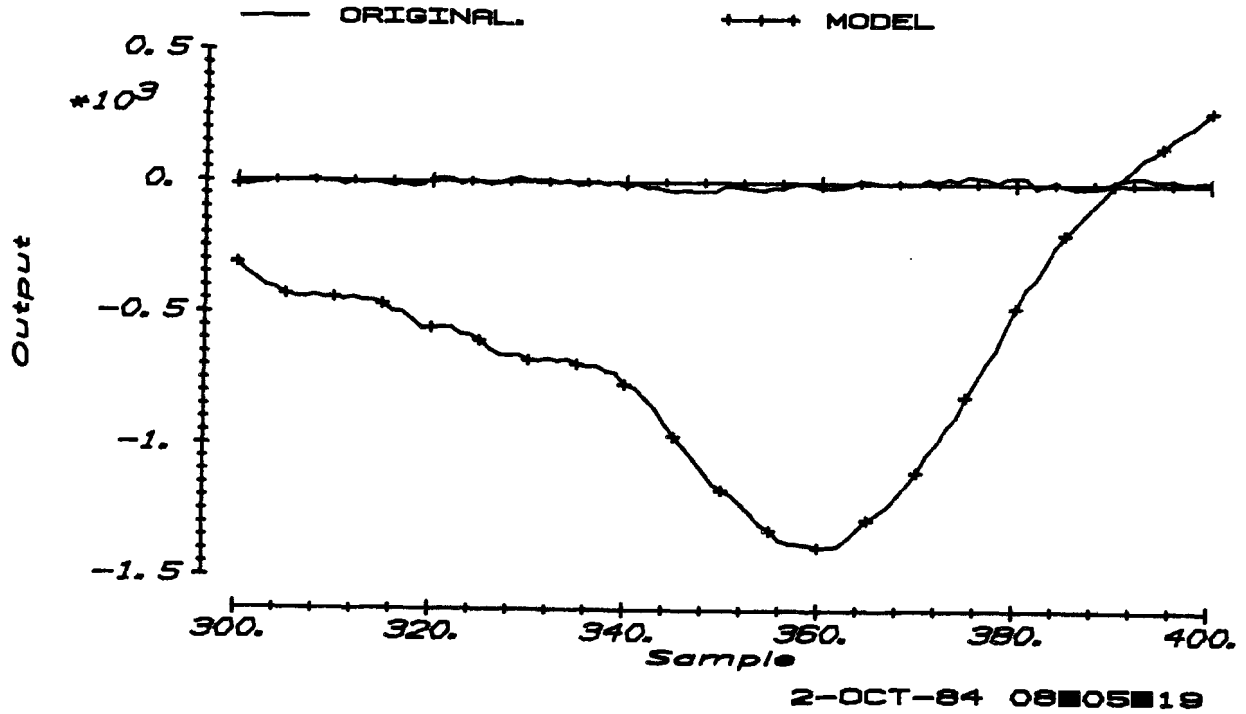
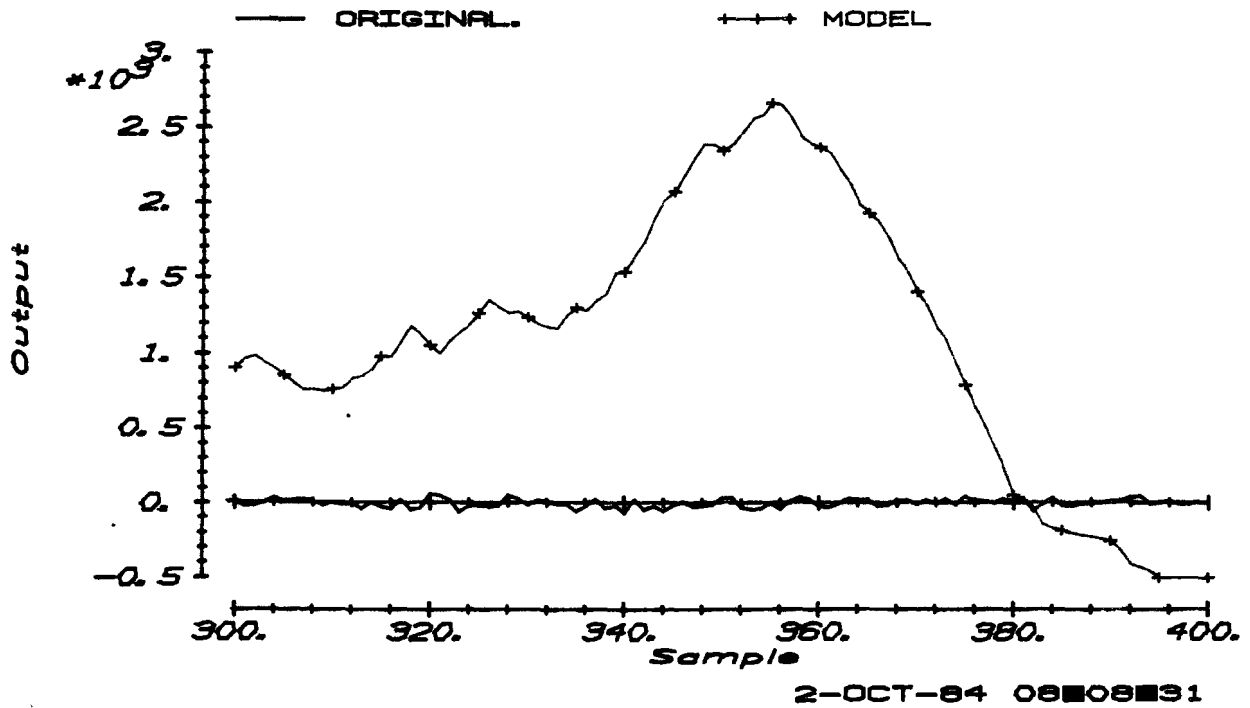
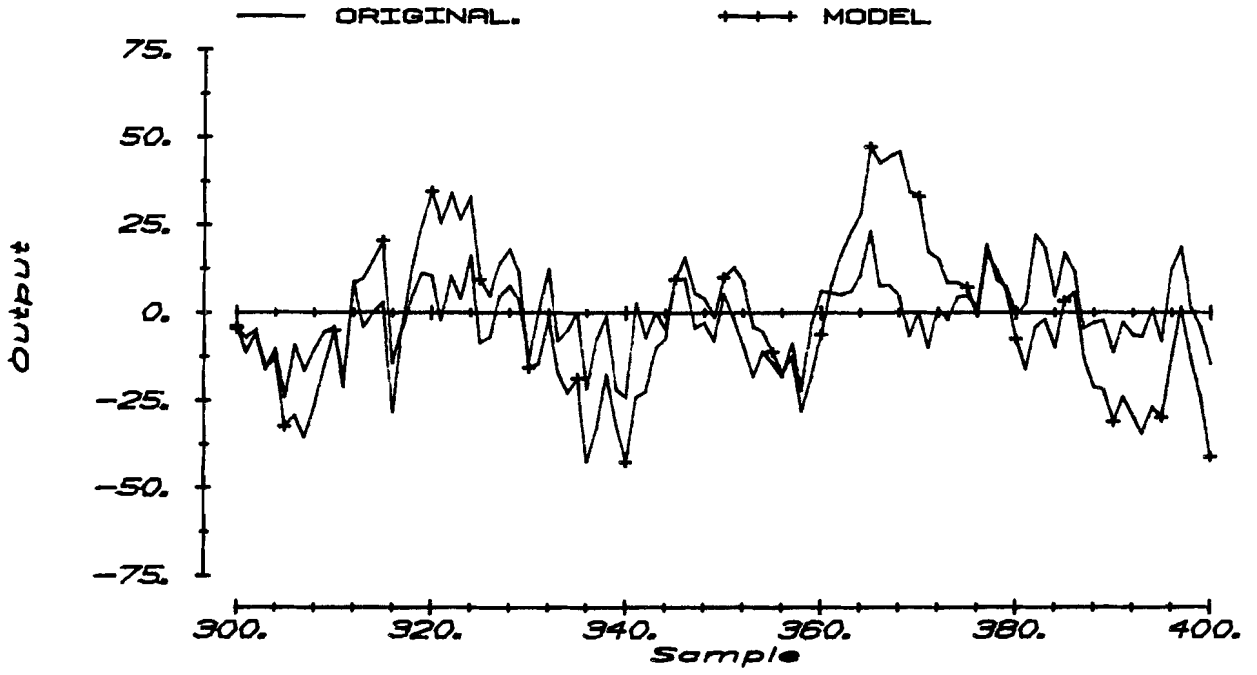


Fig. 7.4.1 a) & b) Original/Model outputs 1 & 2.  
(Rec. errors are 0.1D4 & 0.87D3)

ORIGINAL AND MODEL OUTPUT 2.  
S/N=INF. CAN 1 4. LS.



ORIGINAL AND MODEL OUTPUT 1.  
S/N=40DB. CAN. 1 4. LS.

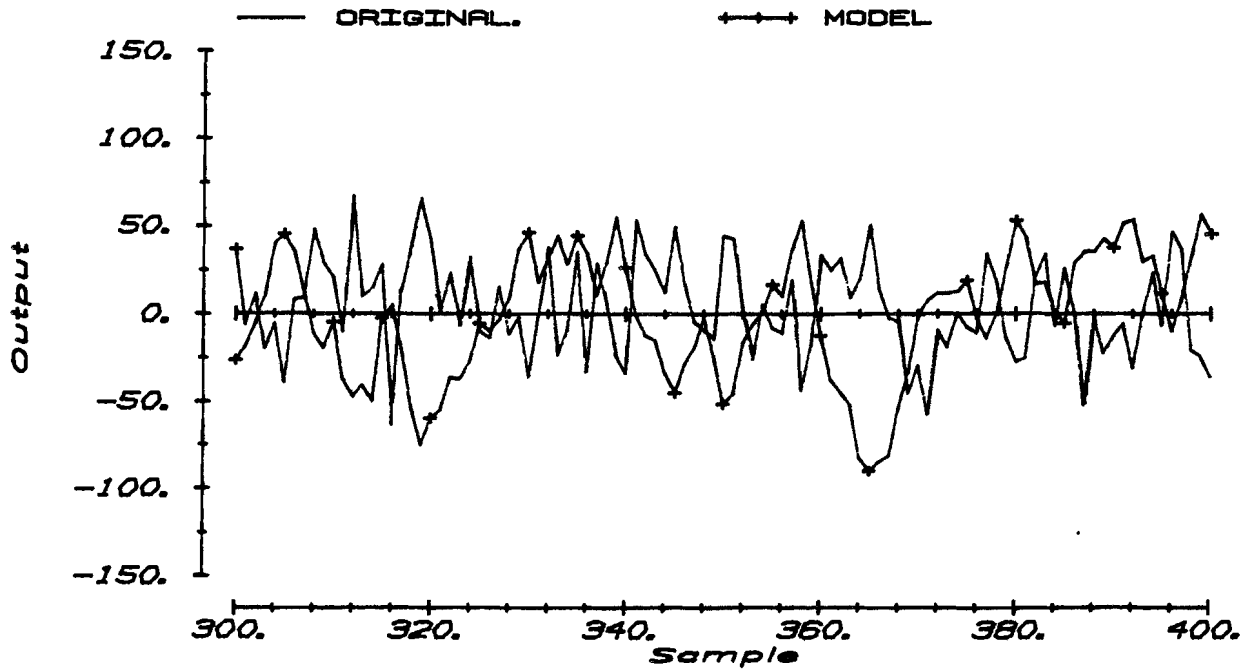


2-OCT-84 08:44:30

Fig. 7.4.2 a) & b) Original/Model outputs 1 & 2.

(Rec. errors are 0.43D1 & 0.61D1)

ORIGINAL AND MODEL OUTPUT 2.  
S/N=40DB. CAN. 1 4. LS.



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ORIGINAL AND MODEL OUTPUT 1.  
S/N=20DB. CAN. 1 4. LS.

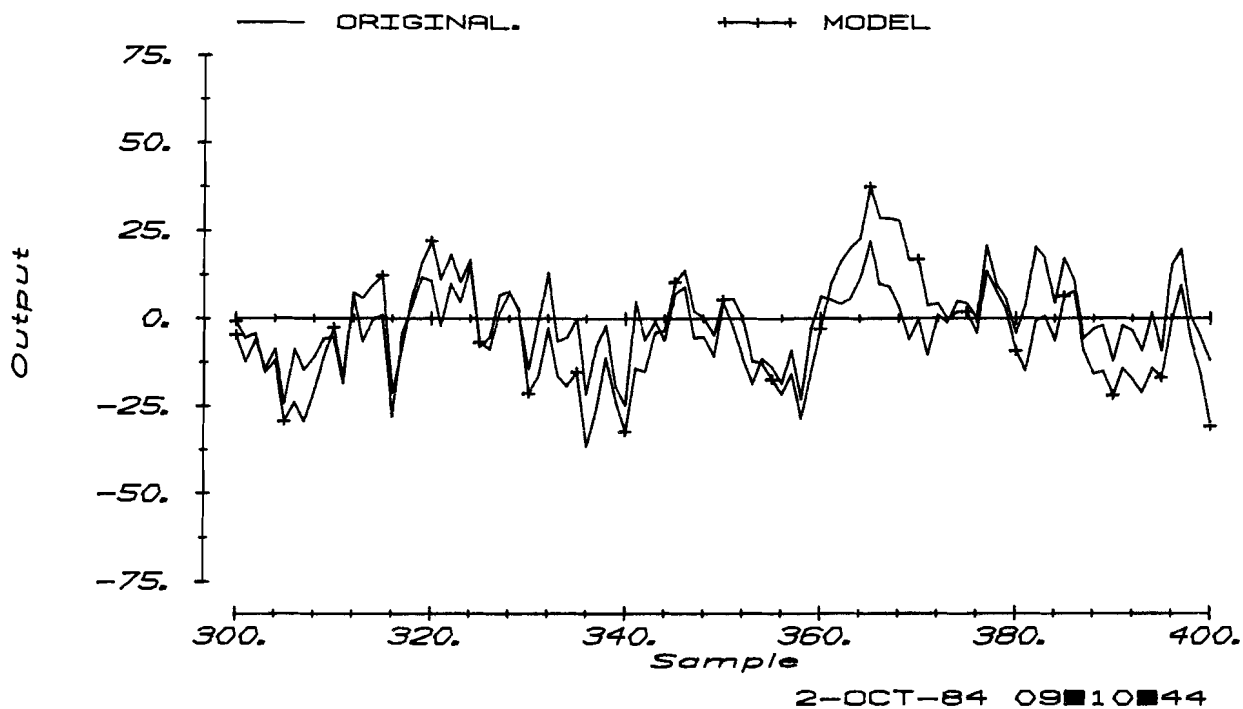
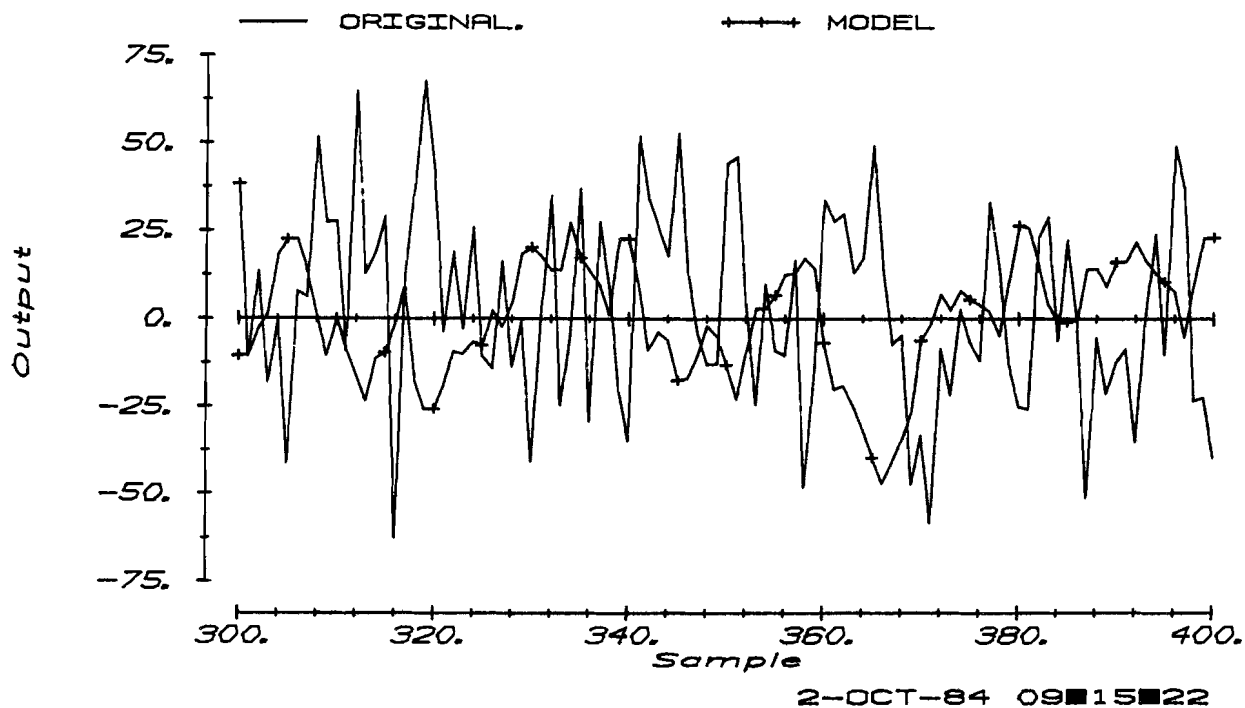


Fig. 7.4.3 a) & b) Original/Model outputs 1 & 2.  
(Rec. errors are 0.92 & 0.21D1)

ORIGINAL AND MODEL OUTPUT 2.  
S/N=20DB. CAN. 1 4. LS.



ORIGINAL AND MODEL OUTPUT 1.  
S/N=20DB. PCAN. 2 3. 20 ITER.

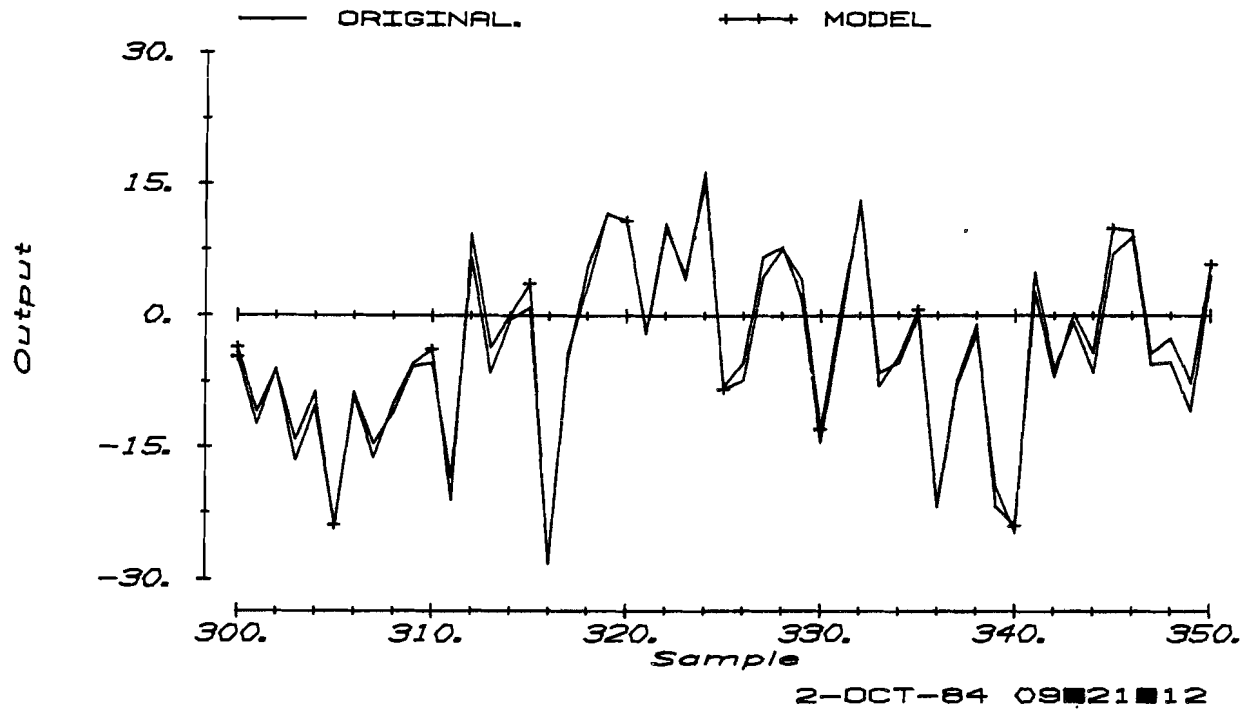
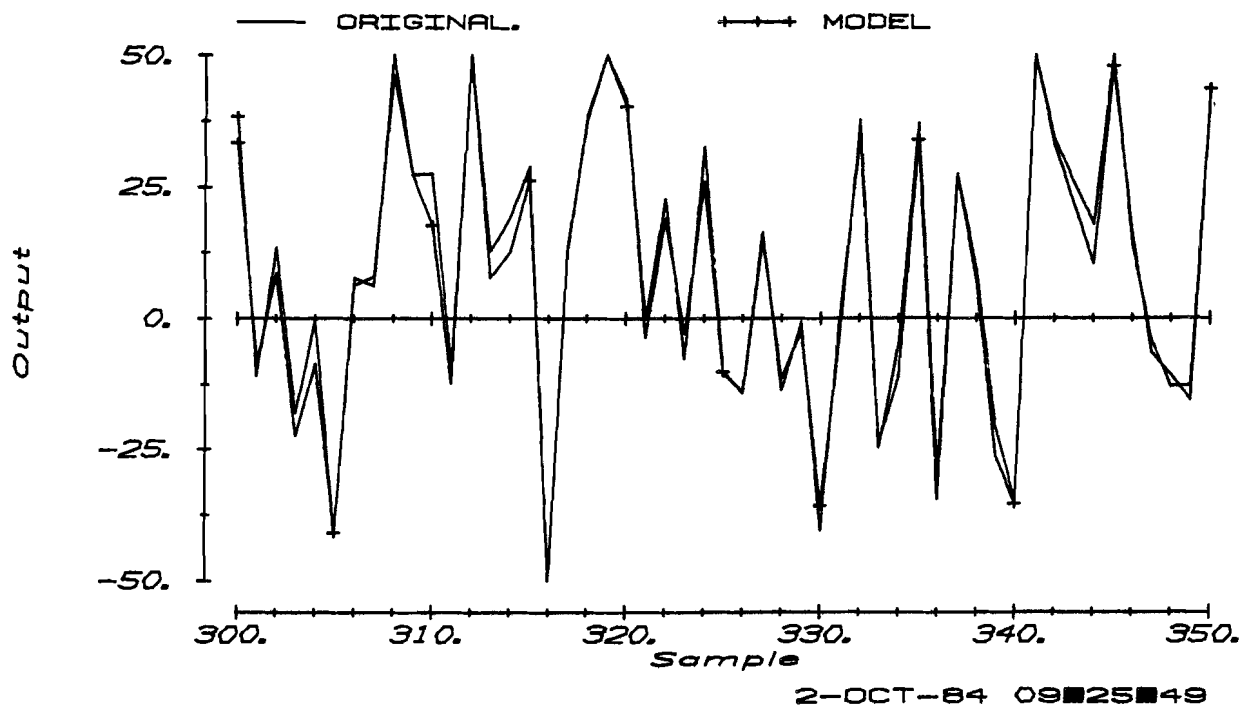


Fig. 7.4.4 a) & b) Original/Model outputs 1 & 2.

(Rec. errors are  $0.11D-1$  &  $0.16D-1$ )

ORIGINAL AND MODEL OUTPUT 2.  
S/N=20DB. PCAN 2 3. 20 ITER.



ORIGINAL AND MODEL OUTPUT 1.  
S/N=20DB. PCAN. 2 3. LS.

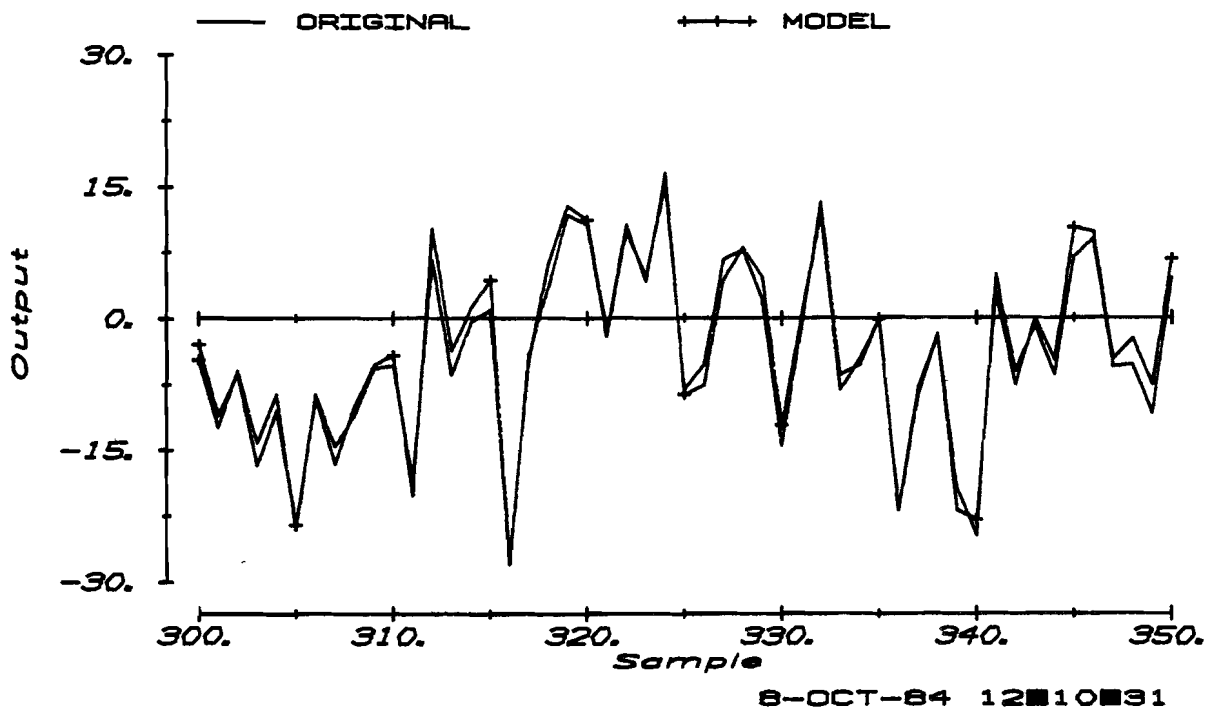
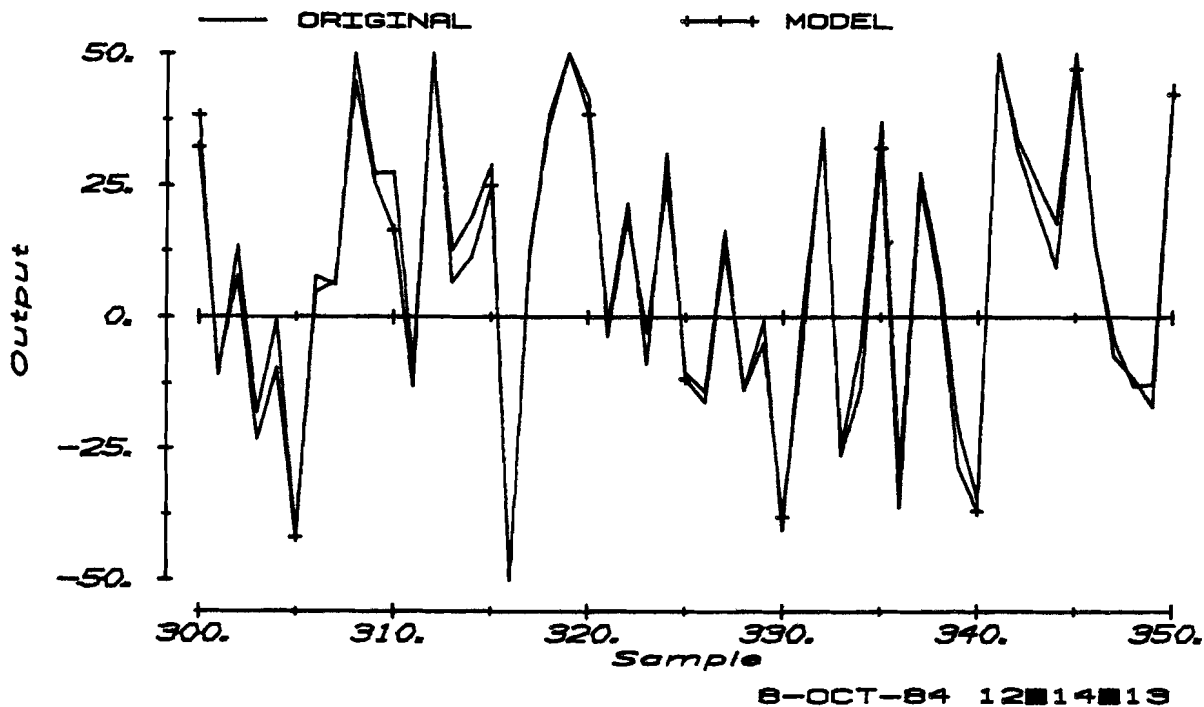


Fig. 7.4.5 a) & b) Original/Model outputs 1 & 2.

(Rec. errors are 0.18D-1 & 0.21D-1)

ORIGINAL AND OUTPUT 2.  
S/N=20DB. PCAN. 2 3. LS.





ORIGINAL AND MODEL OUTPUT 1.  
S/N=0DB, CAN 1 4, LS.

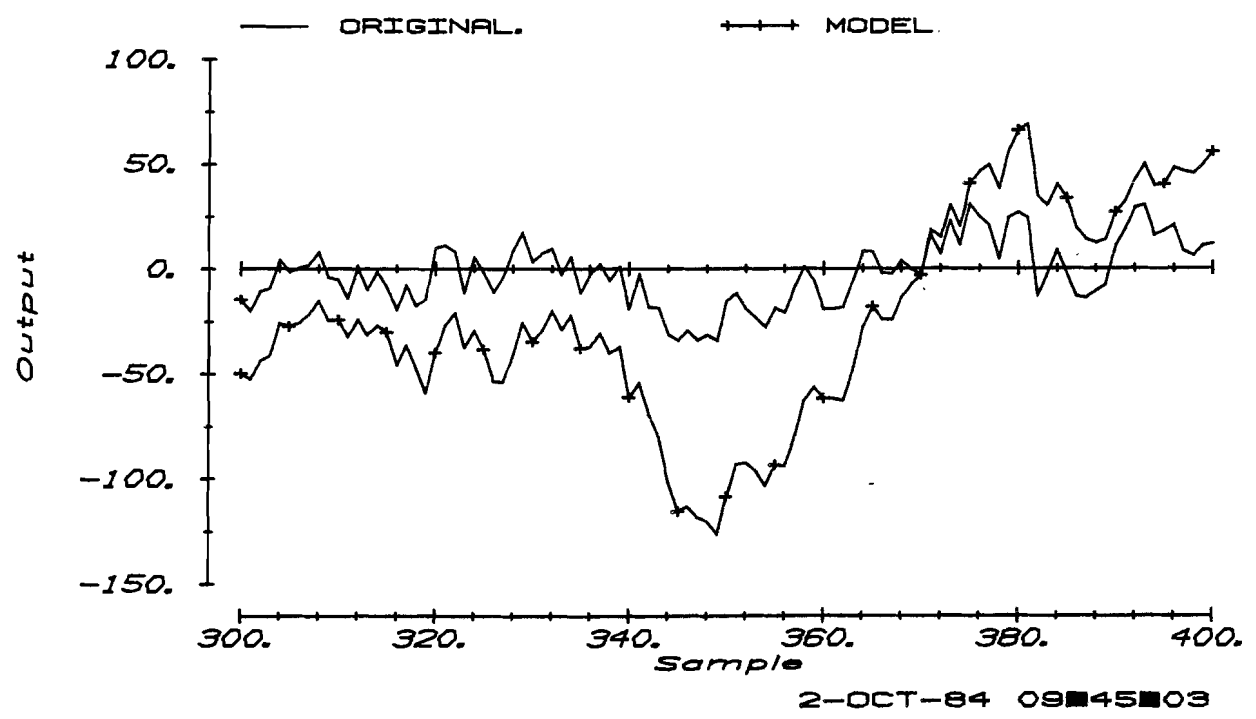
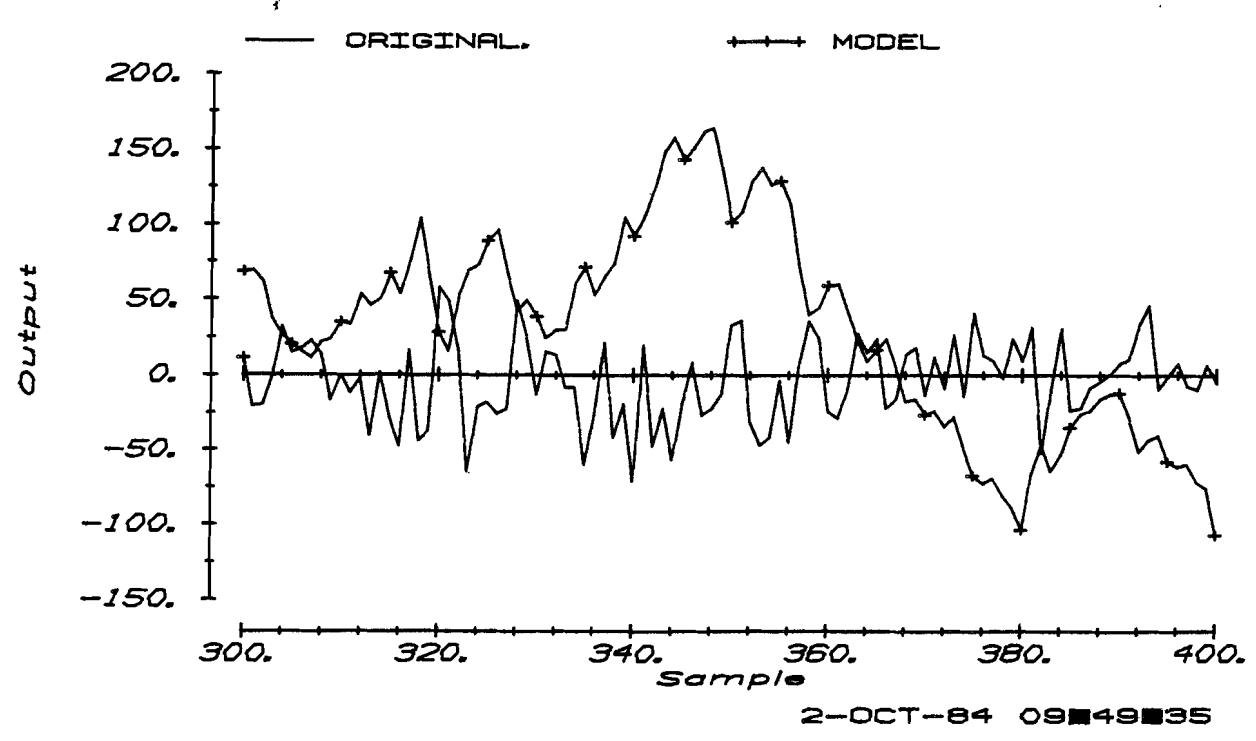


Fig. 7.4.6 a) & b) Original/Model outputs 1 & 2.  
(Rec. errors are 0.37D1 & 0.55D2)

ORIGINAL AND MODEL OUTPUT 2.  
S/N=0DB, CAN. 1 4, LS.



ORIGINAL AND MODEL OUTPUT 1.  
S/N=0DB, CAN. 2 3, 20 ITER.

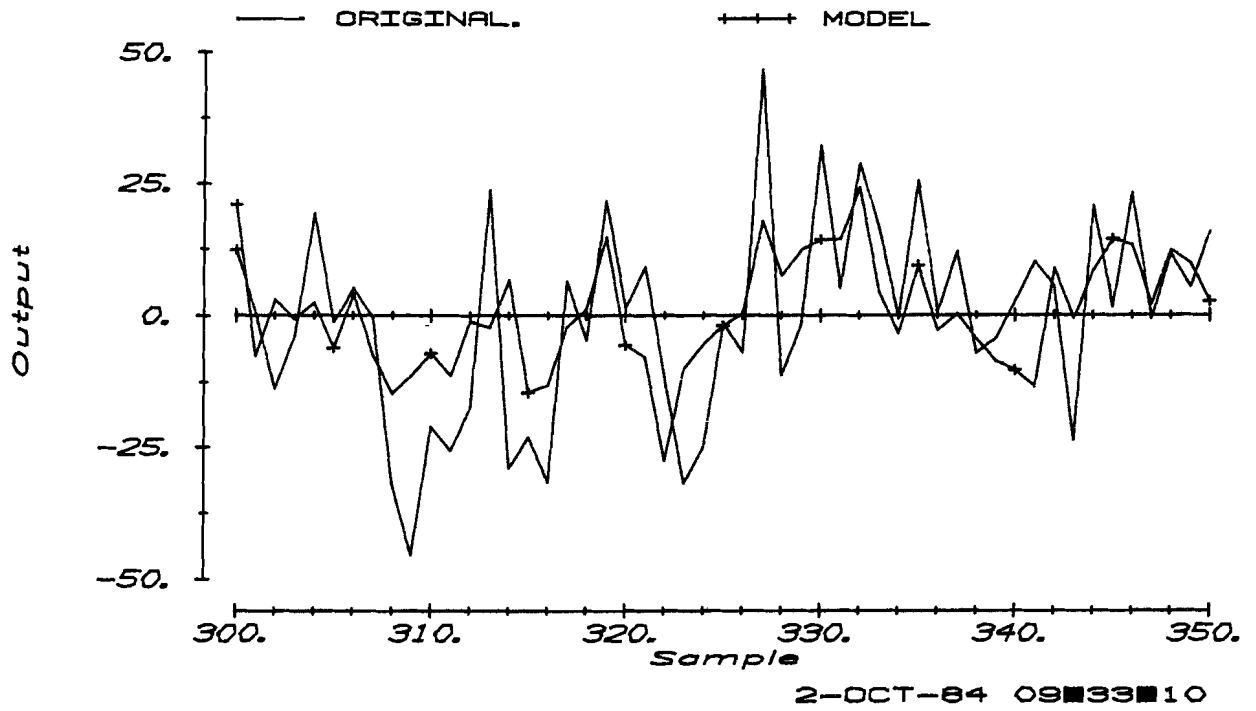
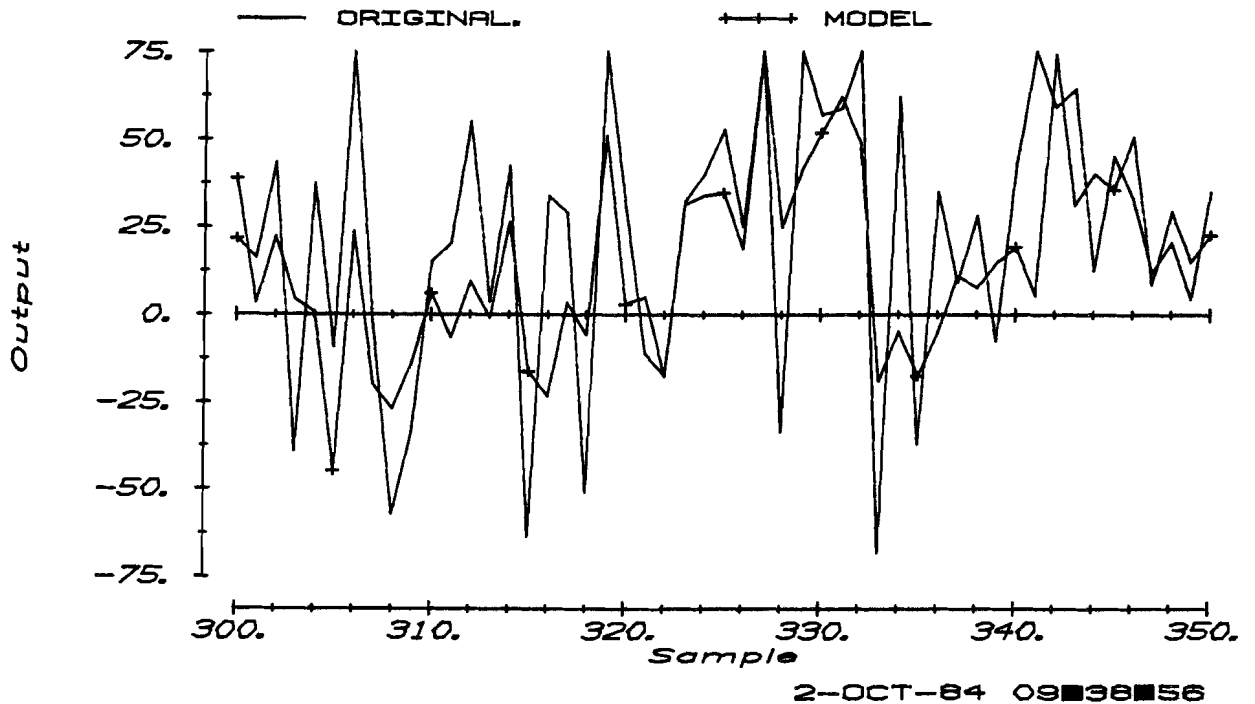


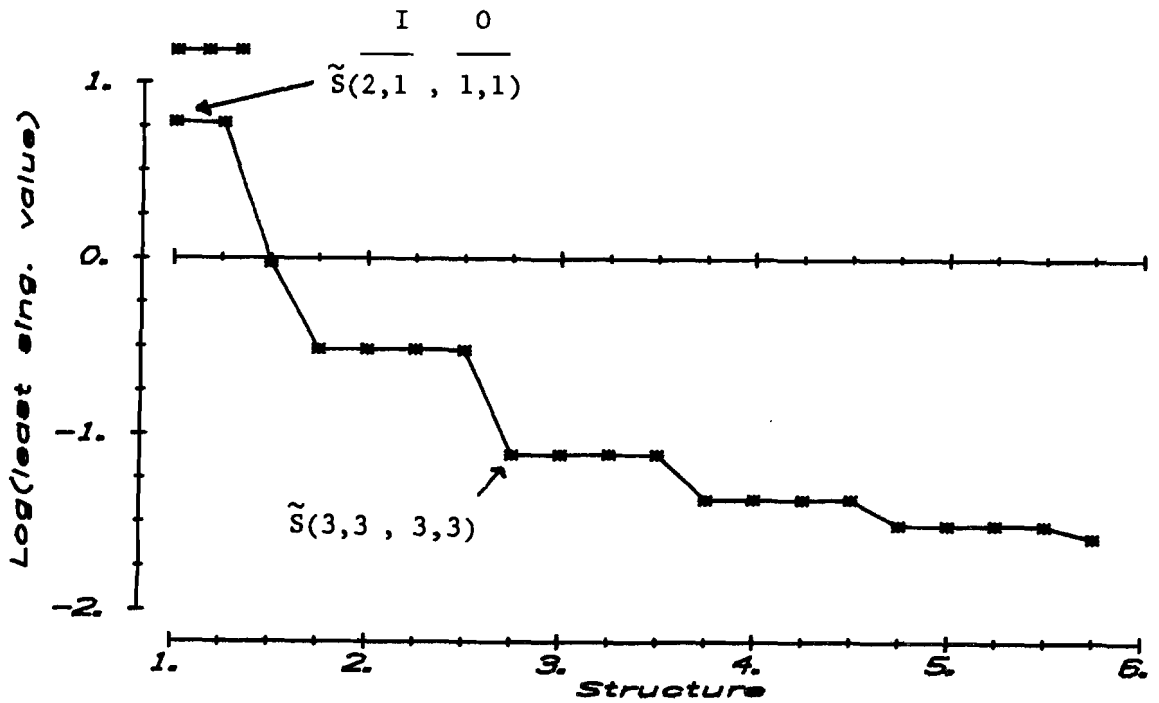
Fig. 7.4.7 a) & b) Original/Model outputs 1 & 2.

(Rec. errors are 0.60 & 0.57)

ORIGINAL AND MODEL OUTPUT 2.  
S/N=0DB, CAN. 2 3, 20 ITER.



STRUCTURAL IDENTIFICATION  
LEAST SINGULAR VALUE METHOD.

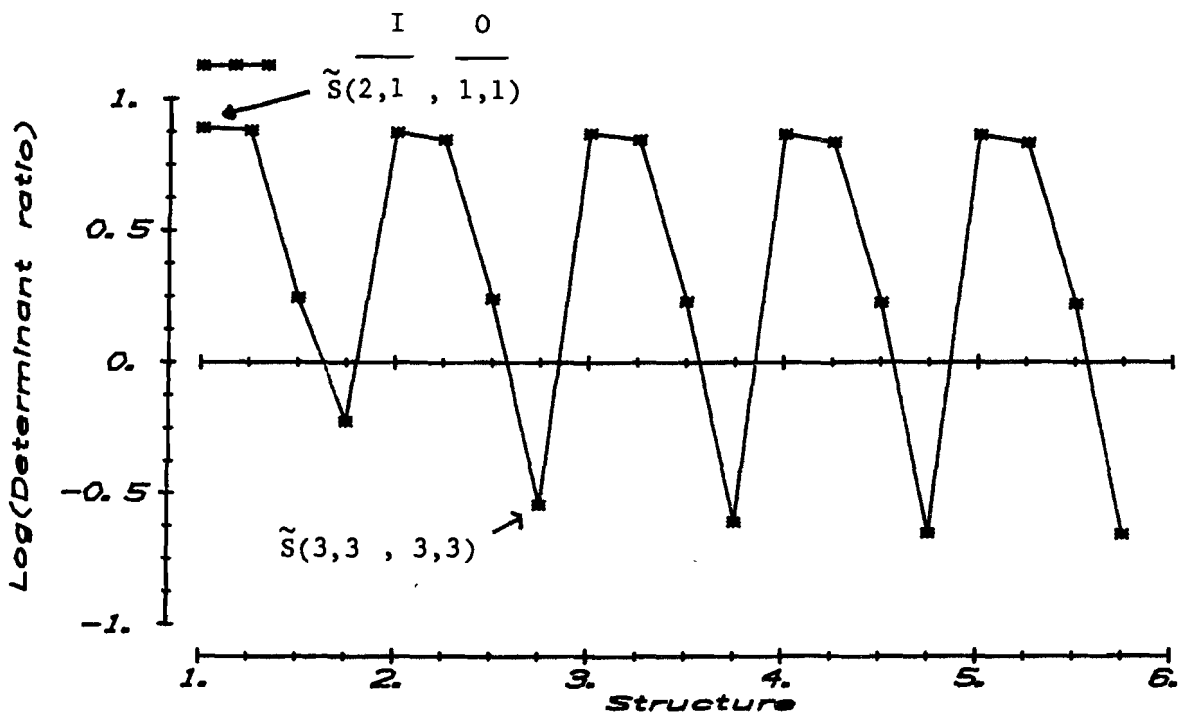


WINDSCHOTEN DATA

Fig. 7.5.1 a) & b) Windschoten data. Structural identification.

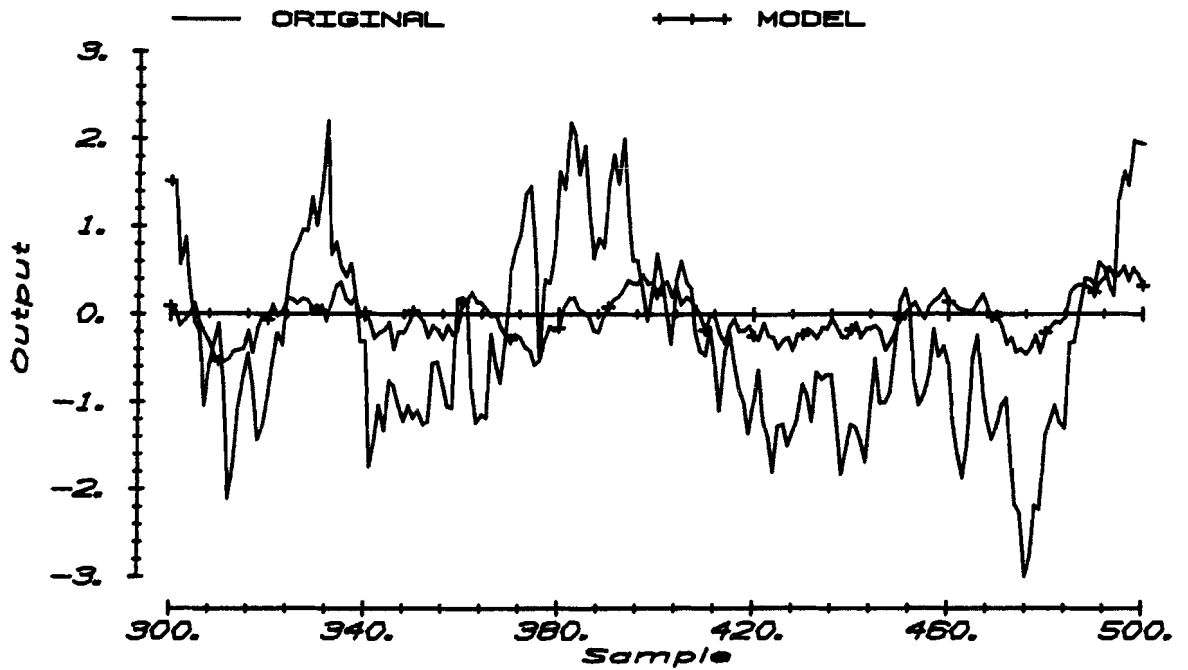
Least Singular Value and determinant method.

STRUCTURAL IDENTIFICATION  
DETERMINANT METHOD



WINDSCHOTEN DATA

ORIGINAL AND MODEL OUTPUT 1.  
PCAN 2 2. 20 ITER., REC. ERROR=0.87

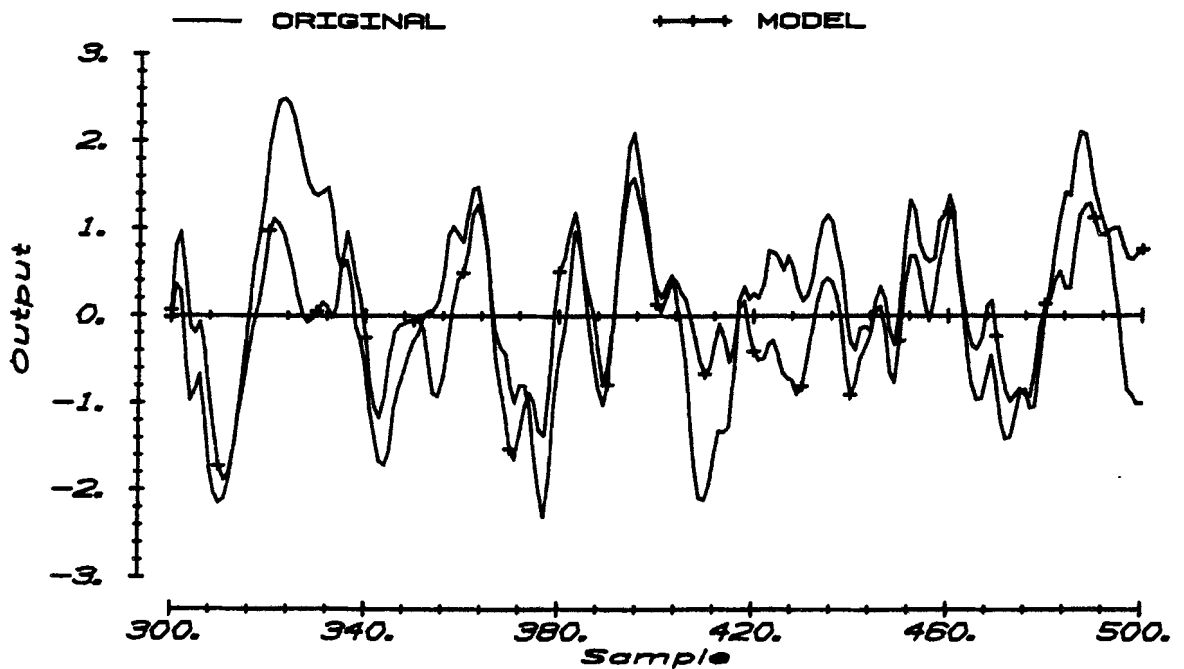


WINDSCHOTEN DATA.

Fig. 7.5.2 a) & b) Windschoten data. Original/Model outputs 1 & 2.

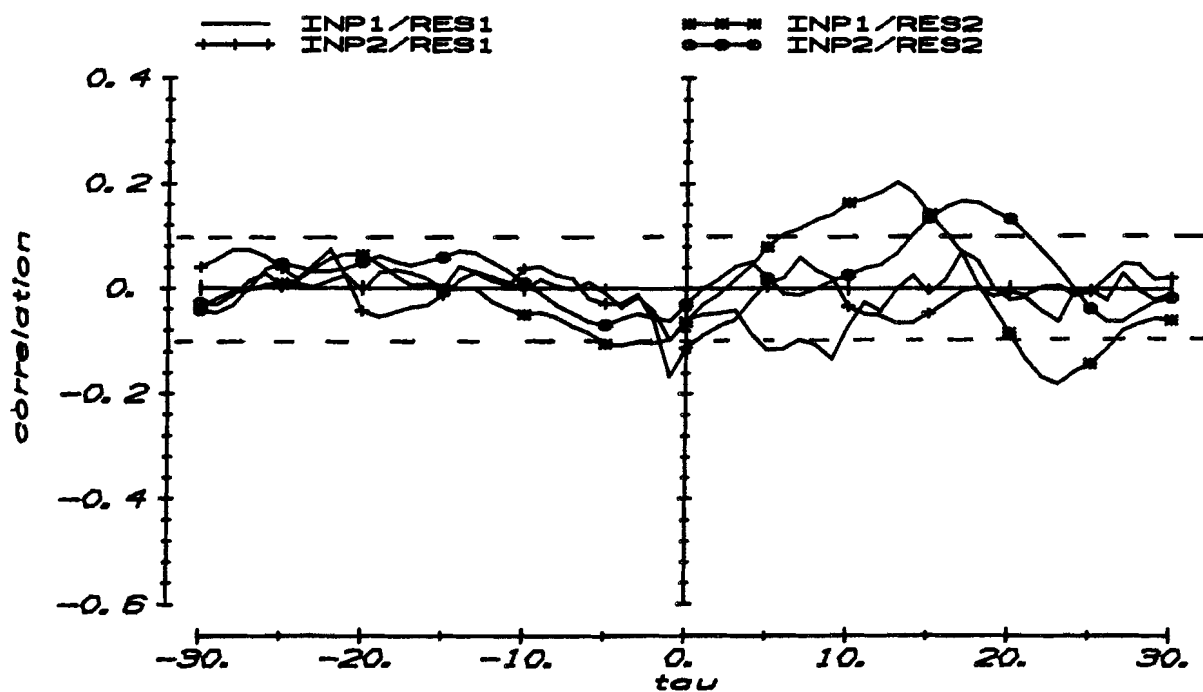
(Rec. errors are 0.87 and 0.54)

ORIGINAL AND MODEL OUTPUT 2.  
PCAN 2 2. 20 ITER., REC. ERR. =0.54



WINDSCHOTEN DATA

INPUTS/RESIDUALS CROSSCORRELATIONS.  
PCAN 2 2. 20 ITER.



WINDSCHOTEN DATA

Fig. 7.5.3 Windschoten data. Crosscorrelations.

(Rec. errors are 0.87 and 0.54)

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