MASTER

A functional Hilbert space approach to frame transforms and wavelet transforms

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MASTER'S THESIS

A functional Hilbert space approach to frame transforms and wavelet transforms

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1 Introduction

A 'frame' in a Hilbert space $\mathcal{H}$ is defined to be a subset $V = \{\phi_x \mid x \in \mathbb{E}\} \subset \mathcal{H}$, with $\mathbb{E}$ an index set, such that the span $\langle V \rangle$ of V is dense in $\mathcal{H}$. Mostly, the set $\mathbb{E}$ is a subset of $\mathbb{C}^n$ or a group. Starting from a frame V we introduce the frame transform

$$W : \mathcal{H} \to \mathbb{C}^{\mathbb{E}} : f \mapsto Wf, \text{ where } Wf : \mathbb{E} \to \mathbb{C} : x \mapsto (\phi_x, f)_{\mathcal{H}}. \tag{1.1}$$

The underlying master's thesis is mainly focused on the questions how and when the above transform defines a unitary map. We succeeded answering these questions in the most general way by using the theory of functional Hilbert spaces (= theory of reproducing kernels). The idea of working with these kind of spaces is inspired by the identity

$$\big|(Wf)(x)\big| \leq \|\psi_x\|_{\mathcal{H}} \|f\|_{\mathcal{H}}. \tag{1.2}$$

This identity states that if the frame transform defines a unitary map from $\mathcal{H}$ onto a Hilbert subspace of $\mathbb{C}^{\mathbb{E}}$, then point evaluation $\delta_x : f \mapsto f(x)$ in the latter space is a continuous linear functional for all $x \in \mathbb{E}$. This means that the Hilbert subspace has a reproducing kernel K and we denote it by $\mathbb{C}^{\mathbb{E}}_K$.

The central observation of this thesis is Theorem 2.3. It states that W defines a unitary map from $\mathcal{H}$ onto the functional Hilbert space $\mathbb{C}^{\mathbb{E}}_K$ where K is the function of positive defined by $K(x, x') = (\psi_x, \psi_{x'})_{\mathcal{H}}$. This is a new and important result. Since the functional Hilbert space $\mathbb{C}^{\mathbb{E}}_K$ is constructed in a rather abstract fashion, we are challenged to find a more tangible alternative description of this space.

As shown in chapter 4, an alternative description comes within sight in case the frame V is constructed from a generating function for orthogonal polynomials. The space $\mathbb{C}^{\mathbb{E}}_K$ then typically consist of analytic functions on (a subset of) the complex plain. The Bargmann-transform is a famous illustration of this phenomenon.

In chapter 5 the well-known Laplace and Fourier transforms are looked upon as frame transforms. Note that the Fourier transform is not a frame transform itself, but with the aid of Gelfand triples we can construct a frame transform which leads to the Fourier transform.

Theorem 2.3 can also be used to construct sampling theorems. A famous example of a sampling theorem concerns the space of functions $f \in \mathbb{L}_2(\mathbb{R})$ for which $\mathcal{F}f$ has support within $(-1, 1)$. Then

$$f(x) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin(\pi(x-n))}{\pi(x-n)}. \tag{1.3}$$

In chapter 6 a general sampling theorem is proved by a simple argument, which covers most of the classical cases. As an excursion, we also mention a construction of functional Hilbert spaces that admit a sampling theorem. In Appendix D an open problem is formulated which is inspired by these results.

A special kind of frame transforms is the wavelet transform. In this case V is constructed from a vector $\psi \in \mathcal{H}$ and a group representation of a group G in $\mathcal{H}$

$$V_\psi = \{\mathcal{U}_g \psi \mid g \in G\}. \tag{1.4}$$

We denote a wavelet transform by W_ψ, where $\psi \in \mathcal{H}$ is called "wavelet". In the last twenty years a lot has been written about these kind of transformations. In 1985 Grossmann, Morlet and Paul published a basic paper [GMP] which can be seen as the fundament of the theory of wavelet transforms. Their main result is that the wavelet transform W_ψ defines a unitary map from a Hilbert space $\mathcal{H}$ onto $\mathbb{L}_2(G)$ for a suitable vector $\psi \in \mathcal{H}$, where G is a locally compact group with a unitary, irreducible and square integrable representation $\mathcal{U}$ of G in $\mathcal{H}$. A square integrable representation is a representation for which a ψ exist such that

$$C_\psi = \frac{1}{\|\psi\|^2_{\mathcal{H}}} \int_G (\mathcal{U}_g \psi, \psi)_{\mathcal{H}} \, d\mu_G(g) < \infty, \tag{1.5}$$

where μ_G is a left invariant Haar measure.

The irreducibility condition is a very strong one. However, many representations of practical interest are not irreducible at all. Therefore it is often suggested, to replace the condition of irreducibility by the condition that the representation is cyclic, i.e. it has a cyclic vector, i.e. a vector for which the span of the orbit under $\mathcal{U}$ is dense in the Hilbert space. But no really successful unitarity results were obtained. For a nice survey of some posed suggestions, see [FM].

Noticeably, our Theorem 2.3 states that W_ψ defines a unitary map from $\overline{\langle V_\psi \rangle}$ onto $\mathbb{C}^G_K$. The conditions we impose on the representation are quite simple: none! Note that $\overline{\langle V_\psi \rangle} = \mathcal{H}$ if and only if ψ is a cyclic vector.

Although the above solves the unitarity questions, the functional Hilbert space, as mentioned before, is not easily characterized. We managed to give an easy to grasp description of the functional Hilbert space in the case $\mathcal{H} = \mathbb{L}_2(S)$ and $G = S \rtimes T$ for an abelian group S and an arbitrary group T which acts on S. As an example we work out the case $S = \mathbb{R}^2$ and $T = \mathbb{T}$ so G is the Euclidean motion group. The idea to consider semi-direct products is not new, several articles have been written on this subject. See for example [FKNP], [FM].

The final chapter concerns a transformation introduced by Sherman in [S]. Although this "Sherman transform" is a frame transform but not a wavelet transform, it has several resemblances with wavelet transforms. In [S], Sherman poses a key lemma concerning a singular integral. This singularity however, can be avoided. Moreover, we suggest an alternative transform which has the advantage that the unitarity relations appear in a more natural way.

2 Frame transforms and functional Hilbert spaces

2.1 Introduction

Denote the space of all complex-valued functions on $\mathbb{E}$ by $\mathbb{C}^{\mathbb{E}}$. We say that a Hilbert space $\mathcal{H}$ consisting of functions on a set $\mathbb{E}$, i.e. a vector subspace of $\mathbb{C}^{\mathbb{E}}$, is a **functional Hilbert space**, if point evaluation at every point is continuous, i.e.

$$\delta_x : \mathcal{H} \to \mathbb{C} : f \mapsto f(x) \tag{2.6}$$

is a continuous linear functional on $\mathcal{H}$ for all $x \in \mathbb{E}$. Then, by the Riesz-representation theorem, there exists a set $\{K_x \mid x \in \mathbb{E}\}$ with

$$(K_x, f)_{\mathcal{H}} = f(x), \tag{2.7}$$

for all $x \in \mathbb{E}$ and $f \in \mathcal{H}$. We use the convention that the inner product is linear in the second entry. It follows that the span of the set $\{K_x \mid x \in \mathbb{E}\}$ is dense in $\mathbb{C}^{\mathbb{E}}_K$. Indeed, if $f \in \mathcal{H}$ is orthogonal to all K_x then $f = 0$ on $\mathbb{E}$.

Then define the function $K : \mathbb{E} \times \mathbb{E} \to \mathbb{C}$ by $K(x, x') = K_{x'}(x) = (K_x, K_{x'})_{\mathcal{H}}$, for all $x, x' \in \mathbb{E}$. The function K is called the **reproducing kernel**. It is obvious that K is a **function of positive type** on $\mathbb{E}$, i.e.,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} K(x_i, x_j)\overline{c_i} c_j \geq 0, \tag{2.8}$$

for all $n \in \mathbb{N},\ c_1, ..., c_n \in \mathbb{C},\ x_1, ..., x_n \in \mathbb{E}$.

So to every functional Hilbert space there belongs a reproducing kernel, which is a function of positive type. Conversely, as Aronszajn pointed out in his paper [Ar], a function K of positive type on a set $\mathbb{E}$, induces uniquely a functional Hilbert space consisting of functions on $\mathbb{E}$ with reproducing kernel K. We will denote this space with $\mathbb{C}^{\mathbb{E}}_K$. Without giving a detailed proof we mention that $\mathbb{C}^{\mathbb{E}}_K$ can be constructed as follows; start with $K : \mathbb{E} \times \mathbb{E} \to \mathbb{C}$, a function of positive type and define $K_x = K(\cdot, x)$. Take the span $\langle\{K_x \mid x \in \mathbb{E}\}\rangle$ and define the inner product on this span as

$$\Big(\sum_{i=1}^{l} \alpha_i K_{x_i}, \sum_{j=1}^{n} \beta_j K_{x_j}\Big)_{\mathbb{C}^{\mathbb{E}}_K} = \sum_{i=1}^{l} \sum_{j=1}^{n} \overline{\alpha_i} \beta_j K(x_i, x_j). \tag{2.9}$$

This is a pre-Hilbert space. After taking the completion we arrive at the functional Hilbert space $\mathbb{C}^{\mathbb{E}}_K$.

There exists a useful characterization of the elements of $\mathbb{C}^{\mathbb{E}}_K$.

Lemma 2.1 *Let K be a function of positive type on $\mathbb{E}$ and F a complex-valued function on $\mathbb{E}$. Then the function F belongs to $\mathbb{C}^{\mathbb{E}}_K$ if and only if there exists a constant $\gamma > 0$ such that*

$$\Big| \sum_{j=1}^{l} \alpha_j \overline{F(x_j)} \Big|^2 \leq \gamma \sum_{k,j=1}^{l} \overline{\alpha_k} \alpha_j K(x_k, x_j), \tag{2.10}$$

for all $l \in \mathbb{N}$ *and* $\alpha_j \in \mathbb{C},\ x_j \in \mathbb{E},\ 1 \le j \le l.$

Proof: See [Ma, Lemma 1.7, pp.31] or [An, Th. II.1.1].

This lemma enables us to give an expression for the norm of an arbitrary element in $\mathbb{C}_K^{\mathbb{E}}$.

Lemma 2.2 *Let* $F \in \mathbb{C}_K^{\mathbb{E}}$. *Then*

$$\|F\|^2_{\mathbb{C}_K^{\mathbb{E}}} = \sup\left\{ \Big|\sum_{j=1}^{l} \alpha_j \overline{F(x_j)}\Big|^2 \Big(\sum_{k,j=1}^{l} \overline{\alpha_k}\alpha_j K(x_k,x_j)\Big)^{-1} \;\Big|\; l \in \mathbb{N},\ \alpha_j \in \mathbb{C},\ x_j \in \mathbb{E},\ \Big\|\sum_{k=1}^{l}\alpha_k K_{x_k}\Big\|_{\mathbb{C}_K^{\mathbb{E}}} \neq 0 \right\}. \tag{2.11}$$

For a detailed discussion of functional Hilbert spaces see [Ar], [An] or [Ma].

2.2 Construction of a frame transform

Starting with some labeled subset V of $\mathcal{H}$, we will construct a functional Hilbert space by means of a function of positive type on the index set, using the construction as described in the introduction. Moreover, there exists a natural unitary map from $\overline{\langle V\rangle}$ to this functional Hilbert space.

Let $\mathcal{H}$ be a Hilbert space. Let $\mathbb{E}$ be an index set and

$$V := \{\phi_x \mid x \in \mathbb{E}\}, \tag{2.12}$$

be a subset of $\mathcal{H}$. We call the set V a **frame**. Define the function $K : \mathbb{E} \times \mathbb{E} \to \mathbb{C}$ of positive type on $\mathbb{E}$ by

$$K(x,x') = (\phi_x, \phi_{x'})_{\mathcal{H}}, \tag{2.13}$$

for all $x, x' \in \mathbb{E}$. From this function of positive type the space $\mathbb{C}_K^{\mathbb{E}}$ can be constructed.

The following theorem is the central observation of this thesis.

Theorem 2.3 (Frame Theorem) *The map*

$$W : \overline{\langle V\rangle} \to \mathbb{C}_K^{\mathbb{E}} : f \mapsto Wf, \ \text{ where } \ Wf : \mathbb{E} \to \mathbb{C} : x \mapsto (\phi_x, f)_{\mathcal{H}}, \tag{2.14}$$

is a unitary map.

Proof:
Here $\overline{\langle V\rangle}$ inherits the inner product from $\mathcal{H}$. First we show that $Wf \in \mathbb{C}_K^{\mathbb{E}}$ for

any element $f \in \overline{\langle V \rangle}$ and that W is bounded (and therefore continuous). If $f \in \overline{\langle V \rangle}$ then

$$\Big|\sum_{j=1}^{l} \alpha_j \overline{(Wf)(x_j)}\Big|^2 = \Big|\sum_{j=1}^{l} \alpha_j \overline{(\phi_{x_j}, f)_{\mathcal{H}}}\Big|^2 = \Big|\Big(\sum_{j=1}^{l} \alpha_j \phi_{x_j}, f\Big)_{\mathcal{H}}\Big|^2$$

$$\leq \Big\|\sum_{j=1}^{l} \alpha_j \phi_{x_j}\Big\|_{\mathcal{H}}^2 \|f\|_{\mathcal{H}}^2 = \Big(\sum_{k,j=1}^{l} \overline{\alpha_k} \alpha_j K(x_k, x_j)\Big)\|f\|_{\mathcal{H}}^2,$$

for all $l \in \mathbb{N}$, $\alpha_1, ..., \alpha_l \in \mathbb{C}$, and $x_1, ..., x_n \in \mathbb{E}$. So $Wf \in \mathbb{C}_K^{\mathbb{E}}$ by Lemma 2.1 and $\|Wf\|^2_{\mathbb{C}_K^{\mathbb{E}}} \leq \|f\|^2_{\mathcal{H}}$, by Lemma 2.2. Next we prove that W is an isometry. Because $\big(\phi_{x'}\big)(x) = K(x, x')$, W maps a linear combination $\sum_i \alpha_i \phi_{x_i}$ onto the linear combination $\sum_i \alpha_i K(\cdot, x_i)$. So $W(\langle V \rangle) = \langle \{K(\cdot, x) | x \in \mathbb{E}\} \rangle$. Moreover, it maps $\langle V \rangle$ isometrically onto $\langle \{K(\cdot, x) | x \in \mathbb{E}\} \rangle$, because

$$\Big(W\big(\sum_i \alpha_i \phi_{x_i}\big), W\big(\sum_j \beta_j \phi_{x'_j}\big)\Big)_{\mathbb{C}_K^{\mathbb{E}}} = \Big(\sum_i \alpha_i K(\cdot, x_i), \sum_j \beta_j K(\cdot, x'_j)\Big)_{\mathbb{C}_K^{\mathbb{E}}}$$

$$= \sum_{i,j} \overline{\alpha}_i \beta_j K(x_i, x'_j) = \sum_{i,j} \overline{\alpha}_i \beta_j (\phi_{x_i}, \phi_{x'_j})_{\mathcal{H}}.$$

Since $\langle V \rangle$ is dense in $\overline{\langle V \rangle}$ and W is bounded on $\overline{\langle V \rangle}$ it follows that W is an isometry. Furthermore, $W[\langle V \rangle]$ is dense in $\mathbb{C}_K^{\mathbb{E}}$. So W is also surjective and therefore unitary. □

We will call the unitary map W a **frame transform**. In the sequel we will mostly introduce a frame transform by '$W : \langle V \rangle \to \mathbb{C}_K^{\mathbb{E}}$ defined by $\big(Wf\big)(x) = (\phi_x, f)_{\mathcal{H}}$ for all $x \in \mathbb{E}$ and $f \in \overline{\langle V \rangle}$', instead of writing '$W : \overline{\langle V \rangle} \to \mathbb{C}_K^{\mathbb{E}} : f \mapsto Wf$, where $Wf : \mathbb{E} \to \mathbb{C} : x \mapsto (\phi_x, f)_{\mathcal{H}}$'. The latter has the advantage that the structure of the objects is more tangible, but it has the disadvantage that it is a bit lengthy for simple calculations. Therefore the first phrase will be used when we deal with an explicit example of a frame transform and the second phrase otherwise.

In most cases we are mainly interested in the case $\overline{\langle V \rangle} = \mathcal{H}$, i.e. V is total in $\mathcal{H}$. To get a feeling for what is happening we deal with two illustrating examples.

Example: The special case $\mathbb{E} = \mathbb{N}$. Let $\mathcal{H}$ be a separable Hilbert space consisting of functions on the set $\mathbb{E} = \mathbb{N}$. Let $V = \{\phi_m \mid m \in \mathbb{N}\}$ consist of an orthonormal basis, so $\overline{\langle V \rangle} = \mathcal{H}$. Then,

$$K(m, m') = (\phi_m, \phi_{m'})_{\mathcal{H}} = \delta_{mm'}, \tag{2.15}$$

for all $m, m' \in \mathbb{N}$. This means that we just get $\mathbb{C}_K^{\mathbb{N}} = l_2(\mathbb{N})$. The unitary map W gives us the sequence of expansion coefficients c_m of a vector $f \in \mathcal{H}$ with respect to the orthonormal basis.

Example: The special case $\mathbb{E} = \mathcal{H}$. Let $\mathbb{E} = \mathcal{H}$ and $V = \{m | m \in \mathcal{H}\} = \mathcal{H}$. The function of positive type is just the inner product

$$K(m, m') = (m, m')_{\mathcal{H}}. \tag{2.16}$$

This means that $\mathbb{C}_K^{\mathbb{E}} = \mathbb{C}_{(\cdot,\cdot)_{\mathcal{H}}}^{\mathcal{H}}$. This is the functional Hilbert representation of an arbitrary Hilbert space. It is equal to the topological dual space $\mathcal{H}'$, the space of all continuous linear functions on $\mathcal{H}$.

The functional Hilbert space $\mathbb{C}_K^{\mathbb{E}}$ is an abstract construction. We are challenged to find alternative characterizations of these functional Hilbert spaces.

In the literature two major classes of functional Hilbert spaces appear, functional Hilbert spaces of Bargmann-type and of Sobolev-type. The first type consists of a nullspace of unbounded operators on $\mathbb{L}_2(\mathbb{E}, \mu)$ and the second of the domain of unbounded operators on $\mathbb{L}_2(\mathbb{E}, \mu)$. For Bargmann-type spaces see [B]. For Sobolev-type, see [EG1] and [EG2].

3 The inverse frame transform

3.1 Inversion using projections

Let $\mathcal{H}$ be a Hilbert space and $V \subset \mathcal{H}$ a subset of $\mathcal{H}$ labeled with elements in a set $\mathbb{E}$

$$V = \{\phi_x \mid x \in \mathbb{E}\}. \tag{3.1}$$

For the sake of simplicity assume that the span is dense in $\mathcal{H}$.Otherwise, replace $\mathcal{H}$ by $\overline{\langle V \rangle}$. Consider the unitary frame transform $W : \mathcal{H} \to \mathbb{C}_K^{\mathbb{E}} : f \mapsto Wf$, where $Wf : \mathbb{E} \to \mathbb{C} : x \mapsto (\phi_x, f)_{\mathcal{H}}$. We will analyze the inverse W^{-1} of W.

Suppose $\mathcal{H} = \mathbb{C}_L^{\mathbb{I}}$ is a functional Hilbert space itself, with reproducing kernel L. Set $\tilde{V} = \{WL_\xi \mid \xi \in \mathbb{I}\}$. Then $\langle \tilde{V} \rangle$ is dense in $\mathbb{C}_K^{\mathbb{E}}$, by unitarity of W. Moreover, $(WL_\xi, WL_{\xi'})_{\mathbb{C}_K^{\mathbb{E}}} = (L_\xi, L_{\xi'})_{\mathbb{C}_K^{\mathbb{I}}} = L(\xi, \xi')$ for all $\xi, \xi' \in \mathbb{I}$. So if

$$\tilde{W} : \mathbb{C}_K^{\mathbb{E}} \to \mathbb{C}_L^{\mathbb{I}} : g \mapsto \tilde{W}g, \;\; \text{where} \;\; \tilde{W}g : \mathbb{I} \to \mathbb{E} : \xi \mapsto (WL_\xi, g)_{\mathbb{C}_K^{\mathbb{E}}}, \tag{3.2}$$

is the associated frame transform, then

$$\big(\tilde{W}g\big)(\xi) = (WL_\xi, g)_{\mathbb{C}_K^{\mathbb{E}}} = (L_\xi, W^{-1}g)_{\mathbb{C}_L^{\mathbb{I}}} = \big(W^{-1}g\big)(\xi), \tag{3.3}$$

for all $g \in \mathbb{C}_K^{\mathbb{E}}$ and $\xi \in \mathbb{I}$. Hence, $W^{-1} = \tilde{W}$ is also a frame transform. Note that $(WL_\xi)(x) = (\phi_x, L_\xi)_{\mathbb{C}_L^{\mathbb{I}}} = \overline{\phi_x(\xi)}$.

Although the second example at the end of section 1 shows that every Hilbert space can be characterized as a functional Hilbert space, this characterization is not always very useful. In some cases, like $\mathbb{L}_2(\mathbb{R})$ or $\mathbb{L}_2(S^1)$, the Hilbert space $\mathcal{H}$ can be regarded as limits of functional Hilbert spaces. The space $\mathbb{L}_2(S^1)$ for example, admits a decomposition in spherical harmonics. This decomposition will be dealt with in section 8.

Definition 3.1 *We say that a sequence of functional Hilbert space $\{\mathbb{C}_{L_n}^{\mathbb{I}}\}_{n \in \mathbb{N}}$ converges to $\mathcal{H}$ if*

1. *each $\mathbb{C}_{L_n}^{\mathbb{I}}$ is a Hilbert subspace of $\mathcal{H}$*
2. *the projections $\mathbb{P}_n$ of $\mathcal{H}$ on $\mathbb{C}_{L_n}^{\mathbb{I}}$ satisfy*
   - *(a)* $\mathbb{P}_n \mathbb{P}_m = \mathbb{P}_{\min\{n,m\}}$
   - *(b)* $\lim_{n\to\infty} \mathbb{P}_n f \to f$ *, for all* $f \in \mathcal{H}$

Suppose the sequence of functional Hilbert space $\mathbb{C}_{L_n}^{\mathbb{I}}$ converges to $\mathcal{H}$. Define

$$V_n = \{\mathbb{P}_n \psi_x \mid x \in \mathbb{E}\}, \tag{3.4}$$

and the function $K_n : \mathbb{E} \times \mathbb{E} \to \mathbb{C}$ of positive type by

$$K_n(x, x') = (\mathbb{P}_n \phi_x, \mathbb{P}\phi_{x'})_{\mathcal{H}} \tag{3.5}$$

for all $n \in \mathbb{N}$ and $x, x' \in \mathbb{E}$. By Theorem 2.3 the map $W_n : \mathbb{C}^{\mathbb{I}}_{L_n} \to \mathbb{C}^{\mathbb{E}}_{K_n} : f \mapsto W_n f$, where $W_n f : \mathbb{E} \to \mathbb{C} : x \mapsto (\mathbb{P}_n \psi_x, f)_{\mathcal{H}}$ is unitary for all $n \in \mathbb{N}$. Note that for the frame transform W_n the inverse frame transform is given by $W_n^{-1} : \mathbb{C}^{\mathbb{E}}_K \to \mathbb{C}^{\mathbb{I}}_L : g \mapsto W_n^{-1} g$ where $W_n^{-1} g : \mathbb{I} \to \mathbb{C} : \xi \mapsto (W_n L_{n;\xi}, g)_{\mathbb{C}^{\mathbb{E}}_K}$, for all $n \in \mathbb{N}$.

Theorem 3.2 *The functional Hilbert spaces $\mathbb{C}^{\mathbb{E}}_{K_n}$ converge to $\mathbb{C}^{\mathbb{E}}_K$.*

Proof:
Let $f \in \mathbb{C}^{\mathbb{E}}_{K_n}$. Then $g = W_n^{-1} f \in \mathbb{C}^{\mathbb{I}}_{L_n} \subset \mathcal{H}$. This means that $\mathbb{C}^{\mathbb{E}}_K \ni Wg = W_n g = f$. By the unitarity of W_n and W we also find $\|f\|_{\mathbb{C}^{\mathbb{E}}_{K_n}} = \|g\|_{\mathbb{C}^{\mathbb{I}}_{L_n}} = \|g\|_{\mathcal{H}} = \|f\|_{\mathbb{C}^{\mathbb{E}}_K}$. Therefore, the spaces $\mathbb{C}^{\mathbb{E}}_{K_n}$ are all closed subspaces of $\mathbb{C}^{\mathbb{E}}_K$.

Finally, to prove condition 2, we remark that the projection $\mathbb{P}'_n$ on the space $\mathbb{C}^{\mathbb{E}}_{K_n}$ is given by $\mathbb{P}'_n = W \mathbb{P}_n W^{-1}$. □

The following theorem gives a formula for the inverse frame transform.

Theorem 3.3

$$\lim_{n \to \infty} W_n^{-1} \mathbb{P}'_n W f = f, \tag{3.6}$$

for all $f \in \mathcal{H}$.

Proof:
Since $\mathbb{P}'_n W f \to W f$ we also find by unitarity $W_n^{-1} \mathbb{P}'_n W f = W^{-1} \mathbb{P}'_n W f \to f$.
□

3.2 Inversion using Gelfand triples

In this section we mention another method to obtain the inverse W^{-1}.

Assume V is a subset of a vector space labeled by a set $\mathbb{E}$. Equip $\langle V \rangle$ with two inner products $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ such that $\|\cdot\|_1 \leq C \|\cdot\|_2$, for some constant $C > 0$. Denote the completions of $\langle V \rangle$ under these two norms by $\mathcal{H}_1$ and $\mathcal{H}_2$. Note that $\mathcal{H}_2 \subset \mathcal{H}_1$. Then by (2.13) we obtain the functional Hilbert spaces $\mathbb{C}^{\mathbb{E}}_{K_1}$ and $\mathbb{C}^{\mathbb{E}}_{K_2}$. The question arises how these two functional Hilbert spaces are related. The answer is straightforward. Since $\|\cdot\|_1 \leq C \|\cdot\|_2$, it is obvious that

$$\begin{aligned}
\sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\alpha_j} K_1(x_i, x_j) &= \sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\alpha_j} (\phi_{x_i}, \phi_{x_j})_1 = \| \sum_{i=1}^n \alpha_i \phi_{x_i} \|_1^2 \\
&\leq C^2 \| \sum_{i=1}^n \alpha_i \phi_{x_i} \|_2^2 = C^2 \sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\alpha_j} K_2(x_i, x_j),
\end{aligned}$$

for all $n \in \mathbb{N}, \alpha_1 \ldots \alpha_n, x_1 \ldots x_n$ and hence $K_2 \geq K_1$. By Lemma 2.1, it follows that $\mathbb{C}^{\mathbb{E}}_{K_1} \subset \mathbb{C}^{\mathbb{E}}_{K_2}$ and $\|\cdot\|_{\mathbb{C}^{\mathbb{E}}_{K_2}} \leq C \|\cdot\|_{\mathbb{C}^{\mathbb{E}}_{K_1}}$ on $\mathbb{C}^{\mathbb{E}}_{K_1}$ by Lemma 2.2.

We return to the problem of inverting the frame transform. Assume $V = \{\phi_x \mid x \in \mathbb{E}\} \subset \mathbb{C}^{\mathbb{I}}$ for some set $\mathbb{I}$. Define $\tilde{\phi}_\xi : \mathbb{E} \to \mathbb{C}$ by $\phi_\xi(x) = \overline{\phi_x(\xi)}$ for all $x \in \mathbb{E}$ and $\xi \in \mathbb{I}$. Recall that if $\mathcal{H}$ is a functional Hilbert space $\mathbb{C}_L^{\mathbb{I}}$ for some function L of positive type, then the inverse W^{-1} equals the frame transform with respect to $\tilde{V} = \{\tilde{\phi}_\xi \mid \xi \in \mathbb{I}\}$. Then in addition $\tilde{V} \subset \mathbb{C}_K^{\mathbb{E}}$. In general L does not exist and $\tilde{V}$ is not a subset of $\mathbb{C}_K^{\mathbb{E}}$. Nevertheless, the functions $\tilde{\phi}_\xi$ are still well-defined. In the sequel we use Gelfand triples to understand the role of these functions.

Let $R \in \mathcal{B}(\mathcal{H})$ such that R^{-1} exists and is a self-adjoint operator on $\mathcal{H}$. Hence R^{-1} is in particular densely defined and closed. Consider the Gelfand triple

$$\mathcal{H}_+ \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_-, \tag{3.7}$$

constructed by this operator R. Recall that $\|\cdot\|_+ = \|R^{-1}\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_- = \|R\cdot\|_{\mathcal{H}}$.

Assume that $\langle V\rangle$ is a dense subspace of $\mathcal{H}_+$, i.e.

$$\overline{\langle\{R^{-1}\phi_x \mid x \in \mathbb{E}\}\rangle} = \mathcal{H}. \tag{3.8}$$

Since R is bounded, this assumption implies that $\langle V\rangle$ is dense in all the space $\mathcal{H}_+, \mathcal{H}$ and $\mathcal{H}_-$ and therefore it makes the assumption $\overline{\langle V\rangle} = \mathcal{H}$ obsolete.

The frame transform W maps $\mathcal{H}$ unitary onto $\mathbb{C}_K^{\mathbb{E}}$. Define $A \in \mathcal{B}(\mathbb{C}_K^{\mathbb{E}})$ by $A = WRW^{-1}$. Then A induces the Gelfand-triple

$$\left(\mathbb{C}_K^{\mathbb{E}}\right)_+ \hookrightarrow \mathbb{C}_K^{\mathbb{E}} \hookrightarrow \left(\mathbb{C}_K^{\mathbb{E}}\right)_-. \tag{3.9}$$

Note that the frame transform W induces by restriction the unitary map $W|_{\mathcal{H}+} : \mathcal{H}_+ \to \left(\mathbb{C}_K^{\mathbb{E}}\right)_+ : f \mapsto Wf$ and after extension it induces in the same way a unitary map from $\mathcal{H}_-$ onto $\left(\mathbb{C}_K^{\mathbb{E}}\right)_-$. Next we introduce two other functions of positive type and frame transforms. Define $K_- : \mathbb{E} \times \mathbb{E} \to \mathbb{C}$ by

$$K_-(x, x') = (\phi_x, \phi_{x'})_+, \tag{3.10}$$

for all $x, x' \in \mathbb{E}$. Note the opposite signs. By Theorem 2.3, the transform $W_+ : \mathcal{H}_+ \to \mathbb{C}_{K_-}^{\mathbb{E}}$ defined by

$$\big(W_+ f\big)(x) = (\phi_x, f)_+, \tag{3.11}$$

for all $x \in \mathbb{E}$ and $f \in \mathcal{H}_+$, is a unitary map from $\mathcal{H}_+$ onto $\mathbb{C}_{K_-}^{\mathbb{E}}$. Equivalently, define $K_+ : \mathbb{E} \times \mathbb{E} \to \mathbb{C}$ by $K_+(x, x') = (\phi_x, \phi_{x'})_-$, for all $x, x' \in \mathbb{E}$. By Theorem 2.3, the transform $W_- : \mathcal{H}_- \to \mathbb{C}_{K_+}^{\mathbb{E}}$ defined by $\big(W_- f\big)(x) = (\phi_x, f)_-$, for all $x \in \mathbb{E}$ and $f \in \mathcal{H}_-$, is a unitary map from $\mathcal{H}_-$ onto $\mathbb{C}_{K_+}^{\mathbb{E}}$.

Lemma 3.4 *$\mathbb{C}_{K_+}^{\mathbb{E}} = \left(\mathbb{C}_K^{\mathbb{E}}\right)_+$ and $\mathbb{C}_{K_-}^{\mathbb{E}} = \left(\mathbb{C}_K^{\mathbb{E}}\right)_-$.*

Proof:
First we show that $\langle\{K_x \mid x \in \mathbb{E}\}\rangle$ is dense in both $\left(\mathbb{C}_K^{\mathbb{E}}\right)_+$ and $\mathbb{C}_{K_+}^{\mathbb{E}}$.

Since $K_x = W\phi_x$ and $\phi_x \in \mathcal{D}(R^{-1})$, we also have $K_x \in \mathcal{D}(A^{-1}) = \left(\mathbb{C}_K^{\mathbb{E}}\right)_+$. Moreover, $\langle\{K_x \mid x \in \mathbb{E}\}\rangle$ is dense in $\left(\mathbb{C}_K^{\mathbb{E}}\right)_+$ by unitarity of $W|_{\mathcal{H}_+}$ and assumption (3.8).

For the space $\mathbb{C}_{K_+}^{\mathbb{E}}$ recall that $\phi_x \in \mathcal{D}(R^{-1})$ and hence $R^{-2}\phi_x \in \mathcal{H}_-$ for all $x \in \mathbb{E}$. Moreover, $\langle\{R^{-2}\phi_x \mid x \in \mathbb{E}\}\rangle$ is dense in $\mathcal{H}_-$. The operator W_- maps $R^{-2}\phi_x$ onto K_x and therefore $\{K_x \mid x \in \mathbb{E}\}$ is a subset of $\mathbb{C}_{K_+}^{\mathbb{E}}$ for which the span is dense in $\mathbb{C}_{K_+}^{\mathbb{E}}$.

Finally we show that the inner products are equal on this dense subspace.

$$\begin{aligned}(K_x, K_y)_{\mathbb{C}_{K_+}^{\mathbb{E}}} &= (R^{-2}\phi_x, R^{-2}\phi_y)_- = (R^{-1}\phi_x, R^{-1}\phi_y)_{\mathcal{H}} \\ &= (WR^{-1}\phi_x, WR^{-1}\phi_y)_{\mathbb{C}_K^{\mathbb{E}}} = (A^{-1}K_x, A^{-1}K_y)_{\mathbb{C}_K^{\mathbb{E}}}\end{aligned}$$

for all $x, y \in \mathbb{E}$. Hence in particular $\|\cdot\|_{\mathbb{C}_{K_+}^{\mathbb{E}}} = \|\cdot\|_{\left(\mathbb{C}_K^{\mathbb{E}}\right)_+}$ on $\langle\{K_x \mid x \in \mathbb{E}\}\rangle$.

In the same way one can prove that $\|\cdot\|_{\mathbb{C}_{K_-}^{\mathbb{E}}} = \|\cdot\|_{\left(\mathbb{C}_K^{\mathbb{E}}\right)_-}$ on $\langle\{K_x \| x \in \mathbb{E}\}\rangle$ which is dense in $\mathbb{C}_{K_-}^{\mathbb{E}}$ and $\left(\mathbb{C}_K^{\mathbb{E}}\right)_-$. □

From now on we assume that R is such that $\mathcal{H}_+$ is a functional Hilbert space with reproducing kernel L.

Theorem 3.5 *The set $\{\tilde{\phi}_\xi \mid \xi \in \mathbb{I}\}$ is contained in $\mathbb{C}_{K_-}^{\mathbb{E}}$. The inverse $W|_{\mathcal{H}_+}^{-1}$ is given by*

$$W|_{\mathcal{H}_+}^{-1} : \mathbb{C}_{K_-}^{\mathbb{E}} \to \mathcal{H}_+ : F \mapsto W|_{\mathcal{H}_+}^{-1}F, \quad \text{where} \quad W|_{\mathcal{H}_+}^{-1}F : \mathbb{E} \to \mathbb{C} : \xi \mapsto \langle\tilde{\phi}_\xi, F\rangle. \tag{3.12}$$

Proof:

The first statement is trivial since $\mathcal{H}_+$ is a functional Hilbert space and hence $\tilde{\phi}_\xi = WL_\xi$ where L is the reproducing kernel of $\mathcal{H}_+$.

For the second statement we first show that $AK_{-;x} = A^{-1}K_x$ for all $x \in \mathbb{E}$.

$$\begin{aligned}\left(AK_{-;x}\right)(y) &= (K_y, AK_{-;x})_{\mathbb{C}_K^{\mathbb{E}}} = (A^{-1}K_y, K_{-;x})_{\mathbb{C}_{K_-}^{\mathbb{E}}} \\ &= \overline{\left(A^{-1}K_y\right)(x)} = (A^{-1}K_y, K_x) = (K_y, A^{-1}K_x) = \left(A^{-1}K_x\right)(y)\end{aligned}$$

for all $x, y \in \mathbb{E}$.

Secondly, we show that $A^{-2}WL_\xi = \tilde{\phi}_\xi$ for all $\xi \in \mathbb{I}$. Denote the reproducing kernel of $\mathcal{H}_-$ by L. Note that $WL_\xi \in \mathbb{C}_{K_+}^{\mathbb{E}}$ hence $A^{-2}WL_\xi$ is well-defined. Then,

$$\begin{aligned}\left(A^{-2}WL_\xi\right)(x) &= (K_{-;x}, A^{-2}WL_\xi)_{\mathbb{C}_{K_-}^{\mathbb{E}}} = (AK_{-;x}, A^{-1}WL_\xi)_{\mathbb{C}_K^{\mathbb{E}}} \\ &= (A^{-1}K_x, A^{-1}WL_\xi)_{\mathbb{C}_K^{\mathbb{E}}} = (WR^{-1}\phi_x, WR^{-1}L_\xi)_{\mathbb{C}_K^{\mathbb{E}}} \\ &= (R^{-1}\phi_x, R^{-1}L_\xi)_{\mathcal{H}} = (\phi_x, L_\xi)_{\mathcal{H}_+} = (W_-\phi_x, W_-L_\xi)_{\mathbb{C}_{K_-}^{\mathbb{E}}} \\ &= (K_{-;x}, \tilde{\phi}_\xi)_{\mathbb{C}_{K_-}^{\mathbb{E}}} = \tilde{\phi}_\xi(x),\end{aligned}$$

for all $x \in \mathbb{E}$.

Finally, we prove the inversion formula:

$$\begin{aligned} f(\xi) &= (L_\xi, f)_{\mathcal{H}_-} = (WL_\xi, Wf)_{\mathbb{C}^{\mathbb{E}}_{K_+}} = (A^{-1}WL_\xi, A^{-1}Wf)_{\mathbb{C}^{\mathbb{E}}_K} \\ &= \langle A^{-2}WL_\xi, Wf\rangle = \langle \tilde{\phi}_\xi, Wf\rangle, \end{aligned}$$

for all $f \in \mathcal{H}_+$ and $\xi \in \mathbb{I}$. □

Let $t \mapsto R_t$ be a strongly continuous contraction semi-group such that the generator is self-adjoint. Then for all $t \geq 0$,

- R_t is self-adjoint,
- R_t^{-1} exists and is self-adjoint (and hence densely defined).

Moreover, assume that

- $\mathcal{H}_t$ is a functional Hilbert space under the norm $\|\cdot\|_t = \|R_t^{-1}\cdot\|_{\mathcal{H}}$,
- $\langle R_t^{-1}V\rangle$ is dense in $\mathcal{H}$

for all $t > 0$. Then each of the operators R_t defines a Gelfand triple, denoted by

$$\mathcal{H}_t \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{-t}, \tag{3.13}$$

for all $t > 0$.

Lemma 3.6 $\|R_t\| \leq \|R_s\|$ *for all* $t > s$.

Proof:
Let $f \in \mathcal{H}$. Then,

$$\|R_t f\|_{\mathcal{H}} = \|R_{t-s}R_s f\|_{\mathcal{H}} \leq \|R_{t-s}\|\|R_s f\|_{\mathcal{H}} \leq \|R_s f\|_{\mathcal{H}}.$$

Hence $\|R_t\| \leq \|R_s\|$. □

Lemma 3.7 *The space* $\mathcal{H}_t$ *is continuous embedded in* $\mathcal{H}_s$ *for all* $t > s$.

Proof:
Let $f \in \mathcal{H}_t$. There exist a $g \in \mathcal{H}$ such that $f = R_t g \in R_t(\mathcal{H})$ and hence $f = R_s R_{t-s} g \in \mathcal{H}_s$. Moreover,

$$\|f\|_{\mathcal{H}_s} = \|R_s^{-1}f\|_{\mathcal{H}} \leq \|R_t^{-1}f\|_{\mathcal{H}} = \|f\|_{\mathcal{H}_t}.$$

Hence the statement follows. □

Now we obtain the following the inversion formula

$$f = \lim_{t\downarrow 0} R_t f = \lim_{t\downarrow 0} \xi \mapsto \langle \tilde{\phi}_\xi, WR_t f\rangle_t = \lim_{t\downarrow 0} \xi \mapsto \langle \tilde{\phi}_\xi, A_t W f\rangle_t. \tag{3.14}$$

This formula is the generalization of the formula Bargmann gives in his article to invert the Bargmann transform. We will return to this subject in the Section 4.2. In that section the assumption (3.8) is equivalent to the assumption that $\overline{\langle V\rangle} = \mathcal{H}$ since in that case $R_t V = V$ for all $t > 0$.

4 Frame transforms constructed from generating functions

4.1 An expansion theorem

Since by Theorem (2.3) a frame transform is a unitary map, it maps an orthonormal basis onto an orthonormal basis. This leads to some nice consequences in case the set V in (2.12) is constructed from a generating function, since it provides a convenient basis for the image space. In this section, the image space typically is a functional Hilbert space consisting of analytic functions. An important result is the following

Theorem 4.1 *Let $K : \mathbb{E} \times \mathbb{E} \to \mathbb{C}$ be a function of positive type and let $\{g_n \mid n \in \mathbb{N}\}$ be an orthonormal set in $\mathbb{C}_K^{\mathbb{E}}$. Then $\sum_{n\in\mathbb{N}} g_n(x)\overline{g_n(y)}$ is absolutely convergent for all $x, y \in \mathbb{E}$. Moreover, the set $\{g_n \mid n \in \mathbb{N}\}$ is a basis for $C_K^{\mathbb{E}}$ if and only if*

$$K(x,y) = \sum_{n\in\mathbb{N}} g_n(x)\overline{g_n(y)}, \tag{4.1}$$

for all $x, y \in \mathbb{E}$.

Proof:
See [Ma, Lemma 1.11]. □

Let $\mathcal{H}$ be a Hilbert space and $V = \{\phi_x \mid x \in \mathbb{E}\}$ a subset of $\mathcal{H}$ labeled by a set $\mathbb{E}$. Define the function $K : \mathbb{E} \times \mathbb{E} \to \mathbb{C}$ of positive type by $K(x, x') = (\phi_{x'}, \phi_x)_{\mathcal{H}}$ for all $x, x' \in \mathbb{E}$. By Theorem 2.3, the frame transform $W : \overline{\langle V \rangle} \to \mathbb{C}_K^{\mathbb{E}} : f \mapsto Wf$, where $Wf : \mathbb{E} \to \mathbb{C} : x \mapsto (\phi_x, f)_{\mathcal{H}}$, is a unitary map.

Suppose $\{g_n \mid n \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{H}$. Then ϕ_x can be expanded in this basis as

$$\phi_x = \sum_{n\in\mathbb{N}} a_n(x) g_n, \tag{4.2}$$

where $a_n(x) = (g_n, \phi_x)$, for all $x \in \mathbb{E}$ and for all $n \in \mathbb{N}$.

Theorem 4.2 (Expansion Theorem) *The reproducing kernel $K : \mathbb{E} \times \mathbb{E} \to \mathbb{C}$ is given by*

$$K(x, x') = \sum_{n\in\mathbb{N}} \overline{a_n(x)} a_n(x'), \tag{4.3}$$

for all $x, x' \in \mathbb{E}$, where the sum is absolutely convergent. Moreover, if $\overline{\langle V \rangle} = \mathcal{H}$ then $\{\overline{a_n} \mid n \in \mathbb{N}\}$ is an orthonormal basis for $\mathbb{C}_K^{\mathbb{E}}$.

Proof:
It is obvious that $\big(Wg_n\big)(x) = \overline{a_n(x)}$ for all $x \in \mathbb{E}$ and $n \in \mathbb{N}$. By unitarity of W it follows that $\{\overline{a_n} \mid n \in \mathbb{N}\}$ is an orthonormal basis for $\mathbb{C}_K^{\mathbb{E}}$. The rest of the statement follows by Theorem 4.1. □

Suppose $f \in V^{\perp}$ and write $f = \sum_{n \in \mathbb{N}} c_n g_n$. Then

$$0 = (\psi_x, f)_{\mathcal{H}} = \sum_{n \in \mathbb{N}} c_n \overline{a_n(x)}, \tag{4.4}$$

for all $x \in \mathbb{E}$. This implies that $\overline{\langle V \rangle} = \mathcal{H}$ if and only if for all $\{c_n\}_{n \in \mathbb{N}} \in \ell_2(\mathbb{N})$

$$\left(\forall_{x \in \mathbb{E}} \sum_{n \in \mathbb{N}} c_n \overline{a_n(x)} = 0 \right) \Rightarrow \forall_{n \in \mathbb{N}} c_n = 0. \tag{4.5}$$

Theorem 4.2 can be used to construct special frame transforms based on generating functions. As an example we deal with the Bargmann-transform and two transforms based on generating functions for Laguerre Polynomials and Gegenbauer polynomials. In these examples $\mathbb{E}$ equals $\mathbb{C}$ or $\{z \in \mathbb{C} \mid |z| < 1\}$ and the generating function is of the form

$$\phi_z = \sum_{n=0}^{\infty} g_n z^n. \tag{4.6}$$

Obviously, the functions $z \mapsto z^n$ satisfy condition (4.5) and therefore $\overline{\langle V \rangle} = \mathcal{H}$.

4.2 Bargmann-transform

In this section the Bargmann-transform as an example to the previous sections. Some proofs, especially the unitarity, will become a lot easier with the aid of the theory developed so far.

The following two results will be used. The proof is omitted.

- A result due to Cramer: $|H_n(x)| < k\sqrt{n} 2^{n/2} \mathrm{e}^{x^2/2}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, where k is a constant. See [C].
- Mehler's formula for Hermite polynomials

$$\sum_{n=0}^{\infty} \frac{H_n(x) H_n(y)}{n!} (w/2)^n = (1 - w^2)^{-\frac{1}{2}} \exp\left[\frac{2xyw - (x^2 + y^2) w^2}{(1 - w^2)} \right], \tag{4.7}$$

  which is valid for all $x, y \in \mathbb{R}$ and $w \in D = \{z \in \mathbb{C} \mid |z| < 1\}$. See [MOS, §5.6, pp.252].

Let $\mathcal{H} = \mathbb{L}_2(\mathbb{R})$. From the theory of special functions it is well-known that the generating function of the Hermite polynomials is given by

$$\mathrm{e}^{-z^2 + 2zx} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n, \tag{4.8}$$

and the sum converges for all $z \in \mathbb{C}$ and $x \in \mathbb{R}$. Then rewrite this identity as

$$\pi^{-1/4} \mathrm{e}^{-\overline{z}^2/2+\sqrt{2}\overline{z}x-x^2/2} = \sum_{n=0}^{\infty} \frac{H_n(x)\mathrm{e}^{-\frac{1}{2}x^2}}{\sqrt{2^n n! \sqrt{\pi}}} \frac{\overline{z}^n}{\sqrt{n!}}. \tag{4.9}$$

for all $z \in \mathbb{C}$ and $x \in \mathbb{R}$. Define the subset $V = \{\phi_z \mid z \in \mathbb{C}\}$ where

$$\phi_z(x) = \pi^{-1/4} \mathrm{e}^{-\overline{z}^2/2+\sqrt{2}\overline{z}x-x^2/2}, \tag{4.10}$$

for almost all $x \in \mathbb{R}$ and all $z \in \mathbb{C}$. Define the function $K : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ of positive type by

$$K(z, w) = \sum_{n=0}^{\infty} \frac{z^n \overline{w^n}}{n!} = \mathrm{e}^{z\overline{w}}, \tag{4.11}$$

for all $z, w \in \mathbb{C}$. By Theorem 4.2, the frame transform $W : \mathcal{H} \to \mathbb{C}_K^{\mathbb{C}}$ defined by

$$\big(Wf\big)(z) = \pi^{-1/4} \int_{\mathbb{R}} \mathrm{e}^{-z^2/2+\sqrt{2}zx-x^2/2} f(x)\ \mathrm{d}x, \tag{4.12}$$

for all $z \in \mathbb{C}$ and $f \in \mathbb{L}_2(\mathbb{R})$ is a unitary map. Moreover the set $\{z \mapsto \frac{z^n}{\sqrt{n}} \mid n \in \mathbb{N}_0\}$ is an orthonormal basis in $\mathbb{C}_K^{\mathbb{C}}$, since $\{g_n : x \mapsto \frac{H_n(x)\mathrm{e}^{-\frac{1}{2}x^2}}{\sqrt{2^n n! \sqrt{\pi}}} \mid n \in \mathbb{N}_0\}$ is an orthonormal basis in $\mathbb{L}_2(\mathbb{R})$. The inner product of the functional Hilbert space is characterized by the integral

$$(\Phi, \Psi)_{\mathbb{C}_K^{\mathbb{E}}} = \frac{1}{\pi} \int_{\mathbb{C}} \overline{\Phi}(z)\Psi(z)\mathrm{e}^{-|z|^2}\ dz, \tag{4.13}$$

for all $\Phi, \Psi \in \mathbb{C}_K^{\mathbb{E}}$, since this inner product makes the set $\{z \mapsto \frac{z^n}{\sqrt{n!}} \mid n \in \mathbb{N}_0\}$ orthonormal. The functional Hilbert space $\mathbb{C}_K^{\mathbb{C}}$ is called the Bargmann-space of dimension 1. It will be denoted by $\mathfrak{B}$. The generalization to higher dimensions is straightforward and therefore we only give the reference [B].

Define the harmonic oscillator operator H on $\mathbb{L}_2(\mathbb{R})$ by

$$\big(Hf\big)(x) = \frac{1}{2}\Big((x^2-1)f(x) - \frac{d^2}{dx^2}f(x)\Big), \tag{4.14}$$

for almost all $x \in \mathbb{R}$ and all $f \in D(H) = \{f \in \mathbb{L}_2(\mathbb{R}) \mid \frac{d^2}{dx^2}f \in \mathbb{L}_2(\mathbb{R}) \wedge Mf \in \mathbb{L}_2(\mathbb{R})\}$, where $\big(Mf\big)(y) = y^2 f(y)$ for all $y \in \mathbb{R}$.. The basis elements g_n are eigenfunctions of H with eigenvalue n for all $n \in \mathbb{N}_0$, i.e. $Hg_n = ng_n$ for all $n \in \mathbb{N}_0$. The operator H is a self-adjoint positive operator and therefore $-H$ is an infinitesimal generator of a strongly continuous contraction semi-group of self-adjoint operators denoted by $t \mapsto \mathrm{e}^{-tH}$. Obviously, $e^{-tH}g_n = e^{-tn}g_n$ for all $t > 0$ and $n \in \mathbb{N}_0$. Denote the inverse of e^{-tH} by e^{tH} for all $t > 0$.

With the operators e^{-tH} and e^{tH} we want to construct a Gelfand triple as in (3.13). Therefore $\mathcal{H}_t = \mathrm{e}^{-tH}(\mathbb{L}_2(\mathbb{R}))$ must be a functional Hilbert space. Strictly speaking,

$\mathcal{H}_t \subset \mathbb{L}_2(\mathbb{R})$. As such, it consists of equivalence classes of measurable functions instead of functions and cannot be a functional Hilbert space. For this moment we will write $[f]$ instead f for elements of $\mathbb{L}_2(\mathbb{R})$. It turns out that every class $[f] \in \mathrm{e}^{-tH}(\mathbb{L}_2(\mathbb{R}))$ has a unique continuous representant $\tilde{f}$. Now $\mathcal{H}_t$ is defined to the linear space of this continuous representants, equipped with the norm $\|h\|_{\mathcal{H}_t} = \|\mathrm{e}^{tH}[h]\|_{\mathbb{L}_2(\mathbb{R})}$. The space $\mathcal{H}_t$ is a Hilbert space of its own right. Moreover, it turns out that $\mathcal{H}_t$ is a functional Hilbert space.

Let $[f] \in \mathcal{H}_t$ and write $[f] = \sum_{n=0}^{\infty} c_n[g_n]$. Since $[f] \in \mathcal{H}_t$, $\|\mathrm{e}^{tH}[f]\|^2_{\mathbb{L}_2(\mathbb{R})} = \sum_{n=0}^{\infty} |c_n|^2 \mathrm{e}^{2tn}$. Define the sequence $\{f_N\}_{N\in\mathbb{N}}$ of functions on $\mathbb{R}$ by

$$f_N(x) = \sum_{n=0}^{N} c_n g_n(x) \tag{4.15}$$

for all $x \in \mathbb{R}$ and $N \in \mathbb{N}$. Then

$$\begin{aligned} |f_N(x) - f_M(x)| &\leq \sum_{n=M}^{N} |c_n g_n(x)| \leq k \sum_{n=M}^{N} |c_n| = \sum_{n=M}^{N} |c_n| \mathrm{e}^{tn} \mathrm{e}^{-tn} \\ &\leq k \sqrt{\sum_{n=M}^{N} |c_n|^2 \mathrm{e}^{2tn}} \sqrt{\sum_{n=M}^{N} \mathrm{e}^{-2tn}} \leq \frac{k}{\sqrt{1-\mathrm{e}^{-2t}}} \|f\|_{\mathcal{H}_t} \end{aligned} \tag{4.16}$$

for all $x \in \mathbb{R}$ and $N > M \in \mathbb{N}$. Hence f_N converges pointwise to a function $\tilde{f}$ on $\mathbb{R}$. Moreover, it converges uniformly on $\mathbb{R}$. Hence it follows that $\tilde{f}$ is the continuous representant of the equivalence-class $[f]$. Naturally this representant must be unique. Moreover, by a same estimate as above

$$|\tilde{f}(x)| \leq \frac{k}{\sqrt{1-\mathrm{e}^{-2t}}} \|\mathrm{e}^{tH}[\tilde{f}]\|_{\mathbb{L}_2(\mathbb{R})} = \frac{k}{\sqrt{1-\mathrm{e}^{-2t}}} \|\tilde{f}\|_{\mathcal{H}_t} \tag{4.17}$$

for all $x \in \mathbb{R}$. In the sequel we will not use the notation $[f]$ for elements in $\mathbb{L}_2(\mathbb{R})$ anymore.

Lemma 4.3 *Let $t > 0$. The space $\mathcal{H}_t = \mathrm{e}^{-tH}(\mathbb{L}_2(\mathbb{R}))$ with inner product $(\cdot,\cdot)_t = (\mathrm{e}^{tH}\cdot, \mathrm{e}^{tH}\cdot)_{\mathbb{L}_2(\mathbb{R})}$ is the functional Hilbert space with reproducing kernel*

$$L_t(x,y) = \sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)}{2^n n!} \mathrm{e}^{-x^2/2-y^2/2} \mathrm{e}^{-2nt} = (1-\mathrm{e}^{-4t})^{-\frac{1}{2}} \exp\left[\frac{2xy - \frac{1}{2}(x^2+y^2)\cosh(2t)}{\sinh(2t)}\right] \tag{4.18}$$

for all $x, y \in \mathbb{R}$.

Proof:
From (4.17) it follows that $\mathcal{H}_t$ is a functional Hilbert space. The set $\{\mathrm{e}^{-tn} g_n \mid n \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{H}_t$, hence the statement follows by Theorem 4.1 and Mehlers formula 4.7. □

By applying the Cauchy-Schwartz inequality one easily obtains the following inequality,

$$|f(x)|^2 \leq \|f\|^2_{\mathcal{H}_t} \|L_x\|^2_{\mathcal{H}_t} = \|f\|^2_{\mathcal{H}_t} (1 - \mathrm{e}^{-4t})^{-\frac{1}{2}} \exp\left[\frac{x^2(2 - \cosh(2t))}{\sinh(2t)}\right], \tag{4.19}$$

for all $x \in \mathbb{R}$. For all $f \in \mathcal{H}_t$ and $x \in \mathbb{R}$.

Lemma 4.4 *Let $t \in \mathbb{R}$, then $V \subset \mathcal{H}_t$. Moreover,*

$$\mathrm{e}^{tH} \phi_z = \phi_{\mathrm{e}^t z}, \tag{4.20}$$

for all $z \in \mathbb{C}$.

Proof:
Let $z \in \mathbb{C}$. Then

$$\mathrm{e}^{tH} \phi_z = \mathrm{e}^{tH} \sum_{n=0}^{\infty} g_n \frac{\overline{z}^n}{\sqrt{n!}} = \sum_{n=0}^{\infty} g_n \mathrm{e}^{tn} \frac{\overline{z}^n}{\sqrt{n!}} = \sum_{n=0}^{\infty} g_n \frac{(\mathrm{e}^t \overline{z})^n}{\sqrt{n!}} = \phi_{\mathrm{e}^t z}$$

This proves the statement. □

The lemma implies that $\mathrm{e}^{-tH} V$ is dense in $\mathbb{L}_2(\mathbb{R})$ since $\mathrm{e}^{-tH} V = V$ and V is dense in $\mathbb{L}_2(\mathbb{R})$. This proves the final assumption in the Gelfand triple (3.13). Analogous to section 3.2, we introduce the operator $A_t = W \mathrm{e}^{-tH} W$.

Corollary 4.5 *Let $t > 0$. The operator $A_t = W \mathrm{e}^{-tH} W^{-1}$ is given by*

$$\big(A_t f\big)(z) = f(\mathrm{e}^{-t} z), \tag{4.21}$$

for all $z \in \mathbb{C}$ and $f \in \mathbb{C}^{\mathbb{C}}_{K_t}$.

Proof:
Let $f \in \mathbb{C}^{\mathbb{C}}_{K_t}$. Then

$$\begin{aligned} \big(A_t f\big)(z) &= \big(W \mathrm{e}^{-tH} W^{-1} f\big)\big(z\big) = (\mathrm{e}^{-tH} W^{-1} f, \phi_z)_{\mathbb{L}_2(\mathbb{R})} = (W^{-1} f, \mathrm{e}^{-tH} \phi_z)_{\mathbb{L}_2(\mathbb{R})} \\ &= (W^{-1} f, \phi_{\mathrm{e}^{-t} z})_{\mathbb{L}_2(\mathbb{R})} = \big(W W^{-1} f\big)(\mathrm{e}^{-t} z) = f(\mathrm{e}^{-t} z), \end{aligned}$$

for all $z \in \mathbb{C}$. □

Since $A_t f$ is only a re-scaling of the functions f, it easily follows that

$$\|f\|^2_{\mathfrak{B}_t} = \|B_t^{-1} f\|^2_{\mathfrak{B}} = \frac{1}{\pi} \int_{\mathbb{C}} |f(\mathrm{e}^t z)|^2 \mathrm{e}^{-|z|^2} \, \mathrm{d}z \tag{4.22}$$

for all $f \in \mathfrak{B}_t$. Moreover, the reproducing kernel $K_t : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is given by

$$K_t(z, w) = \mathrm{e}^{\mathrm{e}^{-2t} z \overline{w}}, \tag{4.23}$$

for all $z, w \in \mathbb{C}$. Hence,

$$|f(z)| \leq \|f\|_{\mathfrak{B}_t} \mathrm{e}^{-\frac{1}{2}\mathrm{e}^{-2t}|z|^2}, \tag{4.24}$$

for all $z \in \mathbb{C}$ and $f \in \mathfrak{B}_t$. This space also frequently occurs in [B], but is not used in the context of a Gelfand triple.

We recall that the function $\tilde{\phi}_x : \mathbb{C} \to \mathbb{C}$ is defined by

$$\tilde{\phi}_x(z) = \overline{\phi_z(x)} = \sum_{n=0}^{\infty} g_n(x) \frac{z^N}{\sqrt{n!}}, \tag{4.25}$$

for all $z \in \mathbb{C}$ and $x \in \mathbb{R}$.

Lemma 4.6 *The functions $\tilde{\phi}_x$ satisfy the following inequality*

$$|\tilde{\phi}_x(z)| \leq A\sqrt{|z|}\mathrm{e}^{-|z|^2/2}, \tag{4.26}$$

for some $A > 0$ which is independent of $x \in \mathbb{R}$.

Proof:
Let $x \in \mathbb{R}$ and $z \in \mathbb{C}$. Then,

$$|\tilde{\phi}_x(z)| = |\phi_z(x)| = \Big|\sum_{n=0}^{\infty} g_n(x) \frac{\overline{z}^n}{\sqrt{n!}}\Big| \leq \sum_{n=0}^{\infty} \Big|g_n(x)\frac{\overline{z}^n}{\sqrt{n!}}\Big| \leq k \sum_{n=0}^{\infty} \frac{|z|^n}{\sqrt{n!}}\Big|.$$

The entire function $F : \mathbb{R} \to \mathbb{C}$ defined by

$$F(r) = \sum_{m=0}^{\infty} \frac{r^m}{\sqrt{m!}} \mathrm{e}^{-r^2/2},$$

for all $r \in \mathbb{R}$, has the asymptotic expansion,

$$F(r) = (8\pi)^{1/4}\sqrt{r}\{1 - \frac{1}{16r^2} + \mathcal{O}(\frac{1}{r^4})\}, \quad r \to \infty.$$

See [O, Ch. 9, §8, pp. 307-309]. Hence the statement follows. □

Theorem 4.7 *Let $t > 0$ and $f \in \mathcal{H}_t$. Then*

$$f(x) = \langle \tilde{\phi}_x, Wf \rangle = \int_{\mathbb{C}} \tilde{\phi}_x(z)\big(Wf\big)(z)\mathrm{e}^{-|z|^2}\mathrm{d}z, \tag{4.27}$$

for all $x \in \mathbb{R}$.

Proof:
Let $f \in \mathcal{H}_t$ and $t > 0$. Then $Wf \in \mathfrak{B}_t$. Let $x \in \mathbb{R}$. By (4.23) and Lemma 4.6,

$$|f(z)\tilde{\phi}_x(z)| \leq A\sqrt{|z|}\mathrm{e}^{-\frac{1}{2}(1+\mathrm{e}^{-2t})|z|^2},$$

for all $z \in \mathbb{C}$. Therefore the integral in (4.27) is well-defined. Moreover,

$$\begin{aligned}\int_{\mathbb{C}} \tilde{\phi}_x(z)\big(Wf\big)(z)\mathrm{e}^{-|z|^2}\mathrm{d}z &= \int_{\mathbb{C}} \lim_{N\to\infty}\sum_{n=0}^{N} g_n(x)\frac{\overline{z}^n}{\sqrt{n!}}\big(Wf\big)(z)\mathrm{e}^{-|z|^2}\mathrm{d}z \\ &= \lim_{N\to\infty}\sum_{n=0}^{N} g_n(x)\int_{\mathbb{C}}\frac{\overline{z}^n}{\sqrt{n!}}\big(Wf\big)(z)\mathrm{e}^{-|z|^2}\mathrm{d}z \\ &= \lim_{N\to\infty}\sum_{n=0}^{N} g_n(x)(g_n, f)_{\mathbb{L}_2(\mathbb{R})} = f(x)\end{aligned}$$

Hence the statement follows. □

By (3.14) we obtain that

$$f = \lim_{t\downarrow 0}\mathrm{e}^{-tH}f = \lim_{t\downarrow 0}\Big(x \mapsto \int_{\mathbb{C}}\tilde{\phi}_x(z)\big(Wf\big)(\mathrm{e}^{-t}z)\mathrm{e}^{-|z|^2}\mathrm{d}z\Big). \tag{4.28}$$

for all $f \in \mathbb{L}_2(\mathbb{R})$.

4.3 Laguerre polynomials

In this section a transform introduced by Bargmann in [B] will be dealt with. The image space frequently occurs in the sequel.

Consider the generalized Laguerre polynomials defined by

$$L_n^{(\alpha)}(x) = \frac{1}{n!}\mathrm{e}^x x^{-\alpha}\left(\frac{d}{dx}\right)^n[\mathrm{e}^{-x}x^{n+\alpha}], \tag{4.29}$$

for all $\alpha > -1, x > 0$ and $n \in \mathbb{N}_0$. The generalized Laguerre polynomials have the following generating function

$$\frac{\exp[-x(1+z)/2(1-z)]}{(1-z)^{\alpha+1}} = \mathrm{e}^{-x/2}\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)z^n, \tag{4.30}$$

for $, \alpha > -1, x > 0$ and $|z| < 1$. For fixed α, the set $\{g_n : x \mapsto \left(\frac{n!}{\Gamma(n+\alpha+1)}\right)^{\frac{1}{2}}\mathrm{e}^{-x/2}L_n^{(\alpha)} \mid n \in \mathbb{N}_0$ is an orthonormal basis in $\mathbb{L}_2((0,\infty), x^\alpha\mathrm{d}x)$.

Let $\alpha > -1$ be fixed and let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disc. Set $V = \{\phi_z^{(\alpha)} \mid z \in D\}$ where $\phi_z^{(\alpha)}$ is defined by

$$\phi_z^{(\alpha)}(x) = \frac{\exp[-x(1+\overline{z})/2(1-\overline{z})]}{(1-\overline{z})^{\alpha+1}} \tag{4.31}$$

for all $x \in (0,\infty)$ and $z \in D$. Define the function $K^{(\alpha)} : D \times D \to \mathbb{C}$ of positive type by

$$K^{(\alpha)}(z,w) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha+1)}{n!}(z\overline{w})^n = \frac{\Gamma(\alpha+1)}{(1-z\overline{w})^{\alpha+1}}, \tag{4.32}$$

for all $z, w \in D$. Then by Theorem 2.3 and 4.2 the frame transform $W_\alpha : \mathbb{L}_2((0,\infty), x^\alpha \mathrm{d}x) \to \mathbb{C}^D_{K^{(\alpha)}}$ defined by

$$\big(W_\alpha f\big)(z) = \int_0^\infty \frac{\exp[-x(1+z)/2(1-z)]}{(1-z)^{\alpha+1}} f(x) x^\alpha \, \mathrm{d}x, \tag{4.33}$$

for all $f \in \mathbb{L}_2((0,\infty), x^\alpha \mathrm{d}x)$, is a unitary map. Moreover, the set $\{a_n : z \mapsto \left(\frac{\Gamma(n+\alpha+1)}{n!}\right)^{\frac{1}{2}} z^n \mid n \in \mathbb{N}_0\}$ is an orthonormal basis for $\mathbb{C}^D_{K^{(\alpha)}}$.

If $\alpha > 0$ there exists a useful characterization of the functional Hilbert space. First we recall an elementary result. By definition of the beta-function,

$$\int_0^1 r^{2m+1}(1-r^2)^{\alpha-1}\mathrm{d}r = \frac{m!\Gamma(\alpha)}{2\Gamma(\alpha+m+1)}, \tag{4.34}$$

for all $\alpha > 0$ and $m \in \mathbb{N}_0$.

Theorem 4.8 *The space $\mathbb{C}^D_{K^{(\alpha)}}$ consists of all analytic functions $f : D \to \mathbb{C}$ for which*

$$\frac{1}{\pi\Gamma(\alpha)} \int_D |f(z)|^2 (1-|z|^2)^{\alpha-1} \, \mathrm{d}\mu(z) < \infty. \tag{4.35}$$

Here $\mathrm{d}\mu$ stands for the normal Lebesgue measure on $\mathbb{C}$. Moreover,

$$(f,g)_{\mathbb{C}^D_{K^{(\alpha)}}} = \frac{1}{\pi\Gamma(\alpha)} \int_D \overline{f(z)} g(z)(1-|z|^2)^{\alpha-1} \, \mathrm{d}\mu(z), \tag{4.36}$$

for all $f, g \in \mathbb{C}^D_{K^{(\alpha)}}$.

Proof:
First, we prove the orthonormality of the set $\{a_n \mid n \in \mathbb{N}_0\}$. Let $n, m \in \mathbb{N}_0$. Then

$$\begin{aligned}
\int_D \overline{a_n(z)} a_m(z)(1-|z|^2)^{\alpha-1} \, \mathrm{d}\mu(z) &= \int_0^1 \int_0^{2\pi} r^{n+m+1} \mathrm{e}^{\mathrm{i}(m-n)\phi}(1-|r|^2)^{\alpha-1} \, \mathrm{d}\phi \mathrm{d}r \\
&= 2\pi\delta_{nm} \int_0^1 r^{n+m+1} \mathrm{e}^{\mathrm{i}(m-n)\phi}(1-|r|^2)^{\alpha-1} \, \mathrm{d}r \\
&= \pi\delta_{nm} \frac{\Gamma(\alpha)m!}{\Gamma(m+\alpha+1)}
\end{aligned}$$

where (4.34) was used in the last step. This proves the orthonormality.

Secondly, let f be an analytic function on D. Write $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for all $z \in D$ and define f_N by $f_N(z) = \sum_{n=0}^{N} c_m z^n$ for all $z \in D$. Then

$$\frac{1}{\pi}\int_D |f_N(z)|^2(1-|z|^2)^{\alpha-1}\,\mathrm{d}\mu(z) = \sum_{n=0}^{N} \frac{\Gamma(\alpha)m!}{\Gamma(m+\alpha+1)}|c_n|^2$$

Hence the integral in (4.35) converges if and only if the sum $\sum_{n=0}^{\infty} \frac{\Gamma(\alpha)m!}{\Gamma(m+\alpha+1)}|c_n|^2$ converges. Moreover, in case of convergence

$$\frac{1}{\pi}\int_D |f(z)|^2(1-|z|^2)^{\alpha-1}\,\mathrm{d}\mu(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha)m!}{\Gamma(m+\alpha+1)}|c_n|^2 = \|f\|_{\mathfrak{F}_0},$$

and therefore the theorem follows. □

In accordance to [B] the space $\mathbb{C}^D_{K(\alpha)}$ will be denoted by $\mathfrak{F}_\alpha$ for $\alpha > 0$.

Finally, consider $\alpha = 0$ which leads to the space $\mathfrak{F}_0 = \mathbb{C}^D_{K(0)}$. Note that the assumption $\alpha > 0$ is crucial in Theorem 4.8 and therefore the theorem is not applicable to $\mathfrak{F}_0$. Since $\{z \mapsto z^n \mid z \in \mathbb{N}_0\}$ is an orthonormal basis for $\mathfrak{F}_0$, the space consists of all analytic functions f that can be represented in power series of the form

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad \{c_n\}_{n\in\mathbb{N}_0} \in \ell_2(\mathbb{N}_0). \tag{4.37}$$

for all $z \in D$.

Lemma 4.9 *Let $f \in \mathfrak{F}_0$. Then*

$$\|f\|^2_{\mathfrak{F}_0} = \frac{1}{2\pi}\lim_{R\uparrow 1}\int_0^{2\pi} |f(R\mathrm{e}^{\mathrm{i}\phi})|^2\mathrm{d}\phi. \tag{4.38}$$

Proof:
Let $\{c_n\}_{n\in\mathbb{N}_0} \in \ell_2(\mathbb{N}_0)$ such that $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for all $z \in D$. Since $\{z \mapsto z^n \mid n \in \mathbb{N}_0\}$ is an orthonormal basis for $\mathfrak{F}_0$, the norm of f is given by $\|f\|^2_{\mathfrak{F}_0} = \sum_{n=0}^{\infty} |c_n|^2$. Moreover

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} |f(R\mathrm{e}^{\mathrm{i}\phi})|^2\mathrm{d}\phi &= \frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \overline{c_n}c_m \int_0^{2\pi} R^{n+m}\mathrm{e}^{\mathrm{i}(m-n)\phi}\,\mathrm{d}\phi \\ &= \sum_{n=0}^{\infty} |a_n|^2 R^{2n}\end{aligned}$$

for all $0 < R < 1$. Take the limit $R \uparrow 1$ on both sides and the statement follows.
□

Note that the above proof also shows that $\mathfrak{F}_0$ consist of all analytic functions on D such that the limit in (4.38) exists.

The reproducing property of the reproducing kernel can be proved directly from Cauchy's integration theorem. By (4.38) the inner product of $f, g \in \mathfrak{F}_0$ is given by

$$\begin{aligned}
(f,g)_{\mathfrak{F}_0} &= \lim_{R\uparrow 1} \frac{1}{2\pi}\int_0^{2\pi} \overline{f(R\mathrm{e}^{\mathrm{i}\phi})}g(R\mathrm{e}^{\mathrm{i}\phi})\mathrm{d}\phi \\
&= \lim_{R\uparrow 1} \frac{1}{2\pi\mathrm{i}}\int_0^{2\pi} \overline{f(R\mathrm{e}^{\mathrm{i}\phi})}g(R\mathrm{e}^{\mathrm{i}\phi})\frac{\mathrm{i}R\mathrm{e}^{\mathrm{i}\phi}}{R\mathrm{e}^{\mathrm{i}\phi}}\mathrm{d}\phi \\
&= \lim_{R\uparrow 1} \frac{1}{2\pi\mathrm{i}}\int_{C_R} \overline{f(w)}g(w)\frac{1}{w}\mathrm{d}w,
\end{aligned}$$

where $C_R = \{z \in D \mid |z| = R\}$ for all $0 < R < 1$ (with positive direction). Let $z \in D$ and $f = K_z^{(0)}$ then

$$\begin{aligned}
(K_z, g)_{\mathfrak{F}_0} &= \lim_{R\uparrow 1} \frac{1}{2\pi\mathrm{i}}\int_{C_R} \frac{1}{1 - z\overline{w}}\ g(w)\frac{1}{w}\ \mathrm{d}w \\
&= \lim_{R\uparrow 1} \frac{1}{2\pi\mathrm{i}}\int_{C_R} \frac{1}{w - zR}\ g(w)\ \mathrm{d}w \\
&= \lim_{R\uparrow 1} g(Rz) = g(z).
\end{aligned}$$

There also exists a characterization of the norm in $\mathfrak{H}_0$ which is similar to the characterization of the norm of $\mathfrak{F}_\alpha$ as formulated in Theorem 4.8.

Define the Euler operator $\mathcal{E}$ on $\mathfrak{F}_0$ by

$$\big(\mathcal{E}f\big)(z) = z\frac{df}{dz}(z), \tag{4.39}$$

for all $z \in D$. It is obvious that the basis elements $z \mapsto z^n$ are eigenvectors with eigenvalue n for all $n \in \mathbb{N}_0$. The domain of $\mathcal{E}$ can therefore be defined by $D(\mathcal{E}) = \{x \mapsto \sum_{n=0}^{\infty} c_n z^n \mid \{c_n n^2\}_{n\in\mathbb{N}} \in \ell_2(\mathbb{N}_0)\}$.

Theorem 4.10 *The space $\mathfrak{F}_0$ consists of all analytic functions f on D for which*

$$\frac{1}{\pi}\int_D \overline{(I+\mathcal{E})f(z)}f(z)\ \mathrm{d}\mu(z) < \infty. \tag{4.40}$$

Here $\mathrm{d}\mu(z)$ stands for the normal Lebesgue measure on $\mathbb{C}$. Moreover,

$$(f,g)_{\mathfrak{F}_0} = \frac{1}{\pi}\int_D \overline{(I+\mathcal{E})f(z)}g(z)\ \mathrm{d}z, \tag{4.41}$$

for all $f, g \in \mathfrak{F}_0$.

Proof:
Let f be analytic function on D. Write $f(z) = \sum_{n=0}^{\infty} c_n z^n$, for all $z \in D$. Define f_N by $f_N(z) = \sum_{n=0}^{N} c_n z^n$. Then

$$\begin{aligned}\frac{1}{\pi}\int_D \overline{(I+\mathcal{E})f_N(z)} f_N(z)\ \mathrm{d}\mu(z) &= \frac{1}{\pi}\sum_{n=0}^{N}\sum_{m=0}^{N}\overline{c_n}c_m\int_D (n+1)\overline{z}^n z^m\ \mathrm{d}\mu(z)\\ &= \sum_{n=0}^{N}\int_0^1 (2n+2)r^{2n+1}\ \mathrm{d}r = \sum_{n=0}^{N}|c_n|^2.\end{aligned}$$

The theorem follows by a same argument as in Theorem 4.8. □

4.4 Gegenbauer polynomials

4.4.1 Introduction

The Gegenbauer polynomials of order λ are defined by the following generating function

$$(1-2xz+z^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{\lambda}(x)z^n, \tag{4.42}$$

which is valid for $-1 < x < 1$, $|z| < 1$ and $\lambda \neq 0$. In this section we consider the Gegenbauer polynomials of order $\lambda > 0$. Apply the operator $z\frac{d}{dz} + 2\lambda$ on both sides to obtain

$$\frac{2\lambda(1-xz)}{(1-2xz+z^2)^{\lambda+1}} = \sum_{n=0}^{\infty} C_n^{\lambda}(x)(n+2\lambda)z^n, \tag{4.43}$$

for all $-1 < x < 1$, $|z| < 1$ and $\lambda \neq 0$.

For fixed $\lambda > 0$, the set $\{g_n : x \mapsto \left(\frac{n!(\lambda+n)}{\Gamma(n+2\lambda)}\right)^{\frac{1}{2}} C_n^{\lambda}(x) \mid n \in \mathbb{N}_0\}$ is an orthonormal basis in $\mathbb{L}_2((-1,1),\mathrm{d}\mu)$ where,

$$\mathrm{d}\mu(x) = \frac{\Gamma(\lambda)^2}{2^{1-2\lambda}\pi}(1-x^2)^{\lambda-\frac{1}{2}}\mathrm{d}x. \tag{4.44}$$

Let $\lambda > 0$ be fixed. Set $V = \{\phi_z^{(\lambda)} \mid z \in D\}$ where ϕ_z is defined by

$$\phi_z^{(\lambda)}(x) = \frac{2\lambda(1-x\overline{z})}{(1-2x\overline{z}+\overline{z}^2)^{\lambda+1}} \tag{4.45}$$

for all $z \in D$ and almost all $x \in (-1,1)$. Obviously $V \subset \mathbb{L}_2((-1,1),\mathrm{d}\mu)$. Define the function $K^{(\lambda)} : D \times D \to \mathbb{C}$ of positive type by

$$K^{(\lambda)}(z,w) = \sum_{n=0}^{\infty}\frac{\Gamma(n+2\lambda)(n+2\lambda)^2}{n!(n+\lambda)}\overline{w}^n z^n \tag{4.46}$$

for all $z, w \in D$. By Theorem 2.3 and 4.2 the frame transform $W_\lambda : \mathbb{L}_2((-1,1),\mu) \to \mathbb{C}^D_{K_\lambda}$ defined by

$$\big(W_\lambda f\big)(z) = \frac{\Gamma(\lambda)^2 2^{1-2\lambda}}{\pi} \int_{-1}^{1} \frac{2\lambda(1-xz)}{(1-2xz+z^2)^{\lambda+1}} f(x)(1-x^2)^{\lambda-\frac{1}{2}}\, \mathrm{d}x, \tag{4.47}$$

for all $f \in \mathbb{L}_2((-1,1),\mu)$ and $z \in D$, is unitary. Moreover, the set $\{a_n \mid n \in \mathbb{N}_0\}$ is an orthonormal basis for $\mathbb{C}^D_{\mathbb{C}^D_{K(\lambda)}}$ where a_n is defined by

$$a_n(z) = \big(W_\lambda g_n\big)(z) = \left(\frac{\Gamma(n+2\lambda)}{n!(n+\lambda)}\right)^{\frac{1}{2}} (n+2\lambda) z^n = \left(\frac{\Gamma(n+2\lambda+1)}{n!}\right)^{\frac{1}{2}} \sqrt{\frac{n+2\lambda}{n+\lambda}}\, z^n, \tag{4.48}$$

for all $z \in D$ and $n \in \mathbb{N}_0$. The following theorem characterizes the space $\mathbb{C}^D_{K(\lambda)}$.

Theorem 4.11 *The space $\mathbb{C}^D_{K(\lambda)}$ equals $\mathfrak{F}_{2\lambda}$ as a set. Moreover,*

$$\|f\|_{\mathfrak{F}_{2\lambda}} < \|f\|_{\mathbb{C}^D_{K(\lambda)}} \leq \sqrt{2}\|f\|_{\mathfrak{F}_{2\lambda}}, \tag{4.49}$$

for all $f \in \mathbb{C}^D_{K(\lambda)}$.

Proof:
Note that $z \mapsto \left(\frac{\Gamma(n+2\lambda+1)}{n!}\right)^{\frac{1}{2}} z^n$ was an orthonormal basis for $\mathfrak{F}_{2\lambda}$. Now the statement easily follows by the estimate

$$1 < \frac{n+2\lambda}{n+\lambda} \leq 2$$

for all $n \in \mathbb{N}$. □

4.4.2 The semi-group generated by the Euler operator

Define the Euler operator $\mathcal{E}$ on $\mathbb{C}^D_{K(\lambda)}$ by

$$\big(\mathcal{E} f\big)(z) = z\frac{df}{dz}(z), \tag{4.50}$$

for all $z \in D$ and that $z \mapsto z^n$ are eigenvectors with eigenvalue n. The domain of $\mathcal{E}$ can therefore be defined as $\{f = \sum_{n=0}^{\infty} c_n a_n \mid \{c_n n\}_{n\in\mathbb{N}_0} \in \ell_2(\mathbb{N}_0)\}$. The following lemma follows by straightforward calculations.

Lemma 4.12

$$(f,g)_{\mathbb{C}^D_{K(\lambda)}} = \big((\mathcal{E}+\lambda)(\mathcal{E}+2\lambda)^{-1} f, g\big)_{\mathfrak{F}_{2\lambda}} \tag{4.51}$$

for all $f, g \in \mathbb{C}^D_{K(\lambda)}$.

Proof:
Obviously $\mathcal{E}+2\lambda$ is a positive operator with the eigenvectors a_n with eigenvalues $n+2\lambda$. Therefore $(\mathcal{E}+2\lambda)^{-1}$ is well-defined and has the same set of eigenvectors a_n with eigenvalue $(n+2\lambda)^{-1}$. The operator $(\mathcal{E}+\lambda)(\mathcal{E}+2\lambda)^{-1}$ eliminates the factor $\sqrt{\frac{n+\lambda}{n+2\lambda}}$ in (4.48). The theorem follows by Theorem 4.8. □

It is easily seen that $-\mathcal{E}$ is a closed dissipative operator and therefore generates a contraction semi-group $t \mapsto \mathrm{e}^{-\mathcal{E}t}$. The (bounded) operator $\mathrm{e}^{-\mathcal{E}t}$ has eigenvectors a_n with eigenvalues e^{-nt} for all $n \in \mathbb{N}_0$ and $t \geq 0$. Let $f \in \mathbb{C}^D_{K^{(\lambda)}}$ and write $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for all $z \in D$. Then,

$$\left(\mathrm{e}^{-t\mathcal{E}} f\right)(z) = \sum_{n=0}^{\infty} c_n \mathrm{e}^{-tn} z^n = f(\mathrm{e}^{-t} z), \tag{4.52}$$

for all $z \in D$. Hence $\mathrm{e}^{-t\mathcal{E}}$ is a scaling of the functions in $\mathbb{C}^D_{K^{(\lambda)}}$. Note that it is crucial that $t > 0$, since only then we obtain $\mathrm{e}^{-t} z \in D$ for all $z \in D$.

From standard theory of evolution equations it follows that

$$(-\mathcal{E} - sI)^{-1} f = \int_0^{\infty} \mathrm{e}^{-st} \mathrm{e}^{-t\mathcal{E}} f \mathrm{d}t, \tag{4.53}$$

for all $f \in \mathbb{C}^D_{K^{(\lambda)}}$. Since $\mathbb{C}^D_{K^{(\lambda)}}$ is a functional Hilbert space, this simplifies to an ordinary integral

$$\left((-\mathcal{E} - sI)^{-1} f\right)(z) = \int_0^{\infty} \mathrm{e}^{-st} \left(\mathrm{e}^{-t\mathcal{E}} f\right)(z) \mathrm{d}t = \int_0^{\infty} \mathrm{e}^{-st} f(\mathrm{e}^{-tz}) \mathrm{d}t, \tag{4.54}$$

for all $f \in \mathbb{C}^D_{K^{(\lambda)}}$ and $z \in D$.

Theorem 4.13

$$(f,g)_{\mathbb{C}^D_{K^{(\alpha)}}} \;=\; (f,g)_{\mathfrak{F}_{2\lambda}} + 2\lambda \int_0^{\infty} \left(\mathrm{e}^{-t\mathcal{E}} f, \mathrm{e}^{-t\mathcal{E}} g\right)_{\mathfrak{F}_{2\lambda}} \mathrm{e}^{-4\lambda t}\, \mathrm{d}t \tag{4.55}$$

for all $f, g \in \mathbb{C}^D_{K^{(\alpha)}}$.

Proof:
Let $f, g \in \mathbb{C}^{\mathbb{E}}_{K^{(\lambda)}}$. Then

$$\begin{aligned}
(f,g)_{\mathbb{C}^D_{K^{(\lambda)}}} &= \left((\mathcal{E}+\lambda)(\mathcal{E}+2\lambda)^{-1} f, g\right)_{\mathfrak{F}_{2\lambda}} \\
&= \left(f,g\right)_{\mathfrak{F}_{2\lambda}} - \lambda\left((\mathcal{E}+2\lambda)^{-1} f, g\right)_{\mathfrak{F}_{2\lambda}} \\
&= \left(f,g\right)_{\mathfrak{F}_{2\lambda}} + \lambda\left((-\mathcal{E}-2\lambda)^{-1} f, g\right)_{\mathfrak{F}_{2\lambda}} \\
&= \left(f,g\right)_{\mathfrak{F}_{2\lambda}} + \lambda\Big(\int_0^{\infty} \mathrm{e}^{-2\lambda t} \mathrm{e}^{-t\mathcal{E}} f \mathrm{d}t, g\Big)_{\mathfrak{F}_{2\lambda}} \\
&= \left(f,g\right)_{\mathfrak{F}_{2\lambda}} + \lambda \int_0^{\infty} \mathrm{e}^{-2\lambda t} \left(\mathrm{e}^{-t\mathcal{E}} f, g\right)_{\mathfrak{F}_{2\lambda}} \mathrm{d}t.
\end{aligned}$$

For the sake of symmetry use the substitution $t' = t/2$ and use the self-adjointness of $\mathrm{e}^{-t\mathcal{E}}$ to obtain the statement. □

Using (4.53) one also obtains an alternative formula for the reproducing kernel.

Theorem 4.14 *The reproducing kernel K is given by the following integral*

$$K(z,w) = \Gamma(2\lambda+1)\int_0^\infty \mathrm{e}^{-\lambda t}\frac{2\lambda + \mathrm{e}^{-tz}\overline{w}}{(1-\mathrm{e}^{-tz}\overline{w})^{2\lambda+2}}\mathrm{d}t \tag{4.56}$$

Proof:
Let $w \in D$. By (4.46),

$$\big((\mathcal{E}+2\lambda)^{-1}(\mathcal{E}+\lambda)K_w\big)(z) = \sum_{n=0}^\infty \frac{\Gamma(n+2\lambda+1)}{n!}\overline{w}^n z^n = \frac{\Gamma(2\lambda+1)}{(1-z\overline{w})^{2\lambda+1}}$$

for all $z \in D$. Therefore,

$$\big((\mathcal{E}+\lambda)K_w\big)(z) = \frac{\Gamma(2\lambda+1)(2\lambda+z\overline{w})}{(1-z\overline{w})^{2\lambda+2}}$$

for all $z \in D$. The statement now follows by applying (4.53). □

5 The classical Fourier and Laplace transforms

5.1 The Fourier transform as a frame transform on a Gelfand triple

In his section the classical Fourier transform on $\mathbb{L}_2(\mathbb{R})$ is interpreted as a special kind of frame transform. Of course the function $x \mapsto \mathrm{e}^{\mathrm{i}\omega x}$ is not an element of $\mathbb{L}_2(\mathbb{R})$ for all $\omega \in \mathbb{R}$. Nevertheless, by making use of a Gelfand triple $\mathcal{H}^I \hookrightarrow \mathbb{L}_2(\mathbb{R}) \hookrightarrow \mathcal{H}^{-I}$ for which the space $\mathcal{H}^{-I}$ does contain this functions, a frame transform can be defined which leads to the Fourier transform.

In this section the following two result will be used, which we will not prove.

- A result due to Cramer: $|H_n(x)| < k\sqrt{n}2^{n/2}\mathrm{e}^{x^2/2}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, where k is a constant. See [C].

- Mehler's formula for Hermite polynomials

$$\sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)}{n!}(w/2)^n = (1-w^2)^{-\frac{1}{2}} \exp\left[\frac{2xyw - (x^2+y^2)w^2}{(1-w^2)}\right], \tag{5.1}$$

which is valid for all $x, y \in \mathbb{R}$ and $w \in D = \{z \in \mathbb{C} \mid |z| < 1\}$. See [MOS, §5.5, pp. 252].

Define the operator R on $\mathbb{L}_2(\mathbb{R})$ by

$$\big(Rf\big)(x) = \big((x^2 - \frac{d^2}{dx^2})f\big)(x), \tag{5.2}$$

for all $f \in \mathcal{D}(R) = \{f \in \mathbb{L}_2(\mathbb{R}) \mid \frac{d^2}{dx^2}f \in \mathbb{L}_2(\mathbb{R}) \wedge Mf \in \mathbb{L}_2(\mathbb{R})\}$, where $\big(Mf\big)(y) = y^2 f(y)$ for all $y \in \mathbb{R}$. It is well-know that this operator has a complete set of eigenvectors $\{g_n \mid n \in \mathbb{N}_0\}$ with eigenvalues $\lambda_n = 2n+1$. Moreover, R is a positive unbounded and self-adjoint (unbounded) operator, with bounded inverse.

Next, we construct the Gelfand triple

$$\mathcal{H}_I(\mathbb{R}) \hookrightarrow \mathbb{L}_2(\mathbb{R}) \hookrightarrow \mathcal{H}_{-I}(\mathbb{R}). \tag{5.3}$$

Note that the sets $\{\frac{g_n}{2n+1} \mid n \in \mathbb{N}_0\}$, $\{g_n \mid n \in \mathbb{N}_0\}$ and $\{(2n+1)g_n \mid n \in \mathbb{N}_0\}$ are orthonormal bases for respectively $\mathcal{H}_I(\mathbb{R}), \mathbb{L}_2(\mathbb{R})$ and $\mathcal{H}_{-I}(\mathbb{R})$.

The following theorem is very easily proved by using the Fourier transform on $\mathbb{L}_2(\mathbb{R})$. But since our goal is to construct the Fourier transform, it would be sloppy to use it in the proof. Therefore the proof is somewhat lengthy and uses the result by Cramer.

Theorem 5.1 *The space $\mathcal{H}_I(\mathbb{R})$ is a functional Hilbert space with reproducing kernel K^I given by*

$$K^I(x,y) = \sum_{n=0}^{\infty} \frac{1}{2^n n!(2n+1)^2\sqrt{\pi}} H_n(x)H_n(y)\mathrm{e}^{-x^2/2}\mathrm{e}^{-y^2/2}, \tag{5.4}$$

for all $x, y \in \mathbb{R}$. Moreover, the sum converges absolutely.

Proof:
Since $\mathcal{H}^I(\mathbb{R})$ as a set is equal to $D(R)$ and hence a subset of $\mathbb{L}_2(\mathbb{R})$. Strictly spoken, its elements are not functions but classes of functions. Therefore it would be better (but tiresome) to write $[h]$ instead of h for its elements. The main purpose of this proof is to prove that making those classes is obsolete. It turns out that every class $[h]$ in $\mathcal{H}^I(\mathbb{R})$ has a unique analytic representant. Instead of considering classes, we only consider the analytic representant.

Let $f \in \mathcal{H}^I(\mathbb{R})$. The set $\{\frac{1}{2n+1} g_n \mid n \in \mathbb{N}_0\}$ is an orthonormal basis for $\mathcal{H}_I(\mathbb{R})$. Hence $f = \sum_{n=0}^{\infty} a_n g_n \in \mathcal{H}^I(\mathbb{R})$ if and only if

$$\sum_{n=0}^{\infty} |a_n|^2 (2n+1)^2 < \infty.$$

Define the function f_N by $f_N(x) = \sum_{n=0}^{N} a_n g_n(x)$, for all $N \in \mathbb{N}$ and $x \in \mathbb{R}$. Since

$$\begin{aligned} |f_N(x) - f_M(x)| &\leq \sum_{n=M}^{N} |a_n g_n(x)| \leq \sqrt{\sum_{n=M}^{N} |a_n|^2(2n+1)^2} \sqrt{\sum_{n=M}^{N} \frac{|g_n(x)|^2}{(2n+1)^2}} \\ &\leq \sqrt{\sum_{n=M}^{N} |a_n|^2(2n+1)^2} \sqrt{\sum_{n=M}^{N} \frac{k^2}{(2n+1)^2}} \\ &\leq \|f_N - f_M\|_I \frac{k\pi}{2\sqrt{2}} \end{aligned}$$

for all $N > M \in \mathbb{N}$ and $x \in \mathbb{R}$, the sequence of analytic functions $\{f_N\}_{N\in\mathbb{N}}$ uniformly on $\mathbb{R}$ to an analytic function $\tilde{f}$. By uniform convergence, it is also true that Rf_N converges uniformly to $R\tilde{f}$. In addition $R\tilde{f} \in \mathbb{L}_2(\mathbb{R})$ and $[\tilde{f}] = [f]$.

By a same estimate as above we obtain

$$|f(x)| \leq \|f\|_I \frac{k\pi}{2\sqrt{2}},$$

for all $x \in \mathbb{R}$. Hence $\mathcal{H}_I$ is a functional Hilbert space. The rest of the statement follows by Theorem 4.1 and Mehler's formula. □

Define for $\alpha \in \mathbb{R}$ the vector $\phi_\alpha \in \mathcal{H}_{-I}$

$$\phi_\alpha := \sum_{n=0}^{\infty} \mathrm{i}^n g_n(\alpha) g_n = \sum_{n=0}^{\infty} \frac{\mathrm{i}^n g_n(\alpha)}{2n+1}\,(2n+1) g_n. \tag{5.5}$$

This vector is well-defined because $\{(2n+1)g_n \mid n \in \mathbb{N}_0\}$ is an orthonormal basis in $\mathcal{H}_{-I}(\mathbb{R})$ and $\{g_n(\alpha)/(2n+1)\}_{n\in\mathbb{N}_0} \in \ell_2(\mathbb{N}_0)$ for all $\alpha \in \mathbb{R}$.

Set $V = \{\phi_\alpha \mid \alpha \in \mathbb{R}\}$. Define the function $K : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ of positive type by

$$K(\alpha,\beta) = (\phi_\alpha, \phi_\beta)_{\mathbb{E}} = \sum_{n=1}^{\infty} \frac{g_n(\alpha) g_n(\beta)}{(2n+1)^2}, \tag{5.6}$$

for all $\alpha, \beta \in \mathbb{R}$. Obviously $K = K^I$ and therefore $\mathbb{C}_K^{\mathbb{R}} = \mathcal{H}_I(\mathbb{R})$. By Theorem 2.3, the frame transform $W : \overline{\langle V \rangle} \to \mathcal{H}_I$ defined by

$$\big(Wf\big)(\alpha) = (\phi_\alpha, f)_{-I}, \tag{5.7}$$

for all $f \in \mathcal{H}_{-I}(\mathbb{R})$ and $x \in \mathbb{R}$ is unitary. With a simple argument it follows that $\overline{\langle V \rangle} = \mathcal{H}_{-I}$. To this end let $f \in V^\perp$, then

$$0 = (\phi_\alpha, f)_{-I} = \sum_{n=0}^{\infty} \frac{\mathrm{i}^n g_n(\alpha)}{2n+1}((2n+1)g_n, f)_{-I}, \tag{5.8}$$

for all α. It is obvious that $\alpha \mapsto (\phi_\alpha, f)_{-I}$ belongs to $\mathcal{H}_I(\mathbb{R})$, by the fact that $\{(2n+1)g_n \mid n \in \mathbb{N}_0\}$ is an orthonormal basis for $\mathcal{H}_{-I}(\mathbb{R})$ and that $\{\frac{g_n}{2n+1} \mid n \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{H}_I(\mathbb{R})$. Moreover, (5.8) implies that $((2n+1)g_n, f)_{-I} = 0$ for all $n \in \mathbb{N}$ and therefore $f = 0$.

Since W is a unitary map from $\mathcal{H}_{-I}(\mathbb{R})$ onto $\mathcal{H}_I(\mathbb{R})$ and R a unitary map from $\mathcal{H}_I$ onto $\mathbb{L}_2(\mathbb{R})$ and from $\mathbb{L}_2(\mathbb{R})$ onto $\mathcal{H}_{-I}(\mathbb{R})$ it follows that RWR is a unitary map from $\mathbb{L}_2(\mathbb{R})$ onto $\mathbb{L}_2(\mathbb{R})$. Note that for $f = \sum_{n=0}^{\infty} a_n g_n$ we have

$$RWRf = \sum_{n=0}^{\infty} \mathrm{i}^n a_n g_n. \tag{5.9}$$

Hence $RWRf$ equals $\alpha \mapsto \langle \phi_\alpha, f\rangle$ for all $f \in \mathcal{H}^I(\mathbb{R})$.

Finally we connect the transform RWR to the classical way of introducing the Fourier transform. Suppose $f \in \mathcal{H}^I(\mathbb{R})$, then $f(x) = f(x)(1+x^2)\frac{1}{1+x^2}$ for almost all $x \in \mathbb{R}$. By definition $Mf \in \mathbb{L}_2(\mathbb{R})$ and therefore $f \in \mathbb{L}_1(\mathbb{R})$ since f is a product of two $\mathbb{L}_2(\mathbb{R})$ functions. Let $\alpha \in \mathbb{R}$. Then

$$\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\alpha x} f(x)\,\mathrm{d}x, \tag{5.10}$$

is well-defined for all $f \in \mathcal{H}^I(\mathbb{R})$. Write $f = \sum_{n=0}^{\infty} a_n g_n$. Then by Mehler's formula

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\alpha x} f(x)\ \mathrm{d}x &= \lim_{t\to 1} \frac{1}{\sqrt{(1+t^2)\pi}} \int_{\mathbb{R}} \mathrm{e}^{\frac{-\mathrm{i}t\alpha x - t^2(x^2+\alpha^2)}{1+t^2}} f(x)\ \mathrm{d}x && (5.11)\\
&= \lim_{t\to 1} \int_{\mathbb{R}} \sum_{n=0}^{\infty} g_n(x) g_n(\alpha) (\mathrm{i}t)^n f(x)\ \mathrm{d}x && (5.12)\\
&= \lim_{t\to 1} \sum_{n=0}^{\infty} g_n(\alpha) \int_{\mathbb{R}} g_n(x) (\mathrm{i}t)^n f(x)\ \mathrm{d}x && (5.13)\\
&= \lim_{t\to 1} \sum_{n=0}^{\infty} a_n g_n(\alpha)(\mathrm{i}t)^n = \sum_{n=0}^{\infty} a_n g_n(\alpha)\mathrm{i}^n = \langle \phi_\alpha, f\rangle. && (5.14)
\end{aligned}$$

Hence we can conclude that ϕ_α equals $x \mapsto \mathrm{e}^{\mathrm{i}\alpha x}$ in distributional sense. In addition,

$$\big(RWRf\big)(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\alpha x} f(x)\ \mathrm{d}x, \tag{5.15}$$

for all $\alpha \in \mathbb{R}$ and $f \in \mathcal{H}^I(\mathbb{R})$.

5.2 The Laplace transform

In this section we interpret the Laplace transform as a frame transform. Let $\mathcal{H} = \mathbb{L}_2((0,\infty))$ and $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$. Moreover define the set $V = \{\phi_z \in \mathbb{L}_2((0,\infty)) \mid z \in \mathbb{C}^+\}$ where

$$\phi_z(t) = \mathrm{e}^{-\overline{z}t}, \tag{5.16}$$

for almost all $t \in (0,\infty)$ and all $z \in \mathbb{C}^+$.

Lemma 5.2 *The set V has a dense span in $\mathbb{L}_2((0,\infty))$.*

Proof:
As a consequence of the approximation theorem of Weierstrass, the set of polynomials is dense in $\mathbb{L}_2((0,1), \mathrm{d}x)$. Suppose $f \in V^\perp$ for some $f \in \mathbb{L}_2((0,\infty))$. Then in particular

$$0 \ = \ \int_0^\infty \mathrm{e}^{-tn} f(t)\ \mathrm{d}t = \int_0^1 x^n f(-\log(x))\ \frac{1}{x}\mathrm{d}x = \int_0^1 x^{n-1} f(-\log(x))\ \mathrm{d}x,$$

for all $n \in \mathbb{N}$. Hence $f = 0$. □

Define the function $K : \mathbb{C}^+ \times \mathbb{C}^+ \to \mathbb{C}$ of positive type by

$$K(z,w) = (\phi_z, \phi_w)_{\mathbb{L}_2((0,\infty))} = \int_0^\infty \mathrm{e}^{-zt}\mathrm{e}^{-\overline{w}t}\mathrm{d}t = \frac{1}{z+\overline{w}}, \tag{5.17}$$

for all $z, w \in \mathbb{C}^+$.

Since $\overline{\langle V \rangle} = \mathcal{H}$, the frame transform $\mathcal{L} : \mathbb{L}_2((0,\infty)) \to \mathbb{C}_K^{\mathbb{C}^+}$ defined by

$$\big(\mathcal{L}f\big)(z) = (\phi_z, f)_{\mathbb{L}_2((0,\infty))} = \int_0^\infty \mathrm{e}^{-zt} f(t)\ \mathrm{d}t, \tag{5.18}$$

for all $f \in \mathbb{L}_2((0,\infty))$ and $z \in \mathbb{C}^+$, is a unitary map. Note that $\mathbb{C}_K^{\mathbb{C}^+}$ consists of analytic functions. Fix $x > 0$ and define $\tilde{f}_x$ by

$$\tilde{f}_x(t) = \begin{cases} \mathrm{e}^{-tx} f(t) & t > 0 \\ 0 & t \leq 0 \end{cases}, \tag{5.19}$$

for all $x \geq 0$ and $f \in \mathbb{L}_2((0,\infty))$. Then it is obvious that for all $x > 0$ we have $\tilde{f}_x \in \mathbb{L}_1(\mathbb{R})$ and $\big(\mathcal{F}\tilde{f}_x\big)(y) = \frac{1}{\sqrt{2\pi}}\big(\mathcal{L}f\big)(x+\mathrm{i}y)$ for almost all $y \in \mathbb{R}$. Next we prove that $\text{l.i.m.}_{x\downarrow 0}\tilde{f}_x = \tilde{f}_0$. Let $\varepsilon > 0$. Then take a $R > 0$ such that

$$\int_{\mathbb{R}/B_{0,R}} |\tilde{f}_0(t)|^2\ \mathrm{d}t < \frac{\varepsilon}{2}. \tag{5.20}$$

Take $x > 0$ such that $1 - \mathrm{e}^{-xR} < \frac{\sqrt{\varepsilon}}{\sqrt{2}\|\tilde{f}_0\|_2}$. Then we see that

$$\begin{aligned} \|\tilde{f}_x - f_0\|_2 &= \int_{\mathbb{R}/B_{0,R}} |(\tilde{f}_0 - \tilde{f}_x)(t)|^2\ \mathrm{d}t + \int_{B_{0,R}} |(\tilde{f}_0 - \tilde{f}_x)(t)|^2\ \mathrm{d}t \\ &= \int_{\mathbb{R}/B_{0,R}} |\tilde{f}_0(t)(1-\mathrm{e}^{-xt})|^2\ \mathrm{d}t + \int_{B_{0,R}} |\tilde{f}_0(t)(1-\mathrm{e}^{-xt})|^2\ \mathrm{d}t \\ &\leq \int_{\mathbb{R}/B_{0,R}} |\tilde{f}_0(t)|^2\ \mathrm{d}t + (1-\mathrm{e}^{-xR})^2 \int_{B_{0,R}} |\tilde{f}_0(t)|^2\ \mathrm{d}t < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \end{aligned} \tag{5.21}$$

and the statement is achieved. Since $\text{l.i.m.}_{x\downarrow 0}\tilde{f}_x = \tilde{f}_0$ we also obtain $\text{l.i.m.}_{x\downarrow 0}\mathcal{F}\tilde{f}_x = \mathcal{F}\tilde{f}_0$ and $\text{l.i.m.}_{x\downarrow 0}\big(y \mapsto \big(\mathcal{L}f\big)(x+\mathrm{i}y)\big) = \mathcal{F}\tilde{f}_0$. Note that $\|\mathcal{F}\tilde{f}_0\|_{\mathbb{L}_2(\mathbb{R})} = \|f\|_{\mathbb{L}_2((0,\infty))}$. As a result we obtain

$$(F, G)_{\mathbb{C}_K^{\mathbb{C}^+}} = \lim_{x\downarrow 0} \frac{1}{2\pi} \int_{\mathbb{R}} \overline{F(x+\mathrm{i}y)} G(x+\mathrm{i}y)\ \mathrm{d}y, \tag{5.22}$$

for all $F, G \in \mathbb{C}_K^{\mathbb{C}^+}$. In particular

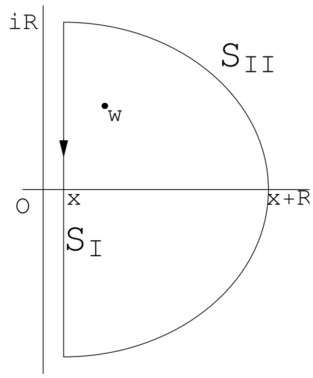
$$F(w) = \lim_{x\downarrow 0} \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{x - \mathrm{i}y + w} F(x+\mathrm{i}y)\ \mathrm{d}y, \tag{5.23}$$

for all $w \in \mathbb{C}^+$ and $F \in \mathbb{C}_K^{\mathbb{C}^+}$.

We now give an alternative proof of (5.23) using the theory of complex functions. Let $F \in \mathbb{C}_K^{\mathbb{C}^+}$, $w \in \mathbb{C}^+$ and $0 < x < \mathrm{Re}(w)$. Let $R > 0$ be sufficiently large, then

$$F(w+x) = \frac{1}{2\pi\mathrm{i}} \int_{S_I \cup S_{II}} \frac{1}{z - w - x} F(z)\mathrm{d}z \tag{5.24}$$

by the residu Theorem and the fact that F is analytic. The contours are given by $S_1 = \{z \in \mathbb{C} \mid \mathrm{Re}(z) = x \wedge |\mathrm{Im}(z)| < R\}$ and $S_{II} = \{z \in \mathbb{C} \mid |z - x| = R \wedge \mathrm{Re}(z - x) > 0\}$.



First we prove that the integral over S_{II} vanishes if R tends to infinity for all $F \in \mathbb{C}_K^{\mathbb{C}^+}$ and all $x > 0$. To this end, we make use of the inequality

$$|F(z)| = |(\phi_z, f)| \leq \frac{1}{\mathrm{Re}(z)}\|f\|_2 = \frac{1}{\mathrm{Re}(z)}\|F\|_{\mathbb{C}_K^{\mathbb{C}^+}}, \tag{5.25}$$

for all $F = \mathcal{L}f$ and $z \in \mathbb{C}^+$, which follows by the Cauchy-Schwartz inequality. Let $F \in \mathbb{C}_K^{\mathbb{C}^+}$. Parameterizing the contour S_{II} gives

$$\frac{1}{2\pi \mathrm{i}}\int_{S_{II}} \frac{1}{z-w-x}F(z)\mathrm{d}z \quad = \quad \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{R\mathrm{e}^{\mathrm{i}\phi}-w}F(x+R\mathrm{e}^{\mathrm{i}\phi})R\mathrm{e}^{\mathrm{i}\phi}\mathrm{d}\phi$$

for all $F \in \mathbb{C}_K^{\mathbb{E}}$. The integrand can be estimated by

$$|\frac{1}{R\mathrm{e}^{\mathrm{i}\phi}-w}F(x+R\mathrm{e}^{\mathrm{i}\phi})R\mathrm{e}^{\mathrm{i}\phi}| \leq \frac{\|F\|_{\mathbb{C}_K^{\mathbb{C}^+}}}{|R\mathrm{e}^{\mathrm{i}\phi}-w|(x+R\cos\phi)}, \tag{5.26}$$

for all $x > 0$, F, $w \in \mathbb{C}^+$ $R > 0$ and $\phi \in [0, 2\pi)$.

Let $\varepsilon > 0$.There exists $\eta > 0, R_1 > 0$ such that $|w - R_1\mathrm{e}^{\mathrm{i}\phi}| > R_1/2$ for all $\phi \in [-\pi/2, \pi/2]$ and $\frac{2\eta}{x} < \frac{2\pi\varepsilon}{3\|F\|_{\mathbb{C}_K^{\mathbb{C}^+}}}$ and $\frac{1}{x+R_1\cos\eta} < \frac{\varepsilon}{3\|F\|_{\mathbb{C}_K^{\mathbb{C}^+}}}$. For $R > R_1$ the same inequalities are satisfied and we find

$$\begin{aligned}
\left|\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{R\mathrm{e}^{\mathrm{i}\phi}-w}F(x+R\mathrm{e}^{\mathrm{i}\phi})R\mathrm{e}^{\mathrm{i}\phi}\mathrm{d}\phi\right| &\leq \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\left|\frac{1}{R\mathrm{e}^{\mathrm{i}\phi}-w}F(x+R\mathrm{e}^{\mathrm{i}\phi})R\mathrm{e}^{\mathrm{i}\phi}\right|\mathrm{d}\phi \\
&= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\frac{\|F\|_{\mathbb{C}_K^{\mathbb{C}^+}}}{|R\mathrm{e}^{\mathrm{i}\phi}-w|(x+R\cos\phi)}\mathrm{d}\phi \\
&\leq \int_{-\pi/2}^{-\pi/2+\eta} |\ | + \int_{-\pi/2+\eta}^{\pi/2-\eta} |\ | + \int_{\pi/2-\eta}^{\pi/2} |\ | \\
&= \frac{\|F\|_{\mathbb{C}_K^{\mathbb{C}^+}}}{2\pi}\left(\frac{4\eta}{x} + \frac{2(\pi-2\eta)}{x+R\cos\eta}\right) \\
&< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned} \tag{5.27}$$

Hence we proved that for all $F \in \mathbb{C}_K^{\mathbb{C}^+}$ we have

$$F(w+x) = \lim_{R\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{S_I} \frac{1}{z-x-w} F(z)\, \mathrm{d}z. \tag{5.28}$$

Parameterizing S_1 one finds

$$F(w+x) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{-\mathrm{i}y+x+w} F(x+\mathrm{i}y)\, \mathrm{d}y. \tag{5.29}$$

Now let $x \downarrow 0$ and by continuity of F we proved the reproducing property.

Note that the structure of the space $\mathbb{C}_K^{\mathbb{C}^+}$ resembles structure of the space $\mathfrak{F}_0$. This resemblance will be clarified by a conformal mapping that relates the two spaces. First we construct an orthonormal basis in $\mathbb{C}_K^{\mathbb{C}^+}$. The set $\{e_n : x \mapsto \mathrm{e}^{-x/2} L_n(x) \mid n \in \mathbb{N}\}$ is an orthonormal base in $\mathbb{L}_2(0,\infty)$.

Lemma 5.3 *The image of e_n is given by*

$$\big(We_n\big)(z) = \frac{(z-\frac{1}{2})^n}{(z+\frac{1}{2})^{n+1}}, \tag{5.30}$$

for all $z \in \mathbb{C}^+$ and $n \in \mathbb{N}_0$. Hence, the set $\{a_n : z \mapsto \frac{(z-\frac{1}{2})^n}{(z+\frac{1}{2})^{n+1}}\}$ is an orthonormal basis for $\mathbb{C}_K^{\mathbb{C}^+}$.

Proof:
Let $n \in \mathbb{N}_0$. Then

$$\begin{aligned}
\big(We_n\big)(z) &= \frac{1}{n!} \int_0^\infty \mathrm{e}^{-zx} \mathrm{e}^{+x/2} \big(\frac{d}{dx}\big)^n [\mathrm{e}^{-x} x^n]\, \mathrm{d}x \\
&= 0 + \frac{1}{n!}(z-\frac{1}{2}) \int_0^\infty \mathrm{e}^{-zx} \mathrm{e}^{+x/2} \big(\frac{d}{dx}\big)^{n-1} [\mathrm{e}^{-x} x^n]\, \mathrm{d}x \\
&= \ldots = \frac{1}{n!}(z-\frac{1}{2})^n \int_0^\infty \mathrm{e}^{-zx} \mathrm{e}^{+x/2} \mathrm{e}^{-x} x^n\, \mathrm{d}x \\
&= \frac{1}{n!}(z-\frac{1}{2})^n (-1)^n \big(\frac{d}{dz}\big)^n \int_0^\infty \mathrm{e}^{-(z+\frac{1}{2})x}\, \mathrm{d}x \\
&= \frac{1}{n!}(z-\frac{1}{2})^n (-1)^n \big(\frac{d}{dz}\big)^n \frac{1}{z+\frac{1}{2}} \\
&= \frac{(z-\frac{1}{2})^n}{(z+\frac{1}{2})^{n+1}}
\end{aligned}$$

for all $z \in \mathbb{C}^+$. □

Corollary 5.4 *For* e^{-zx} *the following decomposition holds*

$$e^{-zx} = \sum_{n=0}^{\infty} L_n(x) e^{-x/2} \frac{(z-\frac{1}{2})^n}{(z+\frac{1}{2})^{n+1}}, \tag{5.31}$$

for all $z \in \mathbb{C}^+$ *and* $x > 0$.

One readily verifies that the Möbius transform defined by $z \mapsto \frac{z-\frac{1}{2}}{z+\frac{1}{2}}$ maps $\mathbb{C}^+$ onto $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Moreover, the inverse is given by $z \mapsto \frac{1+z}{2-2z}$. Define the operator $T : \mathbb{C}_K^{\mathbb{C}^+} \to \mathbb{C}^D$ by

$$\big(Tf\big)(z) = \frac{1}{1-z} f\left(\frac{1+z}{2-2z}\right), \tag{5.32}$$

for all $f \in \mathbb{C}_K^{\mathbb{C}^+}$ and $z \in \mathbb{C}^+$.

Lemma 5.5 *The image of the basis element* a_n *is given by*

$$\big(Ta_n\big)(z) = z^n \tag{5.33}$$

for all $z \in D$ *and* $n \in \mathbb{N}_0$.

Proof:
Let $n \in \mathbb{N}_0$. Then

$$\begin{aligned} \big(Ta_n\big)(z) &= \frac{1}{1-z} a_n\Big(\frac{1+z}{2-2z}\Big) = \frac{1}{1-z} \frac{\big(\frac{1}{2}\frac{1+z}{1-z} - \frac{1}{2}\big)^n}{\big(\frac{1}{2} + \frac{1}{2}\frac{1+z}{1-z}\big)^{n+1}} \\ &= \frac{2}{1-z} \frac{\big(\frac{1+z}{1-z} - 1\big)^n}{\big(1 + \frac{1+z}{1-z}\big)^{n+1}} = 2 \frac{\big(1+z-1+z\big)^n}{\big(1-z+1+z\big)^{n+1}} = z^n \end{aligned}$$

for all $z \in D$. □

Corollary 5.6 *The operator* T *defines a unitary map from* $\mathbb{C}_K^{\mathbb{C}^+}$ *onto* $\mathfrak{F}_0$.

Note the resemblance between (5.22) for $\mathbb{C}_K^{\mathbb{C}^+}$ and (4.38).

6 Sampling Theorems

In the literature sampling theorems are developed to answer a basic question: given a set $H = \mathbb{C}^{\mathbb{E}}$ consisting of complex-valued function on a set $\mathbb{E}$, does there exist a subset $\mathbb{S} \subsetneq \mathbb{E}$ such that each function $f \in H$ is completely determined by its values on $\mathbb{S}$? A famous example is the space of square integrable functions on $\mathbb{R}$ for which the Fourier-transform has support in $[-a, a]$ for some fixed $a > 0$.

6.1 A general sampling theorem

In most of the classical sampling theorems, one construct a space of functions for which a sampling theorem holds, by means of a transformation between two Hilbert spaces. In this section we prove a general sampling theorem based on this idea. Theorem 2.3 will be the key for this theorem.

Let $\mathcal{H}$ be a separable Hilbert space and $V = \{\phi_x \mid x \in \mathbb{E}\}$ a labeled subset of $\mathcal{H}$, for which $\overline{\langle V \rangle} = \mathcal{H}$. Define the function $K : \mathbb{E} \times \mathbb{E} \to \mathbb{C}$ of positive type by $K(x, x') = (\phi_x, \phi_{x'})_{\mathcal{H}}$, for all $x, x' \in \mathbb{E}$. By Theorem 2.3, the frame transform $W : \mathcal{H} \to \mathbb{C}_K^{\mathbb{E}} : f \mapsto Wf$, where $Wf : \mathbb{E} \to \mathbb{C} : x \mapsto (\phi_x, f)_{\mathcal{H}}$ is unitary.

Theorem 6.1 *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence such that $\{\phi_{x_n} \mid n \in \mathbb{N}\}$ is an orthogonal basis for $\mathcal{H}$. Then,*

$$f(x) = \sum_{n\in\mathbb{N}} f(x_n) \frac{K(x, x_n)}{K(x_n, x_n)}, \tag{6.1}$$

for all $x \in \mathbb{E}$ and $f \in \mathbb{C}_K^{\mathbb{E}}$. Moreover,

$$\|f\|^2 = \sum_{n\in\mathbb{N}} \frac{|f(x_n)|^2}{K(x_n, x_n)}, \tag{6.2}$$

for all $f \in \mathbb{C}_K^{\mathbb{E}}$.

Proof:
Since W is unitary, the set $\{K_{x_n} \mid n \in \mathbb{N}\}$ is an orthogonal basis for $\mathbb{C}_K^{\mathbb{E}}$ and hence $\{\frac{K_{x_n}}{\sqrt{K(x_n,x_n)}} \mid n \in \mathbb{N}\}$ is an orthonormal basis for $\mathbb{C}_K^{\mathbb{E}}$. Let $f \in \mathbb{C}_K^{\mathbb{E}}$. Then,

$$f = \sum_{n\in\mathbb{N}} \beta_n K_{x_n} = \sum_{n\in\mathbb{N}} \frac{(K_{x_n}, f)_{\mathbb{C}_K^{\mathbb{E}}}}{\sqrt{K(x_n, x_n)}} \frac{K_{x_n}}{\sqrt{K(x_n, x_n)}} = \sum_{n\in\mathbb{N}} f(x_n) \frac{K_{x_n}}{K(x_n, x_n)}. \tag{6.3}$$

Therefore the second statement follows. For the first statement take the inner product with K_x for all $x \in \mathbb{E}$. □

This is a generalization of a theorem by Kramer, where $\mathcal{H}$ is forced to be equal to $\mathbb{L}_2((a, b))$ for some pair $a < b$. See for example [Ab] or [H].

6.2 A special subspace of $\mathbb{L}_2(\mathbb{R})$

As a first application of Theorem 6.1 we consider a famous sampling theorem on a space of analytic functions. Consider $\mathbb{L}_2((-\pi,\pi))$. Define the subset V as

$$V = \{\omega \mapsto \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\mathrm{i}\omega\overline{z}} \mid z \in \mathbb{C}\} \tag{6.4}$$

Define the function $K : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ of positive type by

$$K(w,z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\omega(w-\overline{z})} \mathrm{d}\omega = \frac{\sin \pi(w-\overline{z})}{\pi(w-\overline{z})}, \tag{6.5}$$

for all $w, z \in \mathbb{C}$ and $w \neq z$ and $K(z,z) = 1$ for all $z \in \mathbb{C}$. Then the frame transform $W : \mathbb{L}_2((-\pi,\pi)) \to \mathbb{C}_K^{\mathbb{C}}$ defined by

$$\big(Wf\big)(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\omega z} f(\omega)\ \mathrm{d}\omega \tag{6.6}$$

for all $f \in \mathbb{L}_2((-\pi,\pi))$ and $z \in \mathbb{C}$ is unitary by Theorem 2.3.

The space $\mathbb{C}_K^{\mathbb{C}}$ is easily characterized. Define the space $\mathbb{C}_K^{\mathbb{R}} = \{F|_{\mathbb{R}} \mid F \in \mathbb{C}_K^{\mathbb{C}}\}$ which is again a functional Hilbert space by [Ma, Lemma 1.14.] or [Ar, §5]. Let $f \in \mathbb{L}_2((-\pi,\pi))$ and define $\tilde{f} \in \mathbb{L}_2(\mathbb{R})$ as f on $(-\pi,\pi)$ and zero outside. Then it is obvious that $Wf_{|\mathbb{R}} = \mathcal{F}^{-1} f$. Hence the functional Hilbert space $\mathbb{C}_K^{\mathbb{R}}$ is a Hilbert subspace of $\mathbb{L}_2(\mathbb{R})$ and it consists of all functions f for which the Fourier-transformed $\mathcal{F}f$ has support in $(-\pi,\pi)$. Note that those functions are analytic and that they can be extended to entire functions. The space $\mathbb{C}_K^{\mathbb{C}}$ precisely consists of the analytic continuations of these functions. The space $\mathbb{C}_K^{\mathbb{C}}$ will play an important role in the sequel and therefore it will be denoted by the symbol $\mathfrak{H}$.

The following Theorem is a direct consequence of Theorem 6.1.

Theorem 6.2 *Let $f \in \mathfrak{H}$. Then*

$$f(z) \;\; = \;\; \sum_{n\in\mathbb{Z}} f(n) K(z,n) = \sum_{n\in\mathbb{Z}} f(n) \frac{\sin \pi(z-n)}{\pi(z-n)}, \tag{6.7}$$

for all $z \in \mathbb{C}$ and

$$\|f\| = \sum_{n\in\mathbb{Z}} |f(n)|^2. \tag{6.8}$$

Proof:
The set $\{a_n : \omega \mapsto \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\mathrm{i}\omega n} \mid n \in \mathbb{Z}\}$ is an orthonormal basis in $\mathbb{L}_2((-\pi,\pi))$.□

The set of normalized Legendre polynomials $\{x \mapsto \sqrt{n+\frac{1}{2}} P_n(x) \mid n \in \mathbb{N}_0\}$ is on orthonormal basis in $\mathbb{L}_2((-1,1))$, where $P_n(x) = C_n^{\frac{1}{2}}(x)$ for all $-\pi < x < \pi$. Therefore

$\{g_n : x \mapsto \sqrt{\frac{n+\frac{1}{2}}{\pi}} P_n(x/\pi) \mid n \in \mathbb{N}_0\}$ is on orthonormal basis in $\mathbb{L}_2((-\pi,\pi))$ The function $x \mapsto \mathrm{e}^{i\omega x}$ admits the following decomposition with respect to this basis

$$\mathrm{e}^{i\omega x} = \sum_{n=0}^{\infty} \mathrm{i}^n (n+\frac{1}{2}) \sqrt{\frac{2}{\omega}} J_{n+\frac{1}{2}}(\pi\omega) P_n(x/\pi), \tag{6.9}$$

for all $-\pi < x < \pi$ and $\omega \in \mathbb{C}$, see [MOS, §5.3 pp. 227]. Note that

$$\sqrt{\frac{2}{\omega}} J_{n+\frac{1}{2}}(\omega) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+1\frac{1}{2})} \left(\frac{\omega}{2}\right)^{2m}, \tag{6.10}$$

defines an entire function on $\mathbb{C}$, for all $n \in \mathbb{N}_0$. By Theorem 4.2, the set

$$\Big\{\omega \mapsto \mathrm{i}^n \sqrt{\frac{(2n+1)\pi}{\omega}} J_{n+\frac{1}{2}}(\pi\omega) \mid n \in \mathbb{N}_0\Big\}, \tag{6.11}$$

is an orthonormal basis in $\mathfrak{H}$.

As a result the following decomposition follows,

$$\frac{\sin \pi(x-y)}{\pi(x-y)} = \sum_{n=0}^{\infty} \frac{\pi(2n+1)}{\sqrt{xy}} J_{n+\frac{1}{2}}(\pi x) J_{n+\frac{1}{2}}(\pi y) \tag{6.12}$$

for all $x, y \in \mathbb{R}$.

Next we introduce an operator which is needed in the following section. Let $0 < d < 1$. Then the operator $\mathcal{B}_d$ defined by

$$\big(\mathcal{B}_d f\big)(\omega) = \begin{cases} \frac{1}{d} f(\omega/d) & \text{if } \omega \in (-\pi d, \pi d) \\ 0 & \text{if } \omega \notin (-\pi d, \pi d) \end{cases}, \tag{6.13}$$

for almost all $x \in (-\pi d, \pi d)$ and all $f \in \mathbb{L}_2((-\pi,\pi))$ maps $\mathbb{L}_2((-\pi,\pi))$ into itself. Moreover,

$$\begin{aligned} \big(W\mathcal{B}_d f\big)(x) &= \frac{1}{d}\int_{-\pi d}^{\pi d} f(\omega/d)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega = \int_{-\pi}^{\pi} f(\omega)\mathrm{e}^{d\mathrm{i}\omega x}\,\mathrm{d}\omega \\ &= \big(Wf\big)(dx) \end{aligned} \tag{6.14}$$

for all $f \in \mathbb{L}_2((-1,1))$ and $x \in \mathbb{C}$. Define the operator $\mathcal{T}_d$ on $\mathfrak{H}$ by

$$\big(\mathcal{T}_d F\big)(x) = f(dx), \tag{6.15}$$

for all $x \in \mathbb{C}$. Then $WB_d = \mathcal{T}_d W$.

6.3 A space of q-functions

In this section the space $\mathfrak{G}_q$ will be introduced for $0 < q < 1$. This space is a functional Hilbert spaces consisting of analytic functions on a open subset of $\mathbb{C}$ containing the interval $(0, \infty)$. The norm of an arbitrary function in $\mathfrak{G}_q$ will turn out be

$$\|f\|_q = \log \frac{1}{q} \sum_{n=-\infty}^{\infty} |f(q^n)|^2 q^n \tag{6.16}$$

for all $f \in \mathfrak{G}_q$. Besides a different constant in front of the summation, this is a well-known expression in the theory of q-functions. Moreover, we provide a unitary map from the space $\mathfrak{H}$ from the preceding section onto $\mathfrak{G}_q$. This map will help us analyzing the space $\mathfrak{G}_q$. In particular we provide a sampling theorem for the space $\mathfrak{G}_q$. For an alternative sampling theorem for special q-functions, see [Ab].

Let $S = \{x \in \mathbb{C} \mid x < 0\}$ be a cut in the complex plain. Define the $\log : \mathbb{C}/S \to \mathbb{C}$ by

$$\log(z) = \int_1^z \frac{1}{w} \mathrm{d}w, \tag{6.17}$$

for all $z \in \mathbb{C}/S$. Define z^α by $z^\alpha = |z|^\alpha \mathrm{e}^{\mathrm{i} \arg z \alpha}$, for all $z \in \mathbb{C}/S$ and $0 < \alpha < 1$, where $-\pi < \arg z < \pi$. The square root $z^{\frac{1}{2}}$ will be denoted by $\sqrt{z}$.

Let $0 < q < 1$. We recall that $\mathfrak{H}$ was the subspace of $\mathbb{L}_2(\mathbb{R})$ consisting of the analytic continuations of functions in $\mathbb{L}_2(\mathbb{R})$ for which the Fourier transform has support within $(-\pi, \pi)$. Define the operator $A_q : \mathfrak{H} \to \mathbb{C}^{\mathbb{C}/S}$ by

$$\big(A_q f\big)(z) = \frac{1}{\sqrt{z}} \, f\left(\frac{\log z}{\log \frac{1}{q}} \right) \tag{6.18}$$

for all $f \in \mathfrak{H}$, $z \in \mathbb{C}/S$.

Lemma 6.3 *Suppose $f \in \mathfrak{H}$. Then $A_q f_{|(0,\infty)} \in \mathbb{L}_2((0,\infty))$. Moreover, $\|A_q f_{|(0,\infty)}\|_{\mathbb{L}_2((0,\infty))} = \sqrt{\log \frac{1}{q}} \|f\|_{\mathfrak{H}}$.*

Proof:
Let $f \in \mathfrak{H}$. Use the substitution $z = q^{-x}$ in

$$\begin{aligned} \log \frac{1}{q} \int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x &= \log \frac{1}{q} \int_0^{\infty} \Big|f\left(\frac{\log z}{\log \frac{1}{q}} \right)\Big|^2 \frac{1}{z \log \frac{1}{q}} \mathrm{d}x \\ &= \int_0^{\infty} \big| \big(A_q f\big)(z) \big|^2 \mathrm{d}z. \end{aligned}$$

This proves the statement. □

Denote the space $A_q(\mathfrak{H})$, endowed with the norm $\|f\|_q = \|f_{|(0,\infty)}\|_{\mathbb{L}_2((0,\infty))}$ for all $f \in A_q(\mathfrak{H})$, by $\mathfrak{G}_q$. Recall that $\mathfrak{H}$ is a functional Hilbert space with reproducing kernel

$$K(w,z) = \frac{\sin \pi(\overline{z} - w)}{\pi(\overline{z} - w)}, \tag{6.19}$$

for all $z, w \in \mathbb{C}$.

Theorem 6.4 *The space $\mathfrak{G}_q$ is a functional Hilbert space. The reproducing kernel L_q of $\mathfrak{G}_q$ is given by*

$$L_z^{(q)}(w) = L^{(q)}(w,z) = \frac{1}{(\log \frac{1}{q})^2 \sqrt{\overline{z}}} \big(A_q K_{\frac{\log z}{\log \frac{1}{q}}}\big)(w) = \frac{\sin \pi \frac{\overline{\log z} - \log w}{\log \frac{1}{q}}}{\log \frac{1}{q} \sqrt{\overline{z} w} (\overline{\log z} - \log w)}, \tag{6.20}$$

for all $w, z \in \mathbb{C}/S$, where K is the reproducing kernel of $\mathfrak{H}$. Moreover, $\frac{1}{\log \frac{1}{q}} A_q$ defines a unitary map from $\mathfrak{H}$ onto $\mathfrak{G}_q$.

Proof:
By Lemma 6.3, the map $\frac{1}{\log \frac{1}{q}} A_q$ defines an isometry from $\mathfrak{H}$ onto $\mathfrak{G}_q$. Moreover, it is surjective by definition. Since $\mathfrak{H}$ is complete it easily follows that $\mathfrak{G}_q$ is also a Hilbert space.

Let $z \in \mathbb{C}/S$. Then

$$\begin{aligned} \left(\frac{1}{\sqrt{\overline{z}} (\log \frac{1}{q})^2} A_q K_{\frac{\log z}{\log \frac{1}{q}}}, A_q f \right)_q &= \frac{1}{\sqrt{z}} (K_{\frac{\log z}{\log \frac{1}{q}}}, f)_{\mathfrak{H}} \\ &= \frac{1}{\sqrt{z}} f\Big(\frac{\log z}{\log \frac{1}{q}}\Big) \\ &= \big(A_q f\big)(z), \end{aligned}$$

for all $f \in \mathfrak{H}$. Hence the statement follows. □

Theorem 6.5 *Suppose $F \in \mathfrak{G}_q$. Then*

$$\|F\|_q^2 = \log \frac{1}{q} \sum_{n=-\infty}^{\infty} |F(q^n)|^2 q^n. \tag{6.21}$$

Moreover,

$$F(z) = \log \frac{1}{q} \sum_{n=-\infty}^{\infty} F(q^n) L_{q^n}(z) q^n \tag{6.22}$$

for all $z \in \mathbb{C}/S$.

Proof:
By definition, there exist a $f \in \mathfrak{H}$ such that $F = A_q f$.

$$\begin{aligned}\|F\|_q^2 &= \int_0^\infty \left|\big(A_q f\big)(z)\right|^2 \mathrm{d}z = \int_0^\infty \left| f\left(\frac{\log z}{\log \frac{1}{q}}\right)\right|^2 \frac{1}{z}\mathrm{d}z \\ &= \log\frac{1}{q}\int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x = \log\frac{1}{q}\sum_{n=-\infty}^{\infty}|f(n)|^2 = \log\frac{1}{q}\sum_{n=-\infty}^{\infty}\left|\big(A_q f\big)(q^n)\right|^2 q^n\end{aligned}$$

This proves the first statement. The second statement easily follows by taking the inner product of $L_z^{(q)}$ with f. □

6.4 Operators on $\mathfrak{G}_q$

Recall that $\mathcal{T}_d f \in \mathfrak{H}$ for all $f \in \mathfrak{H}$ and $0 < d < 1$, where $\mathcal{T}_d$ is the operator defined in (6.15).

Lemma 6.6 $A_q \mathcal{T}_d = A_{q^{\frac{1}{d}}}$.

Proof:
Let $f \in \mathfrak{H}$. Then

$$\big(A_q \mathcal{T}_d f\big)(z) = \frac{1}{\sqrt{z}} f\Big(d\frac{\log z}{\log \frac{1}{q}}\Big) = \frac{1}{\sqrt{z}} f\Big(\frac{\log z}{\log \frac{1}{q^{\frac{1}{d}}}}\Big) = \big(A_{q^{\frac{1}{d}}} f\big)(z),$$

for all $z \in \mathbb{C}/S$. □

Corollary 6.7 *Suppose* $0 < q < s < 1$. *Then* $\mathfrak{G}_q$ *is a Hilbert subspace of* $\mathfrak{G}_s$.

Proof:
Define $d = \frac{\log s}{\log q} < 1$. Then $A_q = \mathcal{T}_d A_s$, since $s^{\frac{\log q}{\log s}} = q$ and Lemma 6.6. It follows that

$$\mathfrak{G}_q = A_q \mathfrak{H} = A_s \mathcal{T}_d \mathfrak{H} \subset A_s \mathfrak{H} = \mathfrak{G}_s,$$

since $\mathcal{T}_d \mathfrak{H} \subset \mathfrak{H}$. Let $f \in \mathfrak{G}_q$, then $\|f\|_q = \|f_{|(0,\infty)}\|_{\mathbb{L}_2((0,\infty))} = \|f\|_s$. □

Let $0 < \alpha < 1$. Suppose $F = A_q f \in \mathfrak{G}_q$. Then

$$F(z^\alpha) = \big(A_q f\big)(z^\alpha) = \frac{1}{\sqrt{z}} f\Big(\frac{\log z^\alpha}{\log \frac{1}{q}}\Big) = \frac{1}{\sqrt{z}} f\Big(\alpha\frac{\log z}{\log \frac{1}{q}}\Big) = \big(A_q \mathcal{T}_\alpha f\big)(z) \tag{6.23}$$

for all $z \in \mathbb{C}/S$. Hence $z \mapsto F(z^\alpha) \in \mathfrak{G}_q$. Define the operator S_α on $\mathfrak{G}_q$ by

$$\big(S_\alpha F\big)(z) = F(z^\alpha), \tag{6.24}$$

for all $F \in \mathfrak{G}_q$ and $z \in \mathbb{C}/S$.

Lemma 6.8 $A_{q^{\frac{1}{d}}} = A_q \mathcal{T}_d = S_d A_q$, *for all* $0 < d < 1$ *and* $0 < q < 1$.

Corollary 6.9 *Let* $0 < q < s < 1$ *and define* $\alpha = \frac{\log q}{\log s}$. *The operator* αS_α *maps* $\mathfrak{G}_q$ *unitarily onto* $\mathfrak{G}_s$.

Proof:
The operator $\frac{1}{\log \frac{1}{t}} A_t$ is a unitary map from $\mathcal{H}$ onto $\mathfrak{G}_t$, for all $0 < t < 1$. The statement now follows by the identity $\alpha S_\alpha = \frac{1}{\log \frac{1}{s}} A_s \big(\frac{1}{\log \frac{1}{q}} A_q\big)^{-1}$. □

6.5 A generalization of Lagrange interpolation

In this section we introduce a general way to construct functional Hilbert spaces of entire functions together with a sampling formula. This construction covers most of the classical sampling formulas.

The order λ of an entire function $\phi : \mathbb{C} \to \mathbb{C}$ is defined by

$$\lambda = \inf \left\{ \mu \in \mathbb{R} \mid \lim_{r\to\infty} \frac{\max_{|z|=r} |f(z)|}{\mathrm{e}^{r^\mu}} = 0 \right\}. \tag{6.25}$$

If $\left\{ \mu \in \mathbb{R} \mid \lim_{r\to\infty} \frac{\max_{|z|=r} |f(z)|}{\mathrm{e}^{r^\mu}} = 0 \right\} = \varnothing$, then set $\lambda = \infty$. Note that λ is the smallest number such that

$$M(r) \leq \mathrm{e}^{r^{\lambda+\varepsilon}}, \tag{6.26}$$

for any given ε as soon as r is sufficiently large, where $M(r)$ stands for the maximum of $|\phi(z)|$ on $|z| = r$.

Let ϕ be an entire function of order $h \leq \lambda \leq h+1$, where h an integer, which has only simple zeros. Denote the set of zeros by $\{a_n \mid n \in \mathbb{N}\}$. For the sake of simplicity assume that 0 is not a zero of f. As a result of a theorem by Hadamard [Ah, §5.3, pp. 205-210],

$$\sum_{n\in\mathbb{N}} |a_n|^{-h} < \infty. \tag{6.27}$$

This result implies in particular that $\lim_{n\to\infty} \frac{1}{a_n} = 0$.

For each zero a_n define the function $L_{a_n} : \mathbb{C} \to \mathbb{C}$ by

$$L_{a_n}(z) = \begin{cases} \frac{\phi(z)}{(z-a_n)} & z \neq a_n \\ \phi'(a_n) & z = a_n \end{cases}, \tag{6.28}$$

for all $z \in \mathbb{C}$. It is clear that the functions L_{a_n} are entire functions and satisfy $L_{a_n}(a_m) = \phi'(a_n)\delta_{mn}$, for all $m, n \in \mathbb{N}$.

Let $\{\lambda_n\}_{n\mathbb{N}}$ be a sequence of positive real numbers such that

$$\sum_{n\in\mathbb{N}} \frac{1}{\lambda_n |a_n|^2} < \infty. \tag{6.29}$$

Note that the sequence defined $\lambda_n = |a_n|^{h-2}$ satisfies this equation. In particular this implies that if $h = 2$ that the sequence defined by $\lambda_n = 1$ satisfies this equation.

Lemma 6.10 *The sum*

$$K(z,w) := \sum_{n\in\mathbb{N}} \frac{1}{\lambda_n} L_{a_n}(z)\overline{L_{a_n}(w)} \tag{6.30}$$

is absolutely convergent for $z, w \in \mathbb{C}$.

Proof:
Let $z, w \in \mathbb{C}$. If z or w equals a zero a_m the statement is trivial, since $L_{a_n}(a_m) = \phi'(a_n)\delta_{mn}$. Now suppose that both are not zeros of f. The summand satisfies

$$|L_{a_n}(z)\overline{L_{a_n}(w)}| = \frac{|\phi(z)||\phi(w)|}{|z-a_n||w-a_n|} \leq \frac{1}{2}\frac{|\phi(z)|^2}{|z-a_n|^2} + \frac{1}{2}\frac{|\phi(w)|^2}{|w-a_n|^2}$$

for all $n \in \mathbb{N}$. It is sufficient to show that $\sum_{n=1}^{\infty} \frac{|\phi(z)|^2}{\lambda_n|z-a_n|^2}$ converges. By (6.27), $\lim_{n\to\infty} \frac{1}{a_n} \to 0$. Therefore there exists a $N \in \mathbb{N}$ such that $|\frac{z}{a_n}| < \frac{1}{2}$ for all $n > N$. Hence

$$\begin{aligned}\sum_{n=N+1}^{M} \frac{|\phi(z)|^2}{\lambda_n|z-a_n|^2} &\leq |\phi(z)|^2 \sum_{n=N+1}^{M} \frac{1}{\lambda_n|a_n|^2}\frac{1}{|\frac{z}{a_n}-1|^2} \\ &\leq |\phi(z)|^2 \sum_{n=N+1}^{M} \frac{1}{\lambda_n|a_n|^2}\frac{1}{(1-|\frac{z}{a_n}|)^2} \leq |\phi(z)|^2 \sum_{n=N+1}^{M} \frac{4}{\lambda_n|a_n|^2}\end{aligned}$$

for all $M > N$. Hence the sum $\sum_{n=1}^{\infty} \frac{|\phi(z)|^2}{\lambda_n|z-a_n|^2}$ converges. This proves the statement. □

The function $K : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ is a function of positive type. Hence it defines the functional Hilbert space $\mathbb{C}_K^{\mathbb{C}}$. As usual define the functions $K_w : \mathbb{C} \to \mathbb{C}$ by $K_w(z) = K(z,w)$ for all $z, w \in \mathbb{C}$ and note that $K_{a_n}(z) = \frac{\overline{\phi'(a_n)}}{\lambda_n} L_{a_n}(z)$.

Theorem 6.11 *The set $\{\frac{1}{\lambda_n^{\frac{1}{2}}} L_{a_n} \mid n \in \mathbb{N}\}$ is an orthonormal basis for $\mathbb{C}_K^{\mathbb{C}}$.*

Proof:
Let $n, m \in \mathbb{N}_0$. Then by Lemma 2.1 it follows that $a_n, a_m \in \mathbb{C}_K^{\mathbb{C}}$. Moreover,

$$\begin{aligned}(L_{a_n}, L_{a_m})_{\mathbb{C}_K^{\mathbb{C}}} &= \frac{\lambda_n\lambda_m}{\phi'(a_m)\overline{\phi'(a_n)}}(K_{a_n}, K_{a_m}) = \frac{\lambda_n\lambda_m}{\phi'(a_m)\overline{\phi'(a_n)}} K(a_n, a_m) \\ &= \lambda_n\delta_{nm}\end{aligned}$$

This proves the orthonormality. By Theorem 4.1 it is also a basis. □

Note that the set $\{\frac{1}{\lambda_n^{\frac{1}{2}}} L_{a_n} \mid n \in \mathbb{N}\}$ consists of entire functions. Hence $\mathbb{C}_K^{\mathbb{C}}$ has a basis of entire functions.

Theorem 6.12 *The space $\mathbb{C}_K^{\mathbb{C}}$ consists of entire functions.*

Proof:
Let $f \in \mathbb{C}_K^{\mathbb{C}}$. By the previous theorem, f can be written limit of a sequence of analytic functions $f = \lim_{n\to\infty} f_n$. Since $\mathbb{C}_K^{\mathbb{C}}$ is a functional Hilbert space, f is also the pointwise limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$, i.e. $f(z) = \lim_{n\to\infty} f_n(z)$ for all $z \in \mathbb{C}$. We will prove that for every $z \in \mathbb{C}$ there exists a compact neighborhood U_z of z such that the pointwise limit converges uniformly on U_z. From this the statement follows, since the uniformity implies that f is also analytic on the interior of U_z and z was arbitrary.

Since

$$|f(z) - f_n(z)| \leq \|K_z\|\|f_n - f\|,$$

for all $z \in \mathbb{C}$ and $n \in \mathbb{N}$ it is sufficient to show that $z \mapsto \|K_z\|^2$ is locally bounded, i.e. for all $z \in \mathbb{C}$ there exists a compact neighborhood U_z and $A_z > 0$ such that $K(w,w) < A_z$ for all $w \in A_z$.

Let $z \in \mathbb{C}$. Separate the cases $z = a_n$ for some $n \in \mathbb{N}$ and $z \neq a_n$ for all $n \in \mathbb{N}$. Start with the last case. Set $U_z = \{w \in \mathbb{C} \mid |w - z| \leq \frac{1}{2} \inf_{n\in\mathbb{N}} |z - a_n|\}$. Note that $\inf_{n\in\mathbb{N}} |z - a_n| > 0$, since the set $\{a_n \mid n \in \mathbb{N}\}$ does not have an accumulation point. Then,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{|\phi(w)|^2}{\lambda_n |w - a_n|^2} &= \sum_{n=1}^{\infty} \frac{|\phi(w)|^2}{\lambda_n |z - a_n|^2} \frac{1}{|1 - \frac{w-z}{a_n - z}|^2} \\
&\leq \sum_{n=1}^{\infty} \frac{|\phi(w)|^2}{\lambda_n |z - a_n|^2} \frac{1}{(1 - |\frac{w-z}{a_n - z}|)^2} \leq 4|\phi(w)|^2 \sum_{n=1}^{\infty} \frac{1}{\lambda_n |z - a_n|^2} \\
&\leq \max_{w \in U_z} |\phi(w)|^2 \sum_{n=1}^{\infty} \frac{1}{\lambda_n |z - a_n|^2}
\end{aligned}$$

for all $w \in U_z$. Hence, $w \mapsto K(w,w)$ is bounded on U_z.

Finally, consider the case $z = a_m$ for a certain $m \in \mathbb{N}$. Set $U_z = \{w \in$

$\mathbb{C} \mid |w-z| \leq \frac{1}{2} \inf_{n\in\mathbb{N}, n\neq m} |z-a_n|\}$.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{|\phi(w)|^2}{\lambda_n |w-a_n|^2} &= \frac{|\phi(w)|^2}{\lambda_n |w-a_m|^2} + \sum_{n=,n\neq m}^{\infty} \frac{|\phi(w)|^2}{\lambda_n |w-a_n|^2} \\
&\leq \max_{w\in U_z} \frac{|\phi(w)|^2}{\lambda_n |w-a_m|^2} + \sum_{n=1,n\neq m}^{\infty} \frac{|\phi(w)|^2}{\lambda_n |z-a_n|^2} \frac{1}{(1-|\frac{w-z}{z-a_n}|)^2} \\
&\leq \max_{w\in U_z} \frac{|\phi(w)|^2}{\lambda_n |w-a_m|^2} + 4 \sum_{n=1,n\neq m}^{\infty} \frac{|\phi(w)|^2}{\lambda_n |z-a_n|^2} \\
&\leq \max_{w\in U_z} \frac{|\phi(w)|^2}{\lambda_n |w-a_m|^2} + 4 \max_{w\in U_z} |\phi(w)|^2 \sum_{n=1,n\neq m}^{\infty} \frac{1}{\lambda_n |z-a_n|^2}
\end{aligned}$$

for all $w \in U_z$. Hence, $w \mapsto K(w,w)$ is bounded on U_z. □

The following theorem is a sampling theorem for the space $\mathbb{C}_K^{\mathbb{C}}$.

Theorem 6.13 *Let $f \in \mathbb{C}_K^{\mathbb{C}}$. Then*

$$\|f\|_{\mathbb{C}_K^{\mathbb{C}}} = \sum_{n=1}^{\infty} |f(a_n)|^2 \frac{\lambda_n}{|\phi'(a_n)|^2}. \tag{6.31}$$

Moreover,

$$f(z) = \sum_{n=1}^{\infty} f(a_n) \frac{L_{a_n}(z)}{\phi'(a_n)} \tag{6.32}$$

for all $z \in \mathbb{C}$.

Proof:
Since $\{\frac{1}{\lambda_n^{\frac{1}{2}}} L_{a_n} \mid n \in \mathbb{N}\}$ is an orthonormal basis for $\mathbb{C}_K^{\mathbb{C}}$, the function f can be decomposed in $f = \sum_{n\in\mathbb{N}} \frac{c_n}{\lambda_n^{\frac{1}{2}}} L_{a_n}$ where

$$c_n = (\frac{1}{\lambda_n^{\frac{1}{2}}} L_{a_n}, f)_{\mathbb{C}_K^{\mathbb{C}}} = \frac{\lambda_n^{\frac{1}{2}}}{\phi'(a_n)} (K_{a_n}, f)_{\mathbb{C}_K^{\mathbb{C}}} = \frac{\lambda_n^{\frac{1}{2}}}{\phi'(a_n)} f(a_n),$$

for all $n \in \mathbb{N}$. Hence the statement follows. □

Example: Consider the case $\phi(z) = \frac{1}{\Gamma(z)}$ for all $z \in \mathbb{C}$. Then ϕ is an entire function with zero's $z = -n$ for all $n \in \mathbb{N}_0$. Moreover $\phi'(-n) = (-1)^n n!$ for all $n \in \mathbb{N}_0$. Define $\lambda_n = 1$ for all $n \in \mathbb{N}_0$.

$$\sum_{n=0}^{\infty} \frac{1}{(z+n)(\overline{w}+n)} = \frac{1}{z-\overline{w}} \sum_{n=0}^{\infty} \frac{1}{\overline{w}+n} - \frac{1}{z+n} = \frac{\psi(z) - \overline{\psi(w)}}{z-\overline{w}}, \tag{6.33}$$

for all $z, w \in \mathbb{C}$, where ψ is the function defined by $\psi(z) = \frac{\Gamma(z)}{\Gamma'(z)}$. Hence the reproducing kernel K is given by

$$K(z,w) = \frac{\psi(z) - \overline{\psi(w)}}{\Gamma(z)\overline{\Gamma(w)}(z - \overline{w})}, \tag{6.34}$$

for all $z, w \in \mathbb{C}$ and $z \neq w$ if $\mathrm{Im}(z) = 0$. For $z \in \mathbb{R}$ we obtain $K(z,z) = \frac{\psi'(z)}{|\Gamma(z)|^2}$. Since $\psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}$ the positiveness is guaranteed as it must be, moreover the singularities $z = -n$ of ψ' are also zeros of $\frac{1}{|\Gamma|^2}$ of the same order, therefore they cancel each other. Let $f \in \mathbb{C}_K^{\mathbb{C}}$ then

$$f(z) = \frac{1}{\Gamma(z)} \sum_{n=0}^{\infty} \frac{(-1)^n f(-n)}{(z+n)n!}, \tag{6.35}$$

for all $z \in \mathbb{C}$ and $z \neq -n$ for all $n \in \mathbb{N}_0$. Moreover,

$$\|f\|^2_{\mathbb{C}_K^{\mathbb{C}}} = \sum_{n=0}^{\infty} \frac{|f(-n)|^2}{n!^2}, \tag{6.36}$$

for all $f \in \mathbb{C}_K^{\mathbb{C}}$. By the identity $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$, we find

$$\frac{L_{-n}(z)}{\Gamma(1-z)} = \frac{1}{\Gamma(z)\Gamma(1-z)(z+n)} = \frac{\sin \pi z}{\pi(z+n)}, \tag{6.37}$$

for all $n \in \mathbb{N}_0$ and $z \in \mathbb{C}/\{-n\}$. Hence $L_n \in \mathbb{L}_2(\mathbb{R}, \frac{1}{\Gamma(1-x)}\mathrm{d}x)$. Moreover,

$$\int_{-\infty}^{\infty} \overline{L_{-n}(x)} L_{-m}(x) \frac{1}{|\Gamma(1-x)|^2}\ \mathrm{d}x = \int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi(x+n)} \frac{\sin \pi x}{\pi(x+m)}\ \mathrm{d}x = \delta_{nm}, \tag{6.38}$$

for all $n, m \in \mathbb{N}_0$. Therefore,

$$\|f\|^2_{\mathbb{C}_K^{\mathbb{C}}} = \int_{-\infty}^{\infty} |f(x)|^2\ \frac{1}{|\Gamma(1-x)|^2}\mathrm{d}x, \tag{6.39}$$

for all $f \in \mathbb{C}_K^{\mathbb{C}}$. Hence $\mathbb{C}_K^{\mathbb{C}}$ is a Hilbert subspace of $\mathbb{L}_2(\mathbb{R}, \frac{1}{|\Gamma(1-x)|^2}\ \mathrm{d}x)$. For details about the Gamma- and the ψ-function, see [Te, Ch. 3].

Example: Let $a, b \in \mathbb{R}$ and $a < b$. Let $w : [a,b] \to (0,\infty)$ be a measurable function such that

$$\int_a^b x^n w(x)\ \mathrm{d}x < \infty, \tag{6.40}$$

for all $n \in \mathbb{N}_0$. Let $\{p_n \mid n \in \mathbb{N}_0\}$ be the orthonormal set in $\mathbb{L}_2((a,b), w(x)\ \mathrm{d}x)$ constructed by the Gramm-Schmidt proces on $\{x \mapsto x^n \mid n \in \mathbb{N}_0\}$. Note that p_n is an polynomial of degree $n \in \mathbb{N}_0$. We recall the formula of Christoffel-Darboux

$$K_n(x,y) = \sum_{k=0}^{n} p_k(x)p_k(y) = \frac{k_n}{k_{n+1}} \frac{p_n(y)p_{n+1}(x) - p_n(x)p_{n+1}(y)}{x-y}, \tag{6.41}$$

for all $x, y \in [a,b]$ with $x \neq y$ and $n \in \mathbb{N}_0$, where k_n is the coefficient in $p_n(x) = k_n x^n + \ldots$, for all $n \in \mathbb{N}_0$. Moreover,

$$K_n(x,x) = \frac{k_n}{k_{n+1}} \big(p_n(x)p'_{n+1}(x) - p'_n(x)p_{n+1}(x)\big), \tag{6.42}$$

for all $x \in [a,b]$. Note that K_n is the reproducing kernel of the subspace $\mathrm{Pol}(n)$ of $\mathbb{L}_2((a,b), w(x)\ \mathrm{d}x)$ consisting of all polynomials of degree n. See [Te, Ch. 6] for more details.

Let $N \in \mathbb{N}$ and set $\phi = p_N$. Let $\{a_k \mid k = 1, \ldots, N\}$ be the set of zeros of p_N. By the formula of Christoffel-Darboux we obtain

$$L_{a_k}(x) = \frac{p_N(x)}{x - a_k} = \frac{k_{N+1}}{k_N p_{N+1}(a_k)} K(x, a_k), \tag{6.43}$$

for all $x \in [a,b]$ and $k = 1, \ldots, N$. Therefore,

$$\begin{aligned}\int_a^b L_{a_k}(x)L_{a_l}(x)w(x)\ \mathrm{d}x &= \left(\frac{k_{N+1}}{k_N p_{N+1}(a_k)}\right)^2 (K_{a_k}, K_{a_l}) = \left(\frac{k_{N+1}}{k_N p_{N+1}(a_k)}\right)^2 K(a_k, a_l) \\ &= -\frac{p'_N(a_k)k_{N+1}}{k_N p_{N+1}(a_k)} \delta_{kl}, \end{aligned} \tag{6.44}$$

for all $k, l = 1, \ldots, N$. Set

$$\lambda_k = -\frac{p'_N(a_k)k_{N+1}}{k_N p_{N+1}(a_k)} > 0, \tag{6.45}$$

for $k = 1, \ldots, N$. It follows that $\{\frac{1}{\lambda_k} L_{a_k} \mid k = 1, \ldots N\}$ is an orthonormal set for the Hilbert subspace $\mathrm{Pol}(N-1)$ of $\mathbb{L}_2((a,b), w(x)\ \mathrm{d}x)$ consisting of all polynomials of degree $\leq N-1$, and hence

$$\sum_{k=1}^{N} \frac{1}{\lambda_k} L_{a_k}(x) L_{a_k}(y) = K_{N-1}(x,y), \tag{6.46}$$

for all $x, y \in [a,b]$.

The following two identities are thus obtained,

$$p(x) = \sum_{k=1}^{n} \frac{p_N(x)p(a_k)}{p'_N(a_k)(x - a_k)}, \tag{6.47}$$

for all $x \in [a,b]$ and $p \in \mathrm{Pol}(N-1)$.

$$(p,q)_{\mathrm{Pol}(N-1)} = \int_a^b p(x)q(x)w(x)\mathrm{d}x = \sum_{k=1}^{N} \frac{1}{\lambda_k |p_N'(a_k)|^2} p(a_k)q(a_k), \tag{6.48}$$

for all $p, q \in \mathrm{Pol}(N-1)$. By substitution of $p(x) = x^k$ and $q(x) = x^l$ for $k, l = 0, 1, \ldots, N-1$ in (6.48), it follows that

$$\int_a^b p(x)w(x)\mathrm{d}x = \sum_{k=1}^{N} \frac{1}{\lambda_k |p_N'(a_k)|^2} p(a_k), \tag{6.49}$$

for all $p \in \mathrm{Pol}(2N-2)$. Moreover, if $r \in \mathrm{Pol}(N-1)$ and $q = p_N$ then (6.49) also holds for $p = qr$, since both terms vanish. Therefore, (6.49) holds for all $p \in \mathrm{Pol}(2N-1)$. This result is known as Gaussian integration.

7 Wavelet transforms

7.1 Construction of V using group representations

From now on we will assume $\mathbb{E}$ to be a group G. Furthermore, we assume the group to have a **representation** on $\mathcal{H}$, i.e. a map $\mathcal{R} : G \to \mathcal{B}(\mathcal{H}) : g \mapsto \mathcal{R}_g$, which satisfies

$$\mathcal{R}_g \mathcal{R}_h = \mathcal{R}_{gh} \quad \forall_{g,h \in G}, \tag{7.1}$$

$$\mathcal{R}_e = I \tag{7.2}$$

where e is the identity element of G. Here $\mathcal{B}(\mathcal{H})$ is the linear space of all bounded operators from $\mathcal{H}$ into $\mathcal{H}$. Given a vector $\psi \in \mathcal{H}$ we can construct the set V in (2.12) as follows

$$V_\psi = \{\mathcal{R}_g \psi \mid g \in G\}. \tag{7.3}$$

We will call ψ a **(generating) wavelet**. Starting with such a set V_ψ we can construct a functional Hilbert subspace $\mathbb{C}_K^G$ and a unitary map W_ψ between $\overline{\langle V_\psi \rangle}$ and this functional Hilbert space, as described in Section 2. The unitary map W_ψ will be called the **wavelet transform**.

We state the following consequence of Theorem 2.3.

Theorem 7.1 *Let $\mathcal{R}$ be a representation of a group G in a Hilbert space $\mathcal{H}$. Let $\psi \in \mathcal{H}$. Define the function $K : G \times G \to \mathbb{C}$ of positive type by*

$$K(g, g') = (\mathcal{R}_g \psi, \mathcal{R}_{g'} \psi)_{\mathcal{H}}. \tag{7.4}$$

Define the set V_ψ by

$$V_\psi = \{\mathcal{R}_g \psi \mid g \in G\}. \tag{7.5}$$

Then the wavelet transform $W_\psi : \overline{\langle V_\psi \rangle} \to \mathbb{C}_K^G$ defined by

$$\big(W_\psi f\big)(g) = (\mathcal{R}_g \psi, f)_{\mathcal{H}}, \tag{7.6}$$

is a unitary map.

Of course, the wavelet transform W_ψ could be defined on the entire space $\mathcal{H}$, but then the unitarity is lost in the case $\overline{\langle V_\psi \rangle} \neq \mathcal{H}$. For a vector $f \perp V_\psi$ we then get $W_\psi f = 0$.

Usually we are interested in the case $\overline{\langle V \rangle} = \mathcal{H}$. If $\overline{\langle V_\psi \rangle} = \mathcal{H}$ for some $\psi \in \mathcal{H}$, we call ψ a **cyclic vector** or a **cyclic wavelet** and the representation is called a **cyclic representation** if a cyclic wavelet exists.

Theorem 7.2 *Let $\mathcal{R}$ be a representation of a group G in a Hilbert space $\mathcal{H}$. Let ψ be a cyclic wavelet. Define a function $K : G \times G \to \mathbb{C}$ of positive type by*

$$K(g, g') = (\mathcal{R}_g \psi, \mathcal{R}_{g'} \psi)_{\mathcal{H}}. \tag{7.7}$$

The wavelet transform $W_\psi : \mathcal{H} \to \mathbb{C}_K^G$ defined by

$$\big(W_\psi f\big)(g) = (\mathcal{R}_g \psi, f)_{\mathcal{H}}, \tag{7.8}$$

is a unitary map.

It is obvious that W_ψ can be defined as a unitary map on the entire space $\mathcal{H}$ if and only if $\mathcal{R}$ is cyclic and ψ is a cyclic wavelet.

Note that up till now there are no restrictions have been imposed on the Hilbert space $\mathcal{H}$, the group G or the representation $\mathcal{R}$. In particular, there are no topological conditions on G and $\mathcal{R}$. There is even no topology on G.

7.2 Unitary representations

The kind of representations, which have our special interest, are **unitary representations**, i.e. representations $\mathcal{R}$ for which the $\mathcal{R}_g$ are unitary for all $g \in G$. These kind of representations have some nice properties. We will use the symbol $\mathcal{U}$ instead of $\mathcal{R}$ to indicate that a representation is unitary.

The function of positive type, from which the functional Hilbert space is constructed, is given by $K(g,h) = (\mathcal{U}_g\psi, \mathcal{U}_h\psi)_\mathcal{H}$. Because the representation is unitary this simplifies to

$$K(g,h) = (\mathcal{U}_g\psi, \mathcal{U}_h\psi)_\mathcal{H} = (\psi, \mathcal{U}_{g^{-1}h}\psi)_\mathcal{H} =: F(g^{-1}h), \tag{7.9}$$

for all $g, h \in G$. In abstract harmonic analysis, the function $F : G \to \mathbb{C}$ is said to be of **positive type** if

$$\sum_{i=1}^{n}\sum_{j=1}^{n} F(g_i^{-1}g_j)c_j\overline{c_i} \geq 0, \tag{7.10}$$

for all $n \in \mathbb{N}$, $c_1, ..., c_n \in \mathbb{C}$ and $g_1, ..., g_n \in G$. Note that not all function of positive type can be written as $K(g,h) = F(g^{-1}h)$ for all $g, h \in G$, for some function $F : G \to \mathbb{C}$ of positive type. A necessary and sufficient condition is that $K(hg_1, hg_2) = K(g_1, g_2)$ for all $h, g_1, g_2 \in G$.

Thus if $\mathcal{U}$ is a unitary representation of G in a Hilbert space $\mathcal{H}$ and $\psi \in \mathcal{H}$, then $F : G \to \mathbb{C}$ defined by $F(g) = (\psi, \mathcal{U}_g\psi)_\mathcal{H}$ is of positive type. The following theorem is a converse.

Let $F : G \to C$ be of positive type and define $K : G \times G \to \mathbb{C}$ by $K(g,h) = F(g^{-1}h)$. Let $g \in G$. Define $\mathcal{U}_g^L : \mathbb{C}_K^G \to \mathbb{C}^G$ by

$$\left(\mathcal{U}_g^L f\right)(h) = f(g^{-1}h), \tag{7.11}$$

for all $f \in \mathbb{C}_K^G$ and $g, h \in G$.

Theorem 7.3 *Let G be a group and $F : G \to \mathbb{C}$ be a function of positive type. Define $K(g,h) = F(h^{-1}g)$ for all $g, h \in G$. Then $\mathcal{U}^L : G \to \mathcal{B}(\mathbb{C}_K^G) : g \mapsto \mathcal{U}_g^L$ is a unitary representation of G in $\mathbb{C}_K^G$. Moreover, $F(g) = (F, \mathcal{U}_g^L F)_{\mathbb{C}_K^G}$ for all $g \in G$.*

Proof:
Let $f \in \mathbb{C}_K^G$. The unitarity of $\mathcal{U}_h^L$ easily follows from Lemma 2.2 and the fact

that $K(hg_1, hg_2) = K(g_1, g_2)$ for all $g_1, g_2, h \in G$:

$$\begin{aligned}
\|f\|^2_{\mathbb{C}^{\mathbb{E}}_K} &= \sup_{l\in\mathbb{N},\, \alpha_j\in\mathbb{C},\, g_j\in G} \left\{ \Big| \sum_{j=1}^{l} \alpha_j \overline{f(g_j)} \Big|^2 \Big(\sum_{k,j=1}^{l} \overline{\alpha_k}\alpha_j K(g_k, g_j) \Big)^{-1} \right\}. \\
&= \sup_{l\in\mathbb{N},\, \alpha_j\in\mathbb{C},\, g_j\in G} \left\{ \Big| \sum_{j=1}^{l} \alpha_j \overline{f(h^{-1}g_j)} \Big|^2 \Big(\sum_{k,j=1}^{l} \overline{\alpha_k}\alpha_j K(h^{-1}g_k, h^{-1}g_j) \Big)^{-1} \right\}. \\
&= \sup_{l\in\mathbb{N},\, \alpha_j\in\mathbb{C},\, g_j\in G} \left\{ \Big| \sum_{j=1}^{l} \alpha_j \overline{f(h^{-1}g_j)} \Big|^2 \Big(\sum_{k,j=1}^{l} \overline{\alpha_k}\alpha_j K(g_k, g_j) \Big)^{-1} \right\},
\end{aligned}$$

for all $h \in G$. Therefore $\mathcal{U}^L_h f \in \mathbb{C}^G_K$ and $\|\mathcal{U}^L_h f\|_{\mathbb{C}^G_K} = \|f\|_{\mathbb{C}^G_K}$. Moreover

$$(F, \mathcal{U}^L_g F)_{\mathbb{C}^G_K} = (K_e, \mathcal{U}^L_g K_e)_{\mathbb{C}^G_K} = K_e(g^{-1}) = F(g),$$

for all $g \in G$. □

The representation $\mathcal{U}^L$ is called the **left regular representation**. Note the intertwining relation $\mathcal{U}^L_g W_\psi = W_\psi \mathcal{U}_g$.

Corollary 7.4 *Let G be a group and $F: G \to \mathbb{C}$ a function of positive type. Define the function $K : G \times G \to \mathbb{C}$ of positive type by $K(g, h) = F(h^{-1}g)$ for all $g, h \in G$. Let $\mathcal{H}$ be a Hilbert space, which is unitarily equivalent to $\mathbb{C}^G_K$. Then there exist a $\psi \in \mathcal{H}$ and a unitary representation $\mathcal{U}$ of G in $\mathcal{H}$ such that*

$$F(g) = (\mathcal{U}_g \psi, \psi)_{\mathcal{H}}, \tag{7.12}$$

for all $g \in G$.

Proof:
By assumption, there exist a unitary map $\mathcal{T}$ from $\mathcal{H}$ to $\mathbb{C}^G_K$. The element $\psi = \mathcal{T}^{-1}F$ and the unitary representation defined by $\mathcal{U}_g = \mathcal{T}^{-1}\mathcal{U}^L_g \mathcal{T}$ for all $g \in G$ do the trick. □

7.3 Topological conditions

Some elementary topological conditions which can be posed on the representation $\mathcal{R}$, are straightforwardly transferred to the wavelet transform.

Let $\mathcal{R}$ be a **bounded representation**, i.e. a representation for which the map $g \mapsto \|\mathcal{R}_g\|$ is a bounded function. Define $\|\mathcal{R}\| = \sup_{g\in G} \|\mathcal{R}_g\|$. Let $f \in \overline{\langle V_\psi \rangle}$. Then,

$$\big|\big(W_\psi f\big)(g)\big| = |(\mathcal{R}_g \psi, f)_{\mathcal{H}}| = \|\mathcal{R}_g \psi\|_{\mathcal{H}} \|f\|_{\mathcal{H}} \le \|\mathcal{R}\| \|\psi\|_{\mathcal{H}} \|f\|_{\mathcal{H}}, \tag{7.13}$$

for all $g \in G$. Hence, the wavelet transform $W_\psi f$ for an arbitrary $f \in \mathcal{H}$ is bounded on G. Also the reproducing kernel is bounded on $G \times G$. A unitary representation is an example of a bounded representation.

Assume G is a **topological group**, i.e. a group on which a topology is defined, such that the group operations, multiplication and inversion, are continuous. Let $\mathcal{R}$ be a **continuous representation**, i.e. a representation for which $\mathcal{R}_g f \to \mathcal{R}_h f$ whenever $g \to h$, for all $h \in G$ and $f \in \mathcal{H}$. Let $f \in \mathcal{H}$. Then $W_\psi f$ is a continuous function on G. Indeed, if $g \to h$ then

$$|\big(W_\psi f\big)(g) - \big(W_\psi f\big)(h)| = |((\mathcal{R}_g - \mathcal{R}_h)\psi, f)_{\mathcal{H}}| \leq \|(\mathcal{R}_g - \mathcal{R}_h)\psi\|_{\mathcal{H}} \|f\|_{\mathcal{H}} \to 0. \quad (7.14)$$

Also the reproducing kernel is a continuous function on $G \times G$.

8 Cyclic representations

Because of Theorem 7.2 the cyclic representations have our special attention. But it is not often straightforward to see whether a representation is cyclic or not. And even if so, one still has to find a cyclic vector. In this section we pose provide criteria for cyclic vectors. Moreover, we work out an example which deals with diffusion on a sphere. For this case we managed, to find an interesting cyclic vector, with the aid of Theorem 8.5.

8.1 Orthogonal sums of functional Hilbert spaces

In this section we analyze the orthogonal direct sum of functional Hilbert spaces. The sequel is based on a part of the article by Aronszajn [Ar, part I, 6]. Theorem 8.1 and Corollary 8.2 are due to Aronszajn.

Let $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$ be a sequence of Hilbert spaces. Then define the **orthogonal direct sum** of the sequence as the Hilbert space

$$\bigoplus_{n=1}^{\infty}\mathcal{H}_n = \left\{ a \in \prod_{n=1}^{\infty}\mathcal{H}_n \mid \sum_{n=1}^{\infty}\|a_n\|^2_{\mathcal{H}_n} < \infty \right\}, \tag{8.1}$$

with the inner product

$$(a,b)_{\oplus} = \sum_{n=1}^{\infty}(a_n,b_n)_{\mathcal{H}_n}. \tag{8.2}$$

Theorem 8.1 *Let $K : \mathbb{E}\times\mathbb{E}\to\mathbb{C}$ and $L : \mathbb{E}\times\mathbb{E}\to\mathbb{C}$ be two functions of positive type. Then*

$$\mathbb{C}^{\mathbb{E}}_{K+L} = \{f_1 + f_2 \mid f_1 \in \mathbb{C}^{\mathbb{E}}_K,\ f_2 \in \mathbb{C}^{\mathbb{E}}_L\} = \mathbb{C}^{\mathbb{E}}_K + \mathbb{C}^{\mathbb{E}}_L. \tag{8.3}$$

Furthermore, if $\mathbb{C}^{\mathbb{E}}_K \cap \mathbb{C}^{\mathbb{E}}_L = \{0\}$ then

$$\|f_1+f_2\|^2_{\mathbb{C}^{\mathbb{E}}_{K+L}} = \|f_1\|^2_{\mathbb{C}^{\mathbb{E}}_K} + \|f_2\|^2_{\mathbb{C}^{\mathbb{E}}_L}. \tag{8.4}$$

Hence it follows that $\mathbb{C}^{\mathbb{E}}_K \perp \mathbb{C}^{\mathbb{E}}_L$ in $\mathbb{C}^{\mathbb{E}}_{K+L}$ and $\|f\|_{\mathbb{C}^{\mathbb{E}}_K} = \|f\|_{\mathbb{C}^{\mathbb{E}}_{K+L}}$ for all $f \in \mathbb{C}^{\mathbb{E}}_K$.

Define the Hilbert space $\mathbb{C}^{\mathbb{E}}_K \oplus \mathbb{C}^{\mathbb{E}}_L$ as the Cartesian product $\mathbb{C}^{\mathbb{E}}_K \times \mathbb{C}^{\mathbb{E}}_L$ with the inner product defined by

$$((f_1,g_1),(f_2,g_2))_{\oplus} = (f_1,f_2)_{\mathbb{C}^{\mathbb{E}}_K} + (g_1,g_2)_{\mathbb{C}^{\mathbb{E}}_L}, \tag{8.5}$$

for all pairs $(f_1,g_1),(f_2,g_2) \in \mathbb{C}^{\mathbb{E}}_K \oplus \mathbb{C}^{\mathbb{E}}_L$. It is obvious that $\mathbb{C}^{\mathbb{E}}_K \oplus \mathbb{C}^{\mathbb{E}}_L$ with the above inner product is a Hilbert space.

The following theorem is a direct consequence of Theorem 8.1.

Corollary 8.2 *Assume* $\mathbb{C}_K^{\mathbb{E}} \cap \mathbb{C}_L^{\mathbb{E}} = \{0\}$*. Then the map defined by*

$$(f_1, f_2) \mapsto f_1 + f_2, \tag{8.6}$$

is a unitary map from $\mathbb{C}_K^{\mathbb{E}} \oplus \mathbb{C}_L^{\mathbb{E}}$ *onto* $\mathbb{C}_{K+L}^{\mathbb{E}}$.

This idea is easily generalized to an infinite sum of functions of positive type. Define the Hilbert space $\bigoplus_{n=1}^{\infty} \mathbb{C}_{K_n}^{\mathbb{E}}$ as in (8.1) and (8.2).

Let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of functions of positive type on a set $\mathbb{E}$ such that

$$\sum_{n=1}^{\infty} K_n(x,x) < \infty, \tag{8.7}$$

for all $x \in \mathbb{E}$. Then by the estimate

$$\begin{aligned} |K_n(x,y)| &= |(K_{n;x}, K_{n;y})_{\mathbb{C}_{K_n}^{\mathbb{E}}}| \le \|K_{n;x}\|_{\mathbb{C}_{K_n}^{\mathbb{E}}} \|K_{n;y}\|_{\mathbb{C}_{K_n}^{\mathbb{E}}} \\ &\le \frac{1}{2}\|K_{n;x}\|^2_{\mathbb{C}_{K_n}^{\mathbb{E}}} + \frac{1}{2}\|K_{n;y}\|^2_{\mathbb{C}_{K_n}^{\mathbb{E}}} = \frac{1}{2}K_n(x,x) + \frac{1}{2}K_n(y,y) \end{aligned} \tag{8.8}$$

for all $x, y \in \mathbb{E}$ and $n \in \mathbb{N}$, the sum

$$K_{\oplus}(x,y) := \sum_{n=1}^{\infty} K_n(x,y), \tag{8.9}$$

converges absolutely on $\mathbb{E} \times \mathbb{E}$. As a result $K_{\oplus}$ is a function of positive type, since K_n is a function of positive type for all $n \in \mathbb{N}$. Moreover, (8.7) implies that $(K_{1;x}, K_{2;x}, \ldots) \in \oplus_{n=1}^{\infty}\mathbb{C}_{K_n}^{\mathbb{E}}$ since

$$\sum_{n=0}^{N} \|K_{n;x}\|^2_{\mathbb{C}_{K_n}^{\mathbb{E}}} = \sum_{n=0}^{N} K_n(x,x) \le \sum_{n=0}^{\infty} K_n(x,x) < \infty \tag{8.10}$$

for all $N \in \mathbb{N}$.

Furthermore, the sequence $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely for all $(f_1, f_2, \ldots) \in \bigoplus_{n=1}^{\infty} \mathbb{C}_{K_n}^{\mathbb{E}}$ and $x \in \mathbb{E}$. Indeed, let $f = (f_1, f_2, \ldots) \in \bigoplus_{n=1}^{\infty} \mathbb{C}_{K_n}^{\mathbb{E}}$ and $x \in \mathbb{E}$, then

$$\begin{aligned} \sum_{n=1}^{N} |f_n(x)| &= \sum_{n=1}^{N} |(K_{n;x}, f_n)_{\mathbb{C}_K^{\mathbb{E}}}| \le \frac{1}{2}\sum_{n=1}^{N} \left\{ \|K_{n;x}\|^2_{\mathbb{C}_K^{\mathbb{E}}} + \|f_n\|^2_{\mathbb{C}_K^{\mathbb{E}}} \right\} \\ &\le \frac{1}{2}\sum_{n=1}^{\infty} \left\{ K_n(x,x)_{\mathbb{C}_K^{\mathbb{E}}} + \|f_n\|^2_{\mathbb{C}_K^{\mathbb{E}}} \right\} < \infty, \end{aligned} \tag{8.11}$$

for all $N \in \mathbb{N}$. Hence $\sum_{n=1}^{\infty} |f_n(x)| < \infty$.

Now we are ready for the following theorem.

Theorem 8.3 *Let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of functions of positive type on a set $\mathbb{E}$ such that*

$$\sum_{n=1}^{\infty} K_n(x,x) < \infty, \tag{8.12}$$

for all $x \in \mathbb{E}$. Define for $x \in \mathbb{E}$ the vector $\psi_x \in \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{K_n}$ as

$$\psi_x = (K_{1;x}, K_{2;x}, \ldots). \tag{8.13}$$

Then the map $\Phi : \overline{\langle\{\psi_x \mid x \in \mathbb{E}\rangle} \to \mathbb{C}^{\mathbb{E}}_{\sum_{n=1}^{\infty} K_n}$ defined by

$$\big[\Phi(f_1, f_2, \ldots)\big](x) = \sum_{n=1}^{\infty} f_n(x), \tag{8.14}$$

is unitary.

Proof:
Since $\big(\Phi f\big)(x) = (\psi_x, f)_{\oplus}$ for all $f \in \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{K_n}$ and by Theorem 2.3 it follows that Φ is a unitary mapping onto the functional Hilbert space of function on the set $\mathbb{E}$ with reproducing kernel

$$(\psi_x, \psi_y)_{\oplus} = \sum_{n=1}^{\infty} (K_{n;x}, K_{n;y})_{\mathbb{C}^{\mathbb{E}}_{K_n}} = \sum_{n=1}^{\infty} K_n(x,y)$$

for all $x, y \in \mathbb{E}$. Hence the statement follows. □

As in the case of the sum of two functional Hilbert spaces we search for a condition such that $\overline{\langle\{\psi_x \mid x \in \mathbb{E}\rangle} = \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{K_n}$. In that case Φ is a unitary map from $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{K_n}$ onto $\mathbb{C}^{\mathbb{E}}_{\sum_{n=1}^{\infty} K_n}$.

Theorem 8.4 *The following statements are equivalent*

1. $\overline{\langle\{\psi_x \mid x \in \mathbb{E}\rangle} = \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{K_n}$
2. $\forall x \in \mathbb{E}\, [\sum_{m=1}^{\infty} f_m(x) = 0] \implies f = 0.$
3. $\mathbb{C}^{\mathbb{E}}_{K_n} \cap \mathbb{C}^{\mathbb{E}}_{\sum_{m=1, m\neq n}^{\infty} K_n} = \{0\}$, *for all $n \in \mathbb{N}$.*

Proof:

1 ⇔ 2 This statement easily follows from $(\psi_x, f)_{\oplus} = 0 \Leftrightarrow \sum_{n=1}^{\infty} f_n(x) = 0$, for all $f = (f_1, f_2, \ldots) \in \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{K_n}$.

2 ⇒ 3.
Let $n \in \mathbb{N}$ and $f \in \mathbb{C}^{\mathbb{E}}_{K_n} \cap \mathbb{C}^{\mathbb{E}}_{\sum_{m=1, m\neq n}^{\infty} K_m}$. Since the map Φ is surjective in

Theorem 8.3, one can write $f = f_n = \sum_{m=1,m\neq n}^{\infty} f_m$, where $f_m \in \mathbb{C}_{K_m}^{\mathbb{E}}$ for all $m \in \mathbb{N}$. Define the element $g \in \bigoplus_{m=1}^{\infty} \mathbb{C}_{K_m}^{\mathbb{E}}$ by

$$g = (f_1, f_2, \ldots, f_{n-1}, -f_n, f_{n+1}, \ldots)$$

Then obviously

$$\forall x \in \mathbb{E} \left[\sum_{m=1}^{\infty} (g_m, K_{m;x}) = 0 \right],$$

hence $g = 0$ and $f = 0$.

$3 \Leftarrow 2$.

First we mention that $\mathbb{C}_{K_n}^{\mathbb{E}} \perp \mathbb{C}_{K_m}^{\mathbb{E}}$ for $m \neq n$ in $\mathbb{C}_{\sum_{n=1}^{\infty} K_n}^{\mathbb{E}}$. If $m \neq n$, $f_1 \in \mathbb{C}_{K_n}^{\mathbb{E}}$ and $f_2 \in \mathbb{C}_{K_2}^{\mathbb{E}}$, then $f_2 \in \mathbb{C}_{\sum_{l=1,l\neq n} K_l}^{\mathbb{E}}$ by Theorem 8.1. By assumption and Theorem 8.1 it follows that $(f_1, f_2)_{\mathbb{C}_{K_\oplus}^{\mathbb{E}}} = 0$. Hence $\mathbb{C}_{K_n}^{\mathbb{E}} \perp \mathbb{C}_{K_m}^{\mathbb{E}}$.

Secondly, let $(f_1, f_2, \ldots) \in \bigoplus_{n=1}^{\infty} \mathbb{C}_{K_n}^{\mathbb{E}}$ satisfy $\sum_{n=1}^{\infty} f_n(x) = 0$, for all $x \in \mathbb{E}$. Then

$$\sum_{n=1}^{\infty} \|f_n\|_{\mathbb{C}_{K_\oplus}^{\mathbb{E}}}^2 = \sum_{n=1}^{\infty} \|f_n\|_{\mathbb{C}_{K_n}^{\mathbb{E}}}^2 < \infty,$$

so $\sum_{n=1}^{\infty} f_n$ exists. Since $\mathbb{C}_{K_\oplus}^{\mathbb{E}}$ s a functional Hilbert space it follows that

$$\Big(\sum_{n=1}^{\infty} f_n\Big)(x) = \sum_{n=1}^{\infty} f_n(x) = 0$$

for all $x \in \mathbb{E}$. □

Now no longer assume that $\sum_{n=1}^{\infty} K_n(x, x) < \infty$. In this case we need a mollifying sequence as in the following theorem.

Theorem 8.5 *Let $\mathbb{E}$ be a set. Let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of functions of positive type on $\mathbb{E}$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ a sequence such that*

1. $\forall\, n \in \mathbb{N} : \quad \lambda_n > 0$

2. $\sup_n \lambda_n < \infty$.

Then $\bigoplus_{n=1}^{\infty} \mathbb{C}_{\lambda_n K_n}^{\mathbb{E}}$ is a dense subspace of $\bigoplus_{n=1}^{\infty} \mathbb{C}_{K_n}^{\mathbb{E}}$. If in addition the sequences satisfy the conditions

3. $\forall\, x \in \mathbb{E} : \quad \sum_{n=1}^{\infty} \lambda_n K_n(x, x) < \infty$

4. $\forall\, n \in \mathbb{N} : \quad \mathbb{C}_{\lambda_n K_n}^{\mathbb{E}} \cap \mathbb{C}_{\sum_{m=1,m\neq n}^{\infty} \lambda_m K_m}^{\mathbb{E}} = \{0\}$.

Then

$$\psi_x = (\lambda_1 K_{1;x}, \lambda_2 K_{2;x}, \ldots) \in \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{K_n} \tag{8.15}$$

for all $x \in \mathbb{E}$*. Furthermore*

$$\overline{\langle\{\psi_x \mid x \in \mathbb{E}\}\rangle} = \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{K_n}. \tag{8.16}$$

Proof:

Assume the first two conditions are satisfied.

First, we remark that from the definition it straightforwardly follows that $\mathbb{C}^{\mathbb{E}}_{K_n} = \mathbb{C}^{\mathbb{E}}_{\lambda_n K_n}$ as a set and $(f,g)_{\mathbb{C}^{\mathbb{E}}_{K_n}} = \lambda_n (f,g)_{\mathbb{C}^{\mathbb{E}}_{\lambda_n K_n}}$ for all $n \in \mathbb{N}$ and $f, g \in \mathbb{C}^{\mathbb{E}}_{K_n}$.

Secondly, write $\|\cdot\|_{\lambda\oplus}$ for the norm of $\bigoplus_{n=1}^{\infty} \mathbb{C}^{G}_{\lambda_n K_n}$. Let $f = (f_1, f_2, \ldots) \in \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{\lambda_n K_n}$. Then, it follows by

$$\begin{aligned} \sum_{n=1}^{\infty} \|f_n\|^2_{\oplus} &= \sum_{n=1}^{\infty} (f_n, f_n)_{\mathbb{C}^{\mathbb{E}}_{K_n}} = \sum_{n=1}^{\infty} \lambda_n (f_n, f_n)_{\mathbb{C}^{\mathbb{E}}_{\lambda_n K_n}} \\ &\leq \sup_n \lambda_n \sum_{n=1}^{\infty} \|f_n\|^2_{\mathbb{C}^{\mathbb{E}}_{\lambda K_n}} = \sup_n \lambda_n \|f\|_{\lambda\oplus}, \end{aligned}$$

that $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{\lambda_n K_n} \subset \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{K_n}$.

Finally, the set

$$\{f \in \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{K_n} \mid \exists N \in \mathbb{N}\ \forall n > N\ [f_n = 0]\}$$

is dense in $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{K_n}$ and contained in $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{\lambda_n K_n}$. Hence $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{\lambda_n K_n}$ is dense in $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{K_n}$.

Now assume in addition that the last two condition are satisfied.

Because $\psi_x \in \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{\lambda_n K_n}$ by (8.10) we have in particular $\psi_x \in \bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{K_n}$ for all $x \in \mathbb{E}$. Then by Theorem 8.3 and Theorem 8.4, the set $\langle\{\psi_x \mid x \in \mathbb{E}\}$ is dense in $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{\lambda_n K_n}$. Moreover, because $\|\cdot\|_{\oplus} \leq \sup_n \lambda_n \|\cdot\|_{\lambda\oplus}$ and $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{\lambda_n K_n}$ is dense in $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{K_n}$, it follows that $\langle\{\psi_x \mid x \in \mathbb{E}\}\rangle$ is dense in $\bigoplus_{n=1}^{\infty} \mathbb{C}^{\mathbb{E}}_{K_n}$ □

It straightforwardly follows that

$$(f, \psi_x)_{\mathcal{H}} = \sum_{n=1}^{\infty} \lambda_n f_n(x), \tag{8.17}$$

for all $x \in \mathbb{E}$, which will turn out to be a useful identity.

8.2 An example: diffusion on a sphere

We now deal with an example concerning the problem of diffusion on a sphere. For a detailed discussion about some statements which we do not prove, see for example [Mu, Ch. 3].

Let S^{q-1} for $q \geq 3$ be the unit sphere in $\mathbb{R}^q$ and $G = SO(q)$ the special orthogonal matrix group. Let the group $SO(q)$ act on S^{q-1} in the usual way, $(A, x) = Ax$. The group acts **transitively** on S^{q-1}, i.e. for all $x, y \in S^{q-1}$ there exists an $A \in SO(q)$ such that $x = Ay$.

Let $\mathcal{H}$ be the Hilbert space $\mathbb{L}_2(S^{q-1})$. Define the representation $\mathcal{U} : SO(q) \to \mathcal{B}(\mathbb{L}_2(S^{q-1}))$: $A \mapsto \mathcal{U}_A$ where $\mathcal{U}_A$ is defined by

$$\big(\mathcal{U}_A f\big)(x) = f(A^{-1}x), \tag{8.18}$$

for all $A \in SO(q)$, $f \in \mathbb{L}_2(S^{q-1})$ and almost all $x \in S^{q-1}$.

First, it is well-known that the space $\mathbb{L}_2(S^{q-1})$ decomposes in $\mathbb{L}_2(S^{q-1}) \cong \bigoplus_{n=1}^{\infty} \mathbb{C}_{K_n}^{S^{q-1}}$ where $\mathbb{C}_{K_n}^{S^{q-1}}$ is the functional Hilbert space of all spherical harmonic polynomials of order n. The reproducing kernel is given by

$$K_{n;x} = \frac{q+2n-2}{q-2} C_N^{q/2-1}((\cdot, x)_2), \tag{8.19}$$

for all $x \in S^{q-1}$, where $C_N^{q/2-1}$ are the Gegenbauer polynomials. Since

$$\|K_{n;x}\|^2_{\mathbb{L}_2(S^q)} = \frac{q+2n-2}{q-2} C_N^{q/2-1}((x, x)_2) = \frac{q+2n-2}{q-2} C_N^{q/2-1}(1) = \frac{q+2n-2}{q-2}, \tag{8.20}$$

for all $x \in S^{q-1}$, it is straightforward to see that this orthogonal sum satisfies the condition of Theorem 8.5 for some sequence $\{\lambda_n\}_{n\in\mathbb{N}}$.

Secondly, we have to choose a sequence $\{\lambda_n\}_{\ n\in\mathbb{N}}$. Let $t > 0$. Then it is obvious that $\lambda_n = \mathrm{e}^{-tn(n+q-2)}$ defines a sequence that satisfies the conditions in Theorem 8.5. Define for all $x \in S^{q-1}$

$$\psi_x = \sum_{n=1}^{\infty} \mathrm{e}^{-tn(n+q-2)} K_{n;x}. \tag{8.21}$$

Then $\psi_x \in \mathbb{L}_2(S^{q-1})$ by Theorem 8.5. Fix $y \in S^{q-1}$. Then,

$$\mathcal{U}_A \psi_y = \psi_{Ay}, \tag{8.22}$$

for all $A \in SO(q)$ by (8.19). Finally, by the transitivity of the action of the group we get by Theorem 8.5

$$\overline{\langle\{\mathcal{U}_A \psi_y \mid A \in SO(q)\}\rangle} = \overline{\langle\{\psi_x \mid x \in S^{q-1}\}\rangle} = \mathbb{L}_2(S^{q-1}). \tag{8.23}$$

Hence ψ_y is a cyclic vector for all $y \in S^{q-1}$ and $\mathcal{U}$ is a cyclic representation.

It is straightforward to see that the stabilizer group y of $H_y = \{A \in SO(q) \mid Ay = y\}$ can be identified with $SO(q-1)$. Moreover, if $Ay = y$ the $\mathcal{U}_A \psi_y = \psi_y$. The quotient space $SO(q)/H_y$ is homeomorphic to S^{q-1}. Moreover, let $t : S^{q-1} \to SO(q)$ be a map such that $t(b)y = b$. Such a map exist by transitivity of the action of $SO(q)$ on S^{q-1}.

Theorem 8.6 *Define the function* $K : S^{q-1} \times S^{q-1} \to \mathbb{C}$ *of positive type by*

$$K(b, b') = (\mathcal{U}_{t(b)}\psi_y, \mathcal{U}_{t(b')}\psi_y)_{\mathcal{H}}. \tag{8.24}$$

Then the frame transform $W_{\psi_a} : \mathbb{L}_2(S^{q-1}) \to \mathbb{C}_K^{S^{q-1}}$ *defined by*

$$\big(W_\psi f\big)(b) = (\mathcal{U}_{t(b)}\psi_y, f)_{\mathbb{L}_2(S^{q-1})} = (\psi_b, f)_{\mathbb{L}_2(S^{q-1})}, \tag{8.25}$$

for all $f \in \mathbb{L}_2(S^{q-1})$ *and* $b \in S^{q-1}$ *is a unitary map.*

The choice $\lambda_n = \mathrm{e}^{-tn(n+q-2)}$ was not without reason. The spherical harmonic polynomials of order n are the eigenvectors of the Laplace-Beltrami operator $\triangle_S$ with eigenvalue $n(n+q-2)$. Therefore the functions of the form $(t, x) \mapsto \mathrm{e}^{-tn(n+q-2)}p_n(x)$ with p_n a spherical harmonic polynomial of order n are solutions of the evolution equation

$$u_t = -\triangle_S u. \tag{8.26}$$

Let $f \in \mathbb{L}_2(S)$. With (8.17) it follows that

$$\big(W_{\psi_y} f\big)(b) = (\psi_b, f)_{\mathbb{L}_2(S^{q-1})} = \sum_{n=1}^{\infty} \mathrm{e}^{-tn(n+q-2)}\big(P_n f\big)(b), \tag{8.27}$$

where P_n stands for the projection operator corresponding to the space of all spherical harmonic polynomials of order n. So we could interpret the above wavelet transform as the solution at time t and point $t(b)y = b$ of the evolution equation (8.26) with initial condition $u(0, \cdot) = f(\cdot)$.

9 Irreducible representations

From now on we do pose a topological condition on G. Recall that a **topological group** is a group on which a topology is defined, such that the group operations, multiplication and inversion, are continuous. We always assume the topology to be Hausdorff. Moreover, we always assume the group G to be a **locally compact group**, i.e. a topological group, in which every group element has a compact neighborhood.

It is well-known that every locally compact group G has a **Haar measure**, which we denote by μ_G. A Haar measure on G is a Radon measure on G which is left invariant, i.e. $\mu_G(gE) = \mu_G(E)$ for all $g \in G$ and Borel sets E. It is unique op to a positive constant.

We call a representation $\mathcal{R}$ of a group G in a Hilbert space $\mathcal{H}$ **irreducible** if the only closed subspaces of $\mathcal{H}$ which are invariant under all $\mathcal{R}_g$ for all $g \in G$ are $\mathcal{H}$ and $\{0\}$. An irreducible representation is in particular cyclic and every nonzero vector is cyclic. Indeed, for every nonzero $\psi \in \mathcal{H}$ the set $\overline{\langle V_\psi \rangle}$ is a subspace which is invariant under all $\mathcal{R}_g$ with $g \in G$ and it is not empty, so $\overline{\langle V_\psi \rangle} = \mathcal{H}$.

The representation $\mathcal{R}$ is called **square integrable** if there exist a $\psi \in \mathcal{H}$ with $\psi \neq 0$ and

$$C_\psi := \frac{1}{(\psi, \psi)_\mathcal{H}} \int_G \left|(\mathcal{R}_g \psi, \psi)_\mathcal{H}\right|^2 \mathrm{d}\mu_G(g) < \infty. \tag{9.1}$$

If the group representation is unitary, irreducible and square integrable, then the functional Hilbert space will always be a closed subspace of $\mathbb{L}_2(G)$, whenever the wavelet $\psi \in \mathcal{H}$ satisfies (9.1). This was first shown by Grossman, Morlet and Paul [GMP] in 1985. In this report we will give a new proof of this theorem. For our proof we need an extension of Schur's lemma, which is presented in Appendix A. Moreover, we need a lemma which is valid for all bounded representations. Hence, let $\mathcal{R}$ be a bounded representation of a group G in a Hilbert space $\mathcal{H}$. Let $\psi \in \mathcal{H}$. First define the linear map $\mathcal{W}_\psi$ as

$$\mathcal{W}_\psi = W_\psi|_\mathcal{D}\,, \tag{9.2}$$

where $\mathcal{D} = \{f \in \mathcal{H} \mid W_\psi f \in \mathbb{L}_2(G)\}$.

Lemma 9.1 *The wavelet transform $\mathcal{W}_\psi : \mathcal{D} \to \mathbb{L}_2(G)$ is a closed operator.*

Proof:
Let $f_1, f_2, \ldots \in D$, $f \in \mathcal{H}$ and assume $f_n \to f$ in $\mathcal{H}$ and $\mathcal{W}_\psi f_n \to \Phi$, for some $\Phi \in \mathbb{L}_2(G)$. Then we have to show that $f \in \mathcal{D}$ and $\mathcal{W}_\psi f = \Phi$. The group G is locally compact, therefore it is sufficient to show that for *any* compact $\Omega \subset G$

$$\int\limits_\Omega |W_\psi f - \Phi|^2 \, \mathrm{d}\mu_\mathrm{G} = 0 \ ,$$

to conclude that $W_\psi f = \Phi$.
Note that by boundedness of the representation

$$\left|\left(W_\psi f\right)(g) - \left(W_\psi f_n\right)(g)\right| = |(\mathcal{R}_g \psi, f - f_n)_\mathcal{H}| \leq \|\mathcal{R}\| \|\psi\|_\mathcal{H} \|f - f_n\|_\mathcal{H},$$

for all $g \in G$ and $n \in \mathbb{N}$.
Now the statement follows from

$$\int_{\Omega} |W_\psi f - \Phi|^2 \ \mathrm{d}\mu_G(g)$$

$$\leq 2 \int_{\Omega} |W_\psi f - \mathcal{W}_\psi f_n|^2 \,\mathrm{d}\mu_G + 2 \int_{\Omega} |\Phi - \mathcal{W}_\psi f_n|^2 \,\mathrm{d}\mu_G$$

$$\leq 2\mu(\Omega) \sup_{g\in G} \big|\big(W_\psi f\big)(g) - \big(\mathcal{W}_\psi f_n\big)(g)\big|^2 + 2 \int_{\Omega} |\Phi - \mathcal{W}_\psi f_n|^2 \,\mathrm{d}\mu_G$$

$$\leq 2\mu(\Omega)\|\mathcal{R}\|^2\|\psi\|^2_{\mathcal{H}}\|f_n - f\|^2_{\mathcal{H}} \ + \|\Phi - \mathcal{W}_\psi f_n\|^2$$

for all $n \in \mathbb{N}$. As $f_n \to f$ and $\mathcal{W}_\psi f_n \to \Phi$ it follows that $W_\psi f = \Phi$ on Ω. Therefore $f \in \mathcal{D}$ and $\mathcal{W}_\psi f = \Phi$. □

The left regular representation $\mathcal{L}$ of G on $\mathbb{L}_2(G)$ is defined by

$$\mathcal{L}_h f(g) = f(h^{-1}g), \tag{9.3}$$

for all $h \in G$, $f \in \mathbb{L}_2(G)$ and almost every $g \in G$.
We now prove a theorem by Morlet, Grossmann and Paul.

Theorem 9.2 *Let $\mathcal{U}$ be an irreducible, unitary and square integrable representation of a locally compact group G on a Hilbert space $\mathcal{H}$. Let $\psi \in \mathcal{H}$ such that (9.1) holds. Then $W_\psi f \in \mathbb{L}_2(G)$ for all $f \in \mathcal{H}$ and the wavelet transform is a linear isometry (up to a constant) from the Hilbert space $\mathcal{H}$ onto a closed subspace $\mathbb{C}^G_K$ of $\mathbb{L}_2(G, \mathrm{d}\mu)$:*

$$\|\mathcal{W}_\psi f\|^2_{\mathbb{L}_2(G)} = C_\psi \|f\|^2_{\mathcal{H}}. \tag{9.4}$$

Here, the space $\mathbb{C}^G_K$ is the functional Hilbert space with reproducing kernel

$$K_\psi(g, g') = \frac{1}{C_\psi}(\mathcal{U}_g\psi, \mathcal{U}_{g'}\psi), \tag{9.5}$$

for all $g, g' \in G$.

Proof:
The domain $\mathcal{D}$ of operator $\mathcal{W}_\psi : \mathcal{D} \to \mathbb{L}_2(G)$ is by definition the set of all $f \in \mathcal{H}$ for which $W_\psi f \in \mathbb{L}_2(G)$. By assumption $\psi \in \mathcal{D}$. Moreover, it follows by the

left-invariance of $\mathrm{d}\mu_G$ that the span $\mathcal{S}_\psi = \langle\{\mathcal{U}_g\psi \mid g \in G\}\rangle$ of the orbit of ψ, is a subspace of $\mathcal{D}$, since for any $\eta = \mathcal{U}_h\psi$, we have

$$\begin{aligned}
\int_G \left|\left(W_\psi\eta\right)(g)\right|^2 \,\mathrm{d}\mu_G(g) &= \int_G |(\mathcal{U}_g\psi,\mathcal{U}_h\psi)|^2 \,\mathrm{d}\mu_G(g) \\
&= \int_G |(\mathcal{U}_{h^{-1}g}\psi,\psi)|^2 \;\mathrm{d}\mu_G(g) \\
&= \int_G |(\mathcal{U}_g\psi,\psi)|^2 \;\mathrm{d}\mu_G(g) \\
&= C_\psi|(\psi,\psi)|^2 = C_\psi|(\mathcal{U}_h\psi,\mathcal{U}_h\psi)|^2 < \infty.
\end{aligned}$$

Obviously $\mathcal{S}_\psi$ is invariant under $\mathcal{U}$ and since $\mathcal{U}$ was assumed to be irreducible, this space is dense in $\mathcal{H}$. By Lemma 9.1 operator $\mathcal{W}_\psi$ is closed, since a unitary representation is bounded. So, $\mathcal{W}_\psi$ is a closed densely defined operator and therefore the operator $\mathcal{W}_\psi^*\mathcal{W}_\psi$ is self-adjoint, by a theorem of J. von Neumann (see [Y, Theorem VII.3.2]).

It is obvious that

$$\left(\mathcal{W}_\psi\mathcal{U}_h f\right)(g) = (\mathcal{U}_g\psi,\mathcal{U}_h f)_{\mathcal{H}} = (\mathcal{U}_{h^{-1}}\mathcal{U}_g\psi, f)_{\mathcal{H}} = (\mathcal{U}_{h^{-1}g}\psi, f)_{\mathcal{H}},$$

for all $g, h \in G$ and $f \in \mathcal{H}$. Therefore, if $f \in \mathcal{D}$ then $\mathcal{U}_h f \in \mathcal{D}$ and $\mathcal{W}_\psi\mathcal{U}_h f = \mathcal{L}_h\mathcal{W}_\psi f$. Hence $\mathcal{W}_\psi\mathcal{U}_h = \mathcal{L}_h\mathcal{W}_\psi$. For the adjoint operator the same is true. If $\Phi \in \mathcal{D}(\mathcal{W}_\psi^*)$, $f \in \mathcal{D}(\mathcal{W}_\psi)$ and $h \in G$

$$\begin{aligned}
(\mathcal{L}_h\Phi, \mathcal{W}_\psi f)_{\mathbb{L}_2(G)} &= (\Phi, \mathcal{L}_{h^{-1}}\mathcal{W}_\psi f)_{\mathbb{L}_2(G)} = (\Phi, \mathcal{W}_\psi\mathcal{U}_{h^{-1}} f)_{\mathbb{L}_2(G)} \\
&= (\mathcal{W}_\psi^*\Phi,\mathcal{U}_{h^{-1}} f)_{\mathcal{H}} = (\mathcal{U}_h\mathcal{W}_\psi^*\Phi, f)_{\mathcal{H}}.
\end{aligned}$$

So for all $\Phi \in \mathcal{D}(\mathcal{W}_\psi^*)$ we have $\mathcal{L}_h\Phi \in \mathcal{D}(\mathcal{W}_\psi^*)$ and furthermore $\mathcal{W}_\psi^*\mathcal{L}_h = \mathcal{U}_h\mathcal{W}_\psi^*$. In particular $\mathcal{W}_\psi^*\mathcal{W}_\psi\mathcal{U}_g = \mathcal{U}_g\mathcal{W}_\psi^*\mathcal{W}_\psi$ for all $g \in G$ and $\mathcal{D}(\mathcal{W}_\psi^*\mathcal{W}_\psi)$ is invariant under $\mathcal{U}$.
By the topological version of Schur's lemma, Theorem A.1, it now follows that there is a $c \in \mathbb{C}$ such that $\mathcal{W}_\psi^*\mathcal{W}_\psi = cI$ on $\mathcal{D}(\mathcal{W}_\psi^*\mathcal{W}_\psi)$. But because $\mathcal{W}_\psi^*\mathcal{W}_\psi$ is closed and bounded on $\mathcal{D}(\mathcal{W}_\psi^*\mathcal{W}_\psi)$ we can conclude from the closed graph theorem that $\mathcal{W}_\psi^*\mathcal{W}_\psi = c\,I$ on the entire Hilbert space $\mathcal{H}$. In particular $D(\mathcal{W}_\psi) = \mathcal{H}$. From $\|\mathcal{W}_\psi\psi\|^2 = C_\psi\|\psi\|^2$ it follows that $c = C_\psi$. □

10 A representation of a semi-direct product $S \rtimes T$ on $\mathbb{L}_2(S)$

In this section we will work out the wavelet construction for the special case $\mathcal{H} = \mathbb{L}_2(S, \mu_S)$ with S some locally compact abelian group. Here μ_S is a Haar measure. Given a locally compact group T we will define a natural unitary representation (not necessarily irreducible) of the semi-direct product $S \rtimes T$ on $\mathbb{L}_2(S)$. From this unitary representation a wavelet transform and a corresponding functional Hilbert space can be constructed for a suitable choice of $\psi \in \mathbb{L}_2(S)$.

10.1 Introduction

We first recall the notion of the semi-direct product of two groups. We also mention some elementary topics from harmonic analysis.

Definition 10.1 *Let S and T be groups and let $\tau : T \to Aut(S)$ be a group homomorphism. The* ***semi-direct product*** *$S \rtimes_\tau T$ is defined to be the group with underlying set $S \times T$ and group operation*

$$(s,t)(s',t') = (s\tau(t)s', tt'), \tag{10.1}$$

for all $(s,t), (s',t') \in S \times T$.

From now on we only consider a group G which is a semidirect product $G = (S, +) \rtimes (T, \cdot)$ for some locally compact group T, a locally compact abelian group S and a group homomorphism $\tau : T \to \mathrm{Aut}(S)$ such that

$$(s,t) \mapsto \tau(t)s \tag{10.2}$$

is a continuous map from $S \rtimes T$ onto S. Since S and T are locally compact, G is also locally compact. Note that $\tilde{S} = \{(s, e_2) \in G \mid s \in S\}$ and $\tilde{T} = \{(e_1, t) \in G \mid t \in T\}$ are closed subgroups of G.

Let μ_T, μ_S, μ_G be Haar measures of resp. T, S, G. There exists a relation which relates these Haar measures. To this end, we need the notion of **modular function**.

Definition 10.2 *Let H be a locally compact group and μ a Haar measure on H. Then for each $h \in H$*

$$\mu_h(E) = \mu(Eh), \qquad E \in \text{ Bor } (H), \tag{10.3}$$

defines a Haar measure, where Bor (H) is the set of Borel sets. Because all Haar measures are equal up to a constant, there exists for all $h \in H$ a $\Delta_H(h) > 0$ such that

$$\mu_h = \Delta_H(h)\mu. \tag{10.4}$$

The function $\Delta_H : h \mapsto \Delta_H(h)$ on H is called the ***modular function****. The modular function is a continuous homomorphism from H into $(\mathbb{R}^+, \cdot)$.*

Now $\tilde{T} = \{(e,t) \in G \mid t \in T\}$ is subgroup of G and it has a Haar measure $\mu_{\tilde{T}}$ corresponding to μ_T. Starting from μ_S, μ_T, the Haar measure μ_G can be chosen such that

$$\int_G f(g)\ \mathrm{d}\mu_G(g) = \int_S \left\{ \int_T f(s,t)\rho^{-1}(t)\ \mathrm{d}\mu_T(t) \right\} \mathrm{d}\mu_S(s), \tag{10.5}$$

for all $f \in \mathbb{L}_1(G)$. Furthermore,

$$\int_S f(\tau(t)^{-1}s)\ \mathrm{d}\mu_S(s) = \rho(t) \int_S f(s)\ \mathrm{d}\mu_S(s), \tag{10.6}$$

for all $t \in T$ and $f \in \mathbb{L}_1(S)$. Here

$$\rho(t) = \frac{\Delta_{\tilde{T}}(e,t)}{\Delta_G(e,t)}, \tag{10.7}$$

for all $t \in T$. It follows that ρ is continuous and strictly positive. For further details, we refer to [RS, (8.1.12) and (8.1.10)] .

In the case $S = \mathbb{R}^n$ we simply get $\rho(t) = |\det \tau(t)|$, which can easily be proved by the transform of variables formula.

We define in a natural way a representation of the semi-direct product $S \rtimes T$ in $\mathbb{L}_2(S)$. Define $\mathcal{U} : G \to \mathcal{B}(\mathbb{L}_2(S)) : (s,t) \mapsto \mathcal{U}_{(s,t)}$ as follows

$$\mathcal{U}_{(s,t)} f = \mathcal{T}_s \mathcal{P}_t f, \tag{10.8}$$

where

$$\big(\mathcal{T}_{s_1} f\big)(s_2) = f(s_2 - s_1), \tag{10.9}$$

for all $s_1 \in S$ and almost all $s_2 \in S$, and

$$\big(\mathcal{P}_t f\big)(s) = \rho^{-\frac{1}{2}}(t) f(\tau(t)^{-1}s), \tag{10.10}$$

for all $t \in T$ and almost all $s \in S$.

Note that $\mathcal{P}_t f \in \mathbb{L}_1(S) \cap \mathbb{L}_2(S)$ for all $t \in T$, if $f \in \mathbb{L}_1(S) \cap \mathbb{L}_2(S)$. It is easily verified that $\mathcal{U}$ is a unitary representation. Moreover, we will prove that it is cyclic. In general this representation need not to be irreducible.

10.2 The wavelet transform

We recall that, with the use of the unitary representation $\mathcal{U}$, for any $\psi \in \mathcal{H}$ we now can define the unitary map $W_\psi : \overline{\langle V_\psi \rangle} \to \mathbb{C}_K^G$ as formulated in Theorem 7.1. In this section present another description of the wavelet transform, making use of Fourier transform for abelian groups. Let $f \in \mathbb{L}_2(S)$ and $\hat{S}$ be the dual group of S. Then $\hat{S}$ exists of all continuous homomorphisms of S into the circle group. The define the Fourier transform as

$$\big(\mathcal{F} f\big)(\gamma) = \int_S f(s) \overline{\langle s, \gamma \rangle} \mathrm{d}\mu_S(s), \tag{10.11}$$

for all $\gamma \in \hat{S}$ and $f \in \mathbb{L}_1(S) \cap \mathbb{L}_2(S)$, where $\langle \cdot, \cdot \rangle$ stands for the dual pairing, $\langle s, \gamma \rangle = \gamma(s)$ for all $s \in S$ and $\gamma \in \hat{S}$. This defines, after extension, a unitary map from $\mathbb{L}_2(S)$ onto $\mathbb{L}_2(\hat{S}, \mathrm{d}\mu_{\hat{S}}(\gamma))$ where the left Haar measure $\mu_{\hat{S}}(\gamma)$ related to μ_S. The inversion is given by

$$\big(\mathcal{F}^{-1}F\big)(s) = \int_{\hat{S}} F(\gamma)\langle s, \gamma \rangle \mathrm{d}\mu_{\hat{S}}(\gamma), \tag{10.12}$$

for all $F \in \mathbb{L}_1(\hat{S}) \cap \mathbb{L}_2(\hat{S})$. For a detailed discussion of the Fourier transform on locally compact abelian groups, see for example [Fo].

Lemma 10.3 *Let $\psi \in \mathbb{L}_1(S) \cap \mathbb{L}_2(S)$. Then, $\big(W_\psi f\big)(\cdot, t) \in \mathbb{L}_2(S)$ for all $f \in \mathbb{L}_2(S)$ and $t \in T$.*

Proof:
Let $f \in \mathbb{L}_2(S)$ and $t \in T$. Then

$$(\mathcal{T}_s \mathcal{P}_t \psi, f)_{\mathbb{L}_2(S)} = \int_S \overline{(\mathcal{P}_t\psi)(s'-s)} f(s') d\mu_S(s'),$$

for all s. So we arrive at a convolution. A convolution of a $\mathbb{L}_1$ function with a $\mathbb{L}_2$ function is again a $\mathbb{L}_2$ function. See [Fo, Proposition 2.39]). □

This means that for all elements Φ of our functional Hilbert space $\mathbb{C}_K^G$, the function $\Phi(\cdot, t)$ will be in $\mathbb{L}_2(S)$ for fixed $t \in T$. Hence, the Fourier transform of $\Phi(\cdot, t)$ is well-defined.

Now use Fourier transform and Plancherel to get a different presentation of the wavelet transform of an arbitrary function $f \in \mathbb{L}_2(S)$

$$\begin{aligned} \big(W_\psi f\big)(s,t) &= (\mathcal{T}_s \mathcal{P}_t \psi, f)_{\mathbb{L}_2(S)} = (\mathcal{F}\mathcal{T}_s\mathcal{P}_t\psi, \mathcal{F}f)_{\mathbb{L}_2(\hat{S})} \\ &= (\overline{\langle s, \cdot \rangle} \mathcal{F}\mathcal{P}_t\psi, \mathcal{F}f)_{\mathbb{L}_2(\hat{S})} = \big(\mathcal{F}^{-1}\left[\overline{\mathcal{F}\mathcal{P}_t\psi}\mathcal{F}f\right]\big)(s), \end{aligned} \tag{10.13}$$

for all $s \in S$ and $t \in T$. We notice that $\mathcal{F}f \in \mathbb{L}_2(\hat{S})$ and $\mathcal{F}\mathcal{P}_t\psi \in \mathbb{L}_\infty(\hat{S})$ for all $t \in T$. Hence $\overline{\mathcal{F}\mathcal{P}_t\psi}\mathcal{F}f \in \mathbb{L}_2(\hat{S})$ and $\big(W_\psi f\big)(\cdot, t) \in \mathbb{L}_2(S)$ for all $t \in T$. Moreover, since $\overline{\mathcal{F}\mathcal{P}_t\psi}\mathcal{F}f \in \mathbb{L}_1(\hat{S})$ we get $\big(W_\psi f\big)(\cdot, t) \in \mathcal{C}_0(S)$ for all $t \in T$ and $f \in \mathbb{L}_2(S)$. With $\mathcal{C}_0(S)$ we denote the space of continuous functions on S which vanish at infinity.

Lemma 10.4 *Let $\psi \in \mathbb{L}_2(S)$. Suppose*

$$\mu_{\hat{S}}\Big(\{\gamma \in \hat{S} \mid \forall t \in T\big[\big(\mathcal{F}\mathcal{P}_t\psi\big)(\gamma) = 0\big]\}\Big) = 0.$$

Then ψ is a cyclic vector.

Proof:
Note that the measure does not depend on the representant. Let $f \in V_\psi^\perp$. Then $W_\psi f = 0$ by the remark after Theorem 7.1. Hence, $\overline{\mathcal{F}\mathcal{P}_t\psi}\mathcal{F}f = 0$ for all $t \in T$, by (10.13). Therefore, $\mathcal{F}f = 0$ by the assumption. □

Corollary 10.5 *Let $\psi \in \mathbb{L}_2(S)$. If $\mathcal{F}\psi \neq 0$ a.e., then ψ is a cyclic vector. Moreover, if S is metrizable then the representation $\mathcal{U}$ is cyclic.*

Proof:
The first statement follows immediately from Lemma 10.4.
If the group S is metrizable, then $\hat{S}$ is σ-compact by [RS, Thm. 4.2.7]. Therefore, there exists a $\psi \in \mathbb{L}_2(S)$ such that $\mathcal{F}\psi > 0$, by the σ-compactness of $\hat{S}$. The conclusion now follows from the first statement. □

10.3 Alternative description of $\mathbb{C}_K^{S \rtimes T}$ for admissible wavelets

In this section we will derive an integral expression for the functional Hilbert space $\mathbb{C}_K^{S \rtimes T}$ for a special kind of wavelets. These wavelets will be called admissible.

Definition 10.6 *Let $\psi \in \mathbb{L}_1(S) \cap \mathbb{L}_2(S)$. Define $M_\psi : \hat{S} \to [0, \infty) \cup \{\infty\}$ as*

$$M_\psi(\gamma) = \int_T \frac{\left|\left(\mathcal{F}\mathcal{P}_t\psi\right)(\gamma)\right|^2}{\rho(t)}\; d\mu_T(t). \tag{10.14}$$

We note that $\mathcal{F}\mathcal{P}_t\psi \in \mathcal{C}_0(\hat{S})$ for all $t \in T$, so M_ψ can be defined pointwise.

Definition 10.7 *We call $\psi \in \mathbb{L}_1(S) \cap \mathbb{L}_2(S)$ an* ***admissible wavelet*** *iff*

$0 < M_\psi < \infty$ *a.e.*

Theorem 10.8 *Assume that T is compact and let $\psi \in \mathbb{L}_1(S) \cap \mathbb{L}_2(S)$ be admissible. Then $M_\psi \in \mathbb{L}_1(\hat{S})$.*

Proof:
For all $t \in T$ the operator $\mathcal{F}\mathcal{P}_t$ is unitary from $\mathbb{L}_2(S)$ onto $\mathbb{L}_2(\hat{S})$ we get

$$\int_{\hat{S}} \frac{|\mathcal{F}\mathcal{P}_t\psi|^2}{\rho(t)}(\gamma)\mathrm{d}\mu_{\hat{S}}(\gamma) = \int_{\hat{S}} |\mathcal{F}\mathcal{P}_t\psi|^2(\gamma)\mathrm{d}\mu_{\hat{S}}(\gamma) = \|\psi\|^2_{\mathbb{L}_2(S)},$$

for all $t \in T$. Hence,

$$\int_{\hat{S}}\int_T \frac{|\mathcal{F}\mathcal{P}_t\psi|^2}{\rho(t)}(\gamma)\;\mathrm{d}\mu_T(t)\mathrm{d}\mu_{\hat{S}}(\gamma) \;=\; \int_T \|\psi\|^2_{\mathbb{L}_2(S)}\mathrm{d}\mu_T(t) = |T|\|\psi\|^2_{\mathbb{L}_2(S)},$$

by changing the order of integration. □

In this section we will assume that $\psi \in \mathbb{L}_1(S) \cap \mathbb{L}_2(S)$ an admissible wavelet. All the admissible wavelets are cyclic, so lead to a unitary map from the entire space $\mathbb{L}_2(S)$ onto $\mathbb{C}_K^{S \rtimes T}$ by Theorem 7.1. This is shown in the following lemma.

Lemma 10.9 *Every admissible wavelet is a cyclic wavelet, i.e.* $\overline{\langle V_\psi \rangle} = \mathbb{L}_2(S)$.

Proof:
If $f \in \mathbb{L}_2(S)$, then with (10.13) we get

$$f \in \langle V_\psi \rangle^\perp \;\; \Leftrightarrow \Big(\forall t \in T \Big[|\overline{\mathcal{F}\mathcal{P}_t\psi}\mathcal{F}f|^2 = 0 \text{ a.e. on } S \Big] \Big). \tag{10.15}$$

Let $f \in \langle V_\psi \rangle^\perp$. Then

$$M_\psi |\mathcal{F}f|^2 = \int_T \frac{\left|\overline{\mathcal{F}\mathcal{P}_t\psi}\mathcal{F}f\right|^2}{\rho(t)} \mathrm{d}\mu_T(t) = 0 \quad \text{a.e. on } \hat{S}.$$

Because ψ is an admissible wavelet, the function $M_\psi > 0$ a.e.. Hence $|\mathcal{F}f|^2 = 0$ a.e. and therefore $f = 0$. $\Box$

Using the function M_ψ we can also give an expression for W_ψ^{-1}.

Lemma 10.10 *Let* $\psi \in \mathbb{L}_1(S) \cap \mathbb{L}_2(S)$ *be an admissible wavelet. Let* $f \in \mathbb{L}_2(S)$. *Then*

$$f = W_\psi^{-1}\Phi = \mathcal{F}^{-1}\Big(\int_T \mathcal{F}[\Phi(\cdot, t)]\mathcal{F}\mathcal{P}_t\psi \; M_\psi^{-1}\rho^{-1}(t) \; \mathrm{d}\mu_T(t) \Big), \tag{10.16}$$

where $\Phi = W_\psi f \in \mathbb{C}_K^{S \rtimes T}$

Proof:
We recall that $0 < M_\psi < \infty$ a.e. on $\hat{S}$, hence also $0 < M_\psi^{-\frac{1}{2}} < \infty$ a.e. on $\hat{S}$. The lemma now easily follows from (10.13) since

$$\begin{aligned}
&\mathcal{F}^{-1}\Big(\int_T \mathcal{F}[\Phi(\cdot, t)]\mathcal{F}\mathcal{P}_t\psi \; M_\psi^{-1}\rho^{-1}(t) \; \mathrm{d}\mu_T(t) \Big) \\
&\qquad = \mathcal{F}^{-1}\Big(M_\psi^{-1} \int_T \mathcal{F}f \; |\mathcal{F}\mathcal{P}_t\psi|^2 \rho^{-1}(t) \; \mathrm{d}\mu_T(t) \Big) \\
&\qquad = \mathcal{F}^{-1}\big(M_\psi^{-1} M_\psi \mathcal{F}f \big) = f.
\end{aligned}$$

$\Box$

We are now able to give an alternative description of the norm of $\mathbb{C}_K^{S \rtimes T}$ using (10.13) and the previous lemma.

Theorem 10.11 *If* $\Phi \in \mathbb{C}_K^{S \rtimes T}$ *then* $M_\psi^{-\frac{1}{2}}\mathcal{F}[\Phi(\cdot, t)] \in \mathbb{L}_2(\hat{S})$ *for almost every* $t \in T$. *Moreover,*

$$\|\Phi\|^2_{\mathbb{C}_K^{S \rtimes T}} = \int_{\hat{S}} \int_T \left| \mathcal{F}[\Phi(\cdot, t)](\gamma) \right|^2 \; M_\psi^{-1}(\gamma)\rho^{-1}(t) \; d\mu_T(t) \mathrm{d}\mu_{\hat{S}}(\gamma). \tag{10.17}$$

Proof:
Let $\Phi \in \mathbb{C}_K^{S \rtimes T}$. Then there exists a function $f \in \mathbb{L}_2(S)$ such that $W_\psi f = \Phi$.

$$\begin{aligned}
(\Phi, \Phi)^2_{\mathbb{C}_K^{S\rtimes T}} &= (f, W_\psi^{-1}\Phi)_{\mathbb{L}_2(S)} = (\mathcal{F}f, \mathcal{F}W_\psi^{-1}\Phi)_{\mathbb{L}_2(S)} \\
&= \int_{\hat{S}} \overline{\mathcal{F}f}(\gamma) \int_T \mathcal{F}[\Phi(\cdot,t)](\gamma)\big(\mathcal{F}\mathcal{P}_t\psi\big)(\gamma)\ M_\psi^{-1}(\gamma)\rho^{-1}(t)\ \mathrm{d}\mu_T(t)\mathrm{d}\mu_{\hat{S}}(\gamma) \\
&= \int_{\hat{S}} \int_T \mathcal{F}[\Phi(\cdot,t)](\gamma)\overline{\mathcal{F}f(\gamma)}\big(\mathcal{F}\mathcal{P}_t\psi\big)(\gamma)\ M_\psi^{-1}(\gamma)\rho^{-1}(t)\ \mathrm{d}\mu_T(t)\mathrm{d}\mu_{\hat{S}}(\gamma) \\
&= \int_{\hat{S}} \int_T \mathcal{F}[\Phi(\cdot,t)](\gamma)\overline{\mathcal{F}[\Phi(\cdot,t)](\gamma)}\ M_\psi^{-1}(\gamma)\rho^{-1}(t)\ \mathrm{d}\mu_T(t)\mathrm{d}\mu_{\hat{S}}(\gamma)
\end{aligned}$$

Therefore,

$$\int_{\hat{S}} \int_T \left|\mathcal{F}[\Phi(\cdot,t)](\gamma)\right|^2\ M_\psi^{-1}\rho^{-1}(t)\ \mathrm{d}\mu_T(t)\mathrm{d}\mu_{\hat{S}}(\gamma) = \|\Phi\|^2_{\mathbb{C}_K^{S\rtimes T}}.$$

The integrand is positive, so by changing the order of integration we find in particular

$$\int_{\hat{S}} |M_\psi \mathcal{F}[\Phi(\cdot,t)]|^2\ \mathrm{d}\mu_{\hat{S}}(\gamma) < \infty,$$

for almost all $t \in T$. Therefore $M_\psi^{-\frac{1}{2}}\mathcal{F}[\Phi(\cdot,t)] \in \mathbb{L}_2(\hat{S})$ for almost all $t \in T$. $\square$

Because of Lemma 10.11 and (10.5) we can define the linear operator $T_{M_\psi} : \mathbb{C}_K^{S\rtimes T} \to \mathbb{L}_2(S \rtimes T)$ by

$$\left(T_{M_\psi}\Phi\right)(s,t) = \left(\mathcal{F}^{-1}[M_\psi^{-\frac{1}{2}}\ \mathcal{F}[\Phi(\cdot,t)]]\right)(s), \tag{10.18}$$

for almost all $(s,t) \in S \rtimes T$.

We summarize the previous in the following theorem.

Theorem 10.12 *Let $\psi \in \mathbb{L}_1(S) \cap \mathbb{L}_2(S)$ be an admissible wavelet. Then the wavelet transform W_ψ defined by*

$$\big(W_\psi f\big)(s,t) = (\mathcal{T}_s\mathcal{P}_t\psi, f)_{\mathbb{L}_2(S)}, \quad f \in \mathbb{L}_2(S),\ (s,t) \in S \rtimes T, \tag{10.19}$$

is a unitary map from $\mathbb{L}_2(S)$ onto $\mathbb{C}_K^{S\rtimes T}$. Here, $\mathbb{C}_K^{S\rtimes T}$ is the functional Hilbert space with reproducing kernel

$$K(g,h) = (\mathcal{U}_g\psi, \mathcal{U}_h\psi)_{\mathbb{L}_2(S)} = (\mathcal{U}_{h^{-1}g}\psi, \psi)_{\mathbb{L}_2(S)}, \tag{10.20}$$

for all $g, h \in S \rtimes T$. The inner product on $\mathbb{C}_K^{S\rtimes T}$ can be written as

$$(\Phi, \Psi)_{\mathbb{C}_K^{S\rtimes T}} = (T_{M_\psi}\Phi, T_{M_\psi}\Psi)_{\mathbb{L}_2(S\rtimes T)}, \tag{10.21}$$

for all $\Phi, \Psi \in \mathbb{C}_K^{S\rtimes T}$.

Corollary 10.13 *If $M_\psi = 1$ on $\hat{S}$, then $\mathbb{C}_K^{S\rtimes T}$ is a closed subspace of $\mathbb{L}_2(S \rtimes T)$.*

By Lemma 10.11, our functional Hilbert space is a closed subspace of

$$\mathbb{H}(S,\mu_S) \otimes \mathbb{L}_2(T,\rho^{-1}\mu_T), \tag{10.22}$$

where

$$\mathbb{H}(S,\mu_S) = \{f \in \mathbb{L}_2(S,\mu_S) \mid M_\psi^{-\frac{1}{2}}\mathcal{F}f \in \mathbb{L}_2(\hat{S})\}. \tag{10.23}$$

The inner product on $\mathbb{H}(S,\mu_S)$ is defined by

$$(f,g)_{\mathbb{H}(S,\mu_s)} = (M_\psi^{-\frac{1}{2}}\mathcal{F}f, M_\psi^{-\frac{1}{2}}\mathcal{F}g)_{\mathbb{L}_2(\hat{S})} \tag{10.24}$$

We recall that $\mathbb{H}(S,\mu_S)$ is a vector subspace of $\mathbb{L}_2(S)$, because of Lemma 10.3. Hence we always arrive at a kind of Sobolev space on S. Now denote the inner product on $\mathbb{H}(S,\mu_S)\otimes \mathbb{L}_2(T,\rho^{-1}\mu_T)$ by $(\cdot,\cdot)_\otimes$. It follows from Lemma 10.11 that $(\cdot,\cdot)_\otimes|_{\mathbb{C}_K^{S\rtimes T}} = (\cdot,\cdot)_{\mathbb{C}_K^{S\rtimes T}}$.

Theorem 10.14 *$\mathbb{C}_K^{S\rtimes T}$ is a closed subspace of $\mathbb{H}(S,\mu_S) \otimes \mathbb{L}_2(T,\rho^{-1}\mu_T)$. The operator $\Phi \mapsto [g \mapsto (K(\cdot,g),\Phi)_\otimes]$ is the projection operator from $\mathbb{H}(S,\mu_S) \otimes \mathbb{L}_2(T,\rho^{-1}\mu_T)$ onto $\mathbb{C}_K^{S\rtimes T}$.*

Proof:
It is obvious that $\mathbb{C}_K^{S\rtimes T}$ is a closed subspace of $\mathbb{H}(S,\mu_S) \otimes \mathbb{L}_2(T,\rho^{-1}\mu_T)$. Let $\Phi \in \mathbb{H}(S,\mu_S) \otimes \mathbb{L}_2(T,\rho^{-1}\mu_T)$. Then it can be written as

$$\Phi = \Phi_1 + \Phi_2,$$

with $\Phi_1 \in \mathbb{C}_K^{S\rtimes T}$ and $\Phi_2 \in (\mathbb{C}_K^{S\rtimes T})^\perp$. Then for all $g \in S \rtimes T$

$$(K(\cdot,g),\Phi)_\otimes = (K(\cdot,g),\Phi_1)_\otimes + (K(\cdot,g),\Phi_2)_\otimes = \Phi_1(g)$$

Therefore Φ is mapped to Φ_1. □

11 $\mathbb{L}_2(\mathbb{R}^2)$ and the Euclidean motion group

11.1 The wavelet transform

We now will work the previous section out in detail for a more explicit example. The circle group $\mathbb{T}$ is defined by the set

$$\mathbb{T} = \{z \mid z \in \mathbb{C} \; |z| = 1\}, \tag{11.25}$$

with complex multiplication. The group $\mathbb{T}$ has the following group homomorphism $\tau : \mathbb{T} \to \mathrm{Aut}(\mathbb{R}^2)$

$$\tau : \; z \mapsto R_z, \tag{11.26}$$

with

$$R_z = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad\qquad \theta = \arg z. \tag{11.27}$$

Using this automorphism we can define the semi-direct product $\mathbb{R}^2 \rtimes \mathbb{T}$. The group product of $\mathbb{R}^2 \rtimes \mathbb{T}$ is given by

$$(x, z_1)(y, z_2) = (x + R_{z_1} y, z_1 z_2). \tag{11.28}$$

for all $(x, z_1), (y, z_2) \in \mathbb{R}^2 \rtimes \mathbb{T}$. The group $\mathbb{R}^2 \rtimes \mathbb{T}$ is called the **Euclidean motion group**.

We normalize the Haar measure on $\mathbb{R}^2$ such that $[0,1]^2$ has measure $\frac{1}{2\pi}$. We normalize the Haar measure on $\mathbb{T}$ such that $\mathbb{T}$ has total measure one. Then we normalize the Haar measure of $\mathbb{R}^2 \rtimes \mathbb{T}$ such that it is equal to the product measure $\mu_{\mathbb{R}^2} \times \mu_{\mathbb{T}}$. Since $\mathbb{T}$ is compact, $\rho(z) = 1$ for all $z \in \mathbb{T}$.

The Euclidean motion groups has the unitary representation $\mathcal{U} : \mathbb{R}^2 \rtimes \mathbb{T} \to \mathcal{B}(\mathbb{L}_2(\mathbb{R}^2) : (y, z) \mapsto \mathcal{U}_{(y,z)}$ where $\mathcal{U}_{(y,z)} \in \mathcal{B}(\mathbb{L}_2(\mathbb{R}^2)$ is defined by

$$\big(\mathcal{U}_{(y,z)} f\big)(x) = \big(\mathcal{T}_y \mathcal{P}_z f\big)(x) = f(R_z^{-1}(x - y))), \tag{11.29}$$

with

$$\big(\mathcal{T}_y f\big)(x) = f(x - y), \;\; \big(\mathcal{P}_z f\big)(x) = f(R_z^{-1} x), \tag{11.30}$$

for all $y \in \mathbb{R}^2$, $z \in \mathbb{T}$, $f \in \mathbb{L}_2(\mathbb{R}^2)$ and almost every $x \in \mathbb{R}^2$. By Corollary 10.5, the representation is cyclic. But it is not irreducible.

Theorem 11.15 *The representation $\mathcal{U}$ is reducible*

Proof:
Let S be the Hilbert subspace of $\mathbb{L}_2(\mathbb{R}^2)$ consisting of all $f \in \mathbb{L}_2(\mathbb{R})$ such that $\big(\mathcal{F} f\big)(\omega) = 0$ for almost all $\omega \in \mathbb{R}^2 / B_{0,1}$. Let $f \in S$ and $(y, z) \in \mathbb{R}^2 \rtimes \mathbb{T}$. Then

$$\big(\mathcal{F}\mathcal{U}_{(y,z)} f\big)(\omega) = \mathrm{e}^{\mathrm{i}(\omega, y)} \big(\mathcal{F} f\big)(R_z^{-1} y) = 0$$

for almost all $\omega \in \mathbb{R}^2 / B_{0,1}$. Hence $\mathcal{U}_{(y,z)} f \in S$ for all $(y, z) \in \mathbb{R}^2 \rtimes \mathbb{T}$ and S is an invariant subspace of $\mathbb{L}_2(\mathbb{R})$. $\square$

We consider the wavelet transform using the representation $\mathcal{U}$, as above, of the group $G = \mathbb{R}^2 \rtimes \mathbb{T}$ in the Hilbert space $\mathbb{L}_2(\mathbb{R}^2)$. The wavelet transform $W_\psi : \mathbb{L}_2(\mathbb{R}^2) \to \mathbb{C}_K^{\mathbb{R}^2 \rtimes \mathbb{T}}$ for cyclic wavelets is defined by

$$\big(W_\psi f\big)(y, z) = \big(\mathcal{T}_y \mathcal{P}_z \psi, f\big)_{\mathbb{L}_2(\mathbb{R}^2)}, \tag{11.31}$$

for all $f \in \mathbb{L}_2(\mathbb{R}^2)$ and $(y, z) \in \mathbb{R}^2 \rtimes \mathbb{T}$.

11.2 Admissible wavelets

First we mention the method involving admissible wavelets. We recall that $\psi \in \mathbb{L}_1(\mathbb{R}^2) \cap \mathbb{L}_2(\mathbb{R}^2)$ is called admissible if

$$0 < M_\psi < \infty \text{ a.e.}$$

where

$$M_\psi(\omega) = \int_{\mathbb{T}} \big|\big(\mathcal{F}\mathcal{P}_z \psi\big)(\omega)\big|^2 \, \mathrm{d}\mu_{\mathbb{T}}(z). \tag{11.32}$$

We can reformulate Theorem 10.12 as follows.

Theorem 11.16 *Let $\psi \in \mathbb{L}_1(\mathbb{R}^2) \cap \mathbb{L}_2(\mathbb{R}^2)$ be an admissible wavelet. Then W_ψ defined by*

$$\big(W_\psi f\big)(x, z) = (\mathcal{T}_x \mathcal{P}_z \psi, f)_{\mathbb{L}_2(\mathbb{R}^2)}, \quad f \in \mathbb{L}_2(\mathbb{R}^2), \;\; (x, z) \in \mathbb{R}^2 \rtimes \mathbb{T}, \tag{11.33}$$

is a unitary map from $\mathbb{L}_2(\mathbb{R}^2)$ onto $\mathbb{C}_K^{\mathbb{R}^2 \rtimes \mathbb{T}}$. Here, $\mathbb{C}_K^{\mathbb{R}^2 \rtimes \mathbb{T}}$ is the functional Hilbert space with reproducing kernel

$$K(g, h) = (\mathcal{U}_g \psi, \mathcal{U}_h \psi)_{\mathbb{L}_2(\mathbb{R}^2)}, \tag{11.34}$$

for all $g, h \in \mathbb{R}^2 \rtimes \mathbb{T}$. The inner product on $\mathbb{C}_K^{\mathbb{R}^2 \rtimes \mathbb{T}}$ can be written as

$$(\Phi, \Psi)_{\mathbb{C}_K^{\mathbb{R}^2 \rtimes \mathbb{T}}} = (T_{M_\psi} \Phi, T_{M_\psi} \Psi)_{\mathbb{L}_2(\mathbb{R}^2 \rtimes \mathbb{T})}, \tag{11.35}$$

for all $\Phi, \Psi \in \mathbb{C}_K^{\mathbb{R}^2 \rtimes \mathbb{T}}$.

We now analyse the function M_ψ, defined in (11.32) a little further. First we mention that $\mathbb{T}$ is a compact group, so $M_\psi \in \mathbb{L}_1(\mathbb{R}^2)$ by Theorem 10.8. Define for $m \in \mathbb{Z}$ the function $\eta_m : [0, 2\pi) \to \mathbb{C}$ by $\eta_m(\phi) = \mathrm{e}^{\mathrm{i}m\phi}$. Because $\mathbb{L}_2(\mathbb{R}^2) \backsimeq \mathbb{L}_2(S^1) \otimes \mathbb{L}_2((0, \infty), r \, \mathrm{d}r)$, we can write all $\psi \in \mathbb{L}_2(\mathbb{R}^2)$ in the following way

$$\psi(r \cos \phi, r \sin \phi) = \sum_{m=-\infty}^{\infty} \eta_m(\phi) \otimes \chi_m(r), \tag{11.36}$$

for almost all $r \in (0,\infty)$ and $\phi \in [0,2\pi)$, where $\chi_m \in \mathbb{L}_2((0,\infty), r\ dr)$ for all $m \in \mathbb{Z}$. For the Fourier transform we can write in polar coordinates

$$\big(\mathcal{F}[\eta_m \otimes \chi_m]\big)(\rho,\phi_\omega) = \mathrm{i}^m \mathrm{e}^{\mathrm{i}m\varphi} \int_0^\infty r\chi_m(r) J_m(\rho r)\ dr \tag{11.37}$$

for all $\rho \in [0,\infty)$ and $\varphi \in [0,2\pi)$, where J_m is the m-th order Bessel function of the first kind. See for example [FH, Ch. II]. (The length of the interval $[0,1]$ is $\frac{1}{\sqrt{2\pi}}$.)

Now M_ψ is easily calculated.

$$\big(P_z\psi\big)(r,\phi) = \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\phi - \arg z)} \chi_m(r), \tag{11.38}$$

for all $r \in (0,\infty)$ and $\phi \in [0,2\pi)$. Hence,

$$\big(\mathcal{F}P_z\psi\big)(\rho,\varphi) = \sum_{m=-\infty}^{\infty} \mathrm{i}^m \mathrm{e}^{\mathrm{i}m(\varphi - \arg z)} \int_0^\infty r\chi_m(r) J_m(\rho r) dr, \tag{11.39}$$

for all $\rho \in [0,\infty)$ and $\varphi \in [0,2\pi)$. Hence M_ψ is given by,

$$M_\psi(\omega) = \sum_{m=-\infty}^{\infty} |\tilde{\chi}_m(|\omega|)|^2, \tag{11.40}$$

for all $\omega \in \mathbb{R}^2$, where $\tilde{\chi}_m$ defined by $\tilde{\chi}_m(\rho) = \int_0^\infty r\chi_m(r) J_m(\rho r) dr$ for all $\rho \in (0,\infty)$ and $m \in \mathbb{Z}$. Thus, the above sum completely determines the inner product. Furthermore, M_ψ only depends on the radius. We have the following relation between a chosen wavelet ψ and M_ψ

$$\int_{\mathbb{R}^2} M_\psi(\omega)\ \mathrm{d}\omega = \|\psi\|^2_{\mathbb{L}_2}. \tag{11.41}$$

This implies that M_ψ^{-1} is unbounded. Because M_ψ only depends on the radius, there exists a function $\tilde{M}_\psi : (0,\infty) \to (0,\infty)$ such that

$$M_\psi(\omega) = \tilde{M}_\psi(|\omega|), \tag{11.42}$$

for almost all $\omega \in \mathbb{R}^2$. Then $\tilde{M}_\psi \in \mathbb{L}_1((0,\infty), r\ dr)$.

We end this subsection with the remark, that $\psi \mapsto M_\psi$ is not injective; several different wavelets ψ can lead to the same M_ψ. If ψ_1 and ψ_2 are different admissible wavelets with the property $M_{\psi_1} = M_{\psi_2}$, then their corresponding functional Hilbert space are different closed subspaces of the same Hilbert space $\mathbb{H}(\mathbb{R}^2) \otimes \mathbb{L}_2(\mathbb{T})$ as defined in (10.22).

11.3 Generalized admissible wavelets

In this section we introduce the notion of a generalized wavelet transform. The aim is to use a Gelfand triple

$$\mathcal{H}_1 \hookrightarrow \mathbb{L}_2(\mathbb{R}^2) \hookrightarrow \mathcal{H}_{-1}, \tag{11.43}$$

to define a transform $\mathfrak{W}_\Psi : \mathcal{H}_1 \to \mathbb{C}^{\mathbb{R}^2 \rtimes \mathbb{T}}$ by

$$\big(\mathfrak{W}_\Psi \phi\big)(x,z) = \langle \Psi, \mathcal{U}_{(x,z)^{-1}} \phi \rangle, \tag{11.44}$$

for all $\phi \in \mathcal{H}_1$, $(x,z) \in \mathbb{R}^2 \rtimes \mathbb{T}$ and a special choice of $\Psi \in \mathcal{H}_{-1}$. For a suitable choice of the Gelfand triple and the generalized wavelet $\Psi \in \mathcal{H}_{-1}$, we will prove that $\mathfrak{W}$ defines an isometry from $\mathcal{H}_1$, but now equipped with the $\mathbb{L}_2$-norm, onto $\mathbb{L}_2(\mathbb{R}^2 \rtimes \mathbb{T})$. Therefore, it has a closure $\overline{\mathfrak{W}}$ which is an isometry from $\mathbb{L}_2(\mathbb{R}^2)$ onto $\mathbb{L}_2(\mathbb{R}^2 \rtimes \mathbb{T})$.

Consider the Gelfand triples $H^{2k,2}(\mathbb{R}^2) \hookrightarrow \mathbb{L}_2(\mathbb{R}^2) \hookrightarrow H^{-2k,2}(\mathbb{R}^2)$, as introduced in appendix B. Since $H^{2k,2}(\mathbb{R}^2) \subset \mathbb{L}_2(\mathbb{R}^2)$ we can restrict the representation $\mathcal{U}$ to $H^{2k,2}$. Because of the special structure of $H^{2k,2}(\mathbb{R}^2)$, this representation is again unitary.

Lemma 11.17 *The operators $\mathcal{U}_g$ and D_k commute for all $k \in \mathbb{N}$ and $g \in \mathbb{R}^2 \rtimes \mathbb{T}$*

Proof:
Since the laplace operator $\triangle$ commutes with translations and rotations, the lemma follows. □

Corollary 11.18 *The restriction $g \mapsto \mathcal{U}_{g|_{H^{2k,2}(\mathbb{R}^2)}}$ defines a unitary representation on $H^{2k,2}(\mathbb{R}^2)$, which will also be denoted by $\mathcal{U}$*

Proof:
Let $\phi \in H^{2k,2}(\mathbb{R}^2)$ and $g \in \mathbb{R}^2 \rtimes \mathbb{T}$. Then

$$\|\mathcal{U}_g \phi\|_{H^{2k,2}(\mathbb{R}^2)} = \|D_k \mathcal{U}_g \phi\|_{\mathbb{L}_2(\mathbb{R}^2)} = \|\mathcal{U}_g D_k \phi\|_{\mathbb{L}_2(\mathbb{R}^2)} = \|D_k \phi\|_{\mathbb{L}_2(\mathbb{R}^2)} = \| \phi\|_{H^{2k,2}(\mathbb{R}^2)},$$

which proves the statement. □

Define $\tilde{\mathcal{T}}_x, \tilde{\mathcal{P}}_z \in \mathcal{B}(H^{-2k,2}(\mathbb{R}^2)$ by

$$\langle \tilde{\mathcal{T}}_x \Psi, \phi \rangle_{H^{-2k,2}(\mathbb{R}^2)} = \langle \Psi, \mathcal{T}_{-x} \phi \rangle_{H^{-2k,2}(\mathbb{R}^2)} \tag{11.45}$$

and

$$\langle \tilde{\mathcal{P}}_z \Psi, \phi \rangle_{H^{-2k,2}(\mathbb{R}^2)} = \langle \Psi, \mathcal{P}_{z^{-1}} \phi \rangle_{H^{-2k,2}(\mathbb{R}^2)} \tag{11.46}$$

for all $\Psi \in H^{-2k,2}(\mathbb{R}^2)$, $\phi \in H^{2k,2}(\mathbb{R}^2)$, $z \in \mathbb{T}$ and $x \in \mathbb{R}^2$. Moreover define the representation $\tilde{\mathcal{U}} : \mathbb{R}^2 \rtimes \mathbb{T} \to \mathcal{B}(H^{-2k,2}(\mathbb{R}^2)) : g \mapsto \tilde{\mathcal{U}}_g$ by

$$\langle \tilde{\mathcal{U}}_g F, \phi \rangle = \langle F, \mathcal{U}_{g^{-1}} \phi \rangle, \tag{11.47}$$

for all $F \in H^{-2k,2}(\mathbb{R}^2)$, $\phi \in H^{2k,2}(\mathbb{R}^2)$ and $g \in \mathbb{R}^2 \rtimes \mathbb{T}$. It follows from the fact that $H^{-2k,2}(\mathbb{R}^2)$ is the anti-dual (and not the dual) of $H^{2k,2}(\mathbb{R}^2)$ that this indeed is a representation. Moreover, this representation is again unitary. With this representation and a wavelet $\Psi \in H^{-2k,2}(\mathbb{R}^2)$ we can define a wavelet transform on $H^{-2k,2}(\mathbb{R}^2)$. But we restrict the set of allowed wavelets.

Definition 11.19 *Let $k \in \mathbb{N}$ and $\Psi \in H^{-2k,2}(\mathbb{R}^2)$. Then Ψ is called an **admissible generalized wavelet** if there exists a measurable complex-valued function ψ such that*

1. *$\omega \mapsto \frac{\psi(\omega)}{1+|\omega|^{2k}}$ is contained in $\mathbb{L}_1(\mathbb{R}^2) \cap \mathbb{L}_2(\mathbb{R}^2)$.*
2. *$\int_{\mathbb{T}} \left|\psi(R_z^{-1}\omega)\right|^2 \mathrm{d}\mu_{\mathbb{T}}(z) = 1$ for almost all $\omega \in \mathbb{R}^2$.*
3. *$\langle \Psi, \phi \rangle = \int_{\mathbb{R}^2} \overline{\psi(\omega)} \big(\mathcal{F}\phi\big)(\omega) d\omega$, for all $\phi \in H^{2k,2}(\mathbb{R}^2)$.*

For a admissible generalized wavelet define the wavelet transform $W_\Psi : H^{-2k,2}(\mathbb{R}^2) \to \mathbb{C}_K^{\mathbb{R}^2 \rtimes \mathbb{T}}$ by

$$\big(W_\Psi F\big)(y,z) = (\mathcal{T}_y \mathcal{P}_z \Psi, F)_{H^{-2k,2}(\mathbb{R}^2)}, \tag{11.48}$$

for all $F \in H^{-2k,2}$ and $y \in \mathbb{R}^2$.

Theorem 11.20 *The map $D_k W_\Psi D_k : \mathbb{L}_2(\mathbb{R}^2) \to \mathbb{L}_2(\mathbb{R}^2 \rtimes \mathbb{T})$ defined by*

$$\big(D_k W_\Psi D_k f\big)(x,z) = D_k\Big(\big(W_\Psi D_k f\big)(\cdot, z)\Big)(x), \tag{11.49}$$

for all $(x,z) \in \mathbb{R}^2 \rtimes \mathbb{T}$ and $f \in \mathbb{L}_2(\mathbb{R}^2)$, is well-defined and an isometry.

Proof:
Rewrite $\big(W_\Psi D_k f\big)(y,z)$ as

$$\begin{aligned} \big(W_\Psi D_k f\big)(x,z) &= (\tilde{\mathcal{U}}_{(x,z)}\Psi, D_k f)_{H^{-2k,2}(\mathbb{R}^2)} = (D_k^{-1}\tilde{\mathcal{U}}_{(x,z)}\Psi, f)_{\mathbb{L}_2(\mathbb{R}^2)} \\ &= (\mathcal{U}_{(x,z)} D_k^{-1}\Psi, f)_{\mathbb{L}_2(\mathbb{R}^2)} = \mathcal{F}^{-1}\big(\overline{\mathcal{F}\mathcal{P}_z D_k^{-1}\Psi}\mathcal{F}f\big)(x) \\ &= \mathcal{F}^{-1}\big(\overline{\mathcal{P}_z \mathcal{F} D_k^{-1}\Psi}\mathcal{F}f\big)(x) \end{aligned}$$

for all $(x,z) \in \mathbb{R}^2 \rtimes \mathbb{T}$. Since

$$(\mathcal{F}D_k^{-1}\Psi, f)_{\mathbb{L}_2(\mathbb{R}^2)} = (D_k^{-1}\Psi, \mathcal{F}^* f)_{\mathbb{L}_2(\mathbb{R}^2)} = \langle \Psi, D_k^{-1}\mathcal{F}^* f\rangle = \int_{\mathbb{R}^2} \frac{\overline{\psi(\omega)}}{1+|\omega|^{2k}} f(\omega)\, \mathrm{d}\omega,$$

for all $f \in \mathbb{L}_2(\mathbb{R}^2)$, it follows that $\big(\mathcal{F}D_k^{-1}\Psi\big)(\omega) = \frac{\psi(\omega)}{1+|\omega|^{2k}}$ for almost all $\omega \in \mathbb{R}^2$. Since $\omega \mapsto \frac{\psi(\omega)}{1+|\omega|^{2k}} \in \mathbb{L}_1(\mathbb{R}^2)$ we obtain that $W_\Psi D_k f(\cdot, z) \in \mathbb{L}_2(\mathbb{R}^2)$ for all

$f \in \mathbb{L}_2(\mathbb{R}^2)$ and $z \in \mathbb{T}$. Moreover,

$$\begin{aligned}\int_{\mathbb{T}}\int_{\mathbb{R}^2} \left|(1+|\omega|^{2k})\big(\mathcal{F}\big(W_\Psi D_k f\big)(\cdot,z)\big)(\omega)\right|^2 \, \mathrm{d}\omega\mathrm{d}\mu_{\mathbb{T}}(z) \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{T}} |\overline{\mathcal{P}_z\psi(\omega)}\mathcal{F}f(\omega)|^2 \, \mathrm{d}\mu_{\mathbb{T}}(z)\mathrm{d}\omega \\ &= \int_{\mathbb{R}^2} |\mathcal{F}f(\omega)|^2 \, \mathrm{d}\omega = \|f\|^2_{\mathbb{L}_2(\mathbb{R}^2)},\end{aligned}$$

and therefore $W_\Psi D_k f(\cdot,z) \in \mathcal{H}^{2k,2}(\mathbb{R}^2)$ for almost all $z \in \mathbb{T}$ and $\|D_k W_\Psi D_k f\|_{\mathbb{L}_2(\mathbb{R}^2 \rtimes \mathbb{T})} = \|f\|_{\mathbb{L}_2(\mathbb{R}^2)}$. □

Note that in the proof it is crucial that D_k and the representations $\mathcal{U}$ and $\tilde{\mathcal{U}}$ commute. We call the transform $D_k W_\Psi D_k$ a **generalized wavelet transform**.

Theorem 11.21 *The unitary transform $D_k W_\Psi D_k$ as formulated in Theorem 11.20 is the closure of the operator $\mathfrak{W} : \mathcal{D}(D_k) \to \mathbb{L}_2(\mathbb{R}^2 \rtimes \mathbb{T})$ defined by*

$$\big(\mathfrak{W}f\big)(g) = \langle \tilde{\mathcal{U}}_g \Psi, f\rangle = \langle \Psi, \mathcal{U}_{g^{-1}} f\rangle, \tag{11.50}$$

for all $f \in \mathcal{D}(D_k)$.

Proof:
Let $g = (x,z) \in \mathbb{R}^2 \rtimes \mathbb{T}$. Then $(x,z)^{-1} = (R_z^{-1}x, \overline{z})$. Let $f \in \mathcal{D}(D_k)$ then,

$$\begin{aligned}\langle \Psi, \mathcal{U}_{g^{-1}} f\rangle &= \int_{\mathbb{R}^2} \overline{\psi(\omega)} \mathrm{e}^{\mathrm{i}(\omega, R_z x)_2} \mathcal{F}f(R_z^{-1}\omega) \, \mathrm{d}\omega \\ &= \int_{\mathbb{R}^2} \overline{\psi(R_z^{-1}\omega)} \mathrm{e}^{\mathrm{i}(\omega,x)_2} \mathcal{F}f(\omega) \, \mathrm{d}\omega \\ &= \mathcal{D}_k \int_{\mathbb{R}^2} \frac{\overline{\psi(R_z^{-1}\omega)}}{1+|\omega|^{2k}} \mathrm{e}^{\mathrm{i}(\omega,x)_2} \mathcal{F}f(\omega) \, \mathrm{d}\omega = \mathcal{F}^{-1}\big(\overline{\mathcal{P}_z \mathcal{F} D_k^{-1}\Psi} \mathcal{F}f\big)(x),\end{aligned}$$

which proves the statement □

11.4 An example of an admissible generalized wavelet

As an example of an admissible generalized wavelet we analyze the function defined by the pointwise limit

$$\psi(x_1,x_2) = \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m!}} \mathrm{e}^{-\frac{1}{2}|z|^2}, \tag{11.51}$$

where $z = x_1 + \mathrm{i}x_2$, which is based on the article [KHV]. Although the sum does converge uniformly on compacta, this function is not contained in $\mathbb{L}_2(\mathbb{R}^2)$. But we will prove that the function on $H^{4,2}(\mathbb{R}^2)$ defined by

$$T_\psi \phi = \lim_{N\to\infty} \int_{\mathbb{R}^2} \sum_{n=0}^{N} \psi_m(x)\phi(x)\ \mathrm{d}x, \tag{11.52}$$

where $\psi_m(x_1, x_2) = \frac{(x_1+\mathrm{i}x_2)^m}{\sqrt{m!}} \mathrm{e}^{-\frac{1}{2}(x_1^2+x_2^2)}$, is a continuous linear functional and hence it has a representant Ψ in $H^{-4,2}(\mathbb{R}^2)$. Moreover, it turns out to be an admissible generalized wavelet. To prove this statement, we need some asymptotics.

The entire function defined by

$$F(r) = \sum_{m=0}^{\infty} \frac{r^m}{\sqrt{m!}} \mathrm{e}^{-r^2/2}, \tag{11.53}$$

for all $r \in \mathbb{R}$, has the asymptotic expansion,

$$F(r) = (8\pi)^{1/4}\sqrt{r}\{1 - \frac{1}{16r^2} + \mathcal{O}(\frac{1}{r^4})\}, \quad r \to \infty. \tag{11.54}$$

See [O, 2.Ch. 9, §8, pp. 307-309]. From this expansion and the continuity of ψ it easily follows that there exists a constant A such that

$$|\psi(x)| < A(1 + \sqrt{|x|}), \tag{11.55}$$

for all $x \in \mathbb{R}^2$. Moreover, since $F(r)$ converges uniformly on every finite interval, we also find that ψ converges uniformly on compacta.

It is clear that the functions ψ_m are eigenvectors of the Fourier transform with eigenvalues $(-\mathrm{i})^m$ for all $m \in \mathbb{N} \cup \{0\}$. Hence, we can rewrite (11.52) into,

$$\begin{aligned} T_\psi \phi &= \lim_{N\to\infty} \int_{\mathbb{R}^2} \sum_{m=0}^{N} \psi_m(x)\phi(x)\ \mathrm{d}x = \lim_{N\to\infty} \int_{\mathbb{R}^2} \sum_{m=0}^{N} (-\mathrm{i})^m \psi_m(\omega)\mathcal{F}\phi(\omega))\ \mathrm{d}x \\ &= \lim_{N\to\infty} \int_{\mathbb{R}^2} \sum_{m=0}^{N} \frac{\psi_m(R_{\mathrm{i}^{-1}}\omega)}{\sqrt{1+|\omega|^4}} \sqrt{1+|\omega|^4}\mathcal{F}\phi(\omega)\ \mathrm{d}x, \end{aligned} \tag{11.56}$$

for all $\phi \in H^{4,2}$. It is now obvious that $\omega \mapsto \sum_{m=0}^{N} \frac{\psi_m(R_{\mathrm{i}}^{-1}\omega)}{\sqrt{1+|\omega|^4}}$ converges in the mean to $\omega \mapsto \frac{\psi(R_{\mathrm{i}}^{-1}\omega)}{\sqrt{1+|\omega|^4}}$ in $\mathbb{L}_2(\mathbb{R}^2)$. Hence,

$$T_\psi \phi = \int_{\mathbb{R}^2} \psi(R_{-\mathrm{i}}\omega)\mathcal{F}\phi(\omega)\ \mathrm{d}\omega, \tag{11.57}$$

for all $\phi \in H^{4,2}(\mathbb{R}^2)$ and T_ψ has a representant $\Psi \in H^{-4,2}(\mathbb{R}^2)$. We now only need to verify the integral condition in definition 11.19.

$$\begin{aligned}
\int_{\mathbb{T}} \left|\psi(R_{\mathrm{i}z}^{-1}\omega)\right|^2 \mathrm{d}\mu_{\mathbb{T}}(z) &= \int_{\mathbb{T}} \left|\psi(R_{z}^{-1}\omega)\right|^2 \mathrm{d}\mu_{\mathbb{T}}(z) \\
&= \frac{1}{2\pi}\sum_{m=0}^{\infty}\sum_{m'=0}^{\infty}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(m'-m)\theta}\frac{z^m\overline{z}^{m'}}{\sqrt{m!m'!}}\mathrm{e}^{-|z|^2}\mathrm{d}\theta \\
&= \sum_{m=0}^{\infty}\frac{|z|^{2m}}{m!}\mathrm{e}^{-|z|^2} = 1, \qquad (11.58)
\end{aligned}$$

for all $\omega \in \mathbb{R}^2$. Hence, Ψ related to T_ψ is an admissible generalized wavelet and we can apply Theorem 11.20 and its Corollary 11.21.

12 A transform by Sherman revisited

12.1 Preliminaries

Let $q \geq 2$. In this section we consider the space $\mathbb{L}_2(S^{q-1})$ with inner product

$$(f,g)_q = \frac{1}{\sigma_{q-1}} \int_{S_{q-1}} \overline{f(\xi)} g(\xi)\ \mathrm{d}\sigma_{q-1}(\xi), \tag{12.59}$$

for all $f, g \in \mathbb{L}_2(S^{q-1})$.

Denote the space of all harmonic polynomials of degree n on $\mathbb{R}^q$ by $\mathrm{HarmPol}(\mathbb{R}^q, n)$, the space of all homogeneous polynomials of degree N by $\mathrm{HomPol}(\mathbb{R}^q, n)$ and the space of all harmonic homogeneous polynomials of degree n on $\mathbb{R}^q$ by $\mathrm{HarmHomPol}(\mathbb{R}^q, n)$ for all $n \in \mathbb{N}_0$. Let $\mathcal{H}_{q,n}$ be the subspace of $\mathbb{L}_2(S^{q-1})$ consisting of all $p \in \mathrm{HarmHomPol}(\mathbb{R}^q, n)$ restricted to S^{q-1}, named spherical harmonics of degree n, for all $n \in \mathbb{N}_0$. It is well-known that $\mathbb{L}_2(S^{q-1}) \simeq \oplus_{n=0}^{\infty} \mathcal{H}_{q,n}$. The spaces $\mathcal{H}_{q,n}$ are all functional Hilbert spaces. The corresponding reproducing kernels are denoted by Q_n^q and can be expressed in suitable normalized Gegenbauer polynomials as follows

$$Q_n^q(s,s') = \frac{q+2n-2}{n-2} C_n^{q/2-1}((s,s')), \tag{12.60}$$

for all $s, s' \in S^{q-1}$. For a polynomial $p \in \mathrm{Pol}(\mathbb{R}^q)$ we define the operator $p(D) : \mathrm{Pol}(\mathbb{R}^q) \to \mathrm{Pol}(\mathbb{R}^q)$ by $p(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$. For each spherical harmonic p_n of degree n there exists a homogeneous harmonic polynomial $\tilde{p}_n$ such that $\tilde{p}_n|_{S^{q-1}} = p_n$. Moreover,

$$(p_n, r_n)_q = \frac{\Gamma(\frac{1}{2}q)}{2^n \Gamma(n + \frac{1}{2}q)} \big(\overline{\tilde{p}_n(D)} \tilde{r}_n\big)(0), \tag{12.61}$$

for all $p_n, r_n \in \mathcal{H}_{q,n}$.

The space $\mathrm{Pol}(S^{q-1}, n) = \sum_{k=0}^{n} \mathcal{H}_{q,n}$ has the reproducing kernel

$$R_n^q(s,s') = \sum_{k=0}^{n} Q_k^q(s,s'), \tag{12.62}$$

for all $s, s' \in S^{q-1}$. This space consist of all harmonic polynomials on $\mathbb{R}^q$ of degree $\leq n$ restricted to S^{q-1}.

12.2 The transform by Sherman

In [S], Sherman introduces a transform on $\mathbb{L}_2(S^{q-1})$ which he regards as a Fourier transform on a sphere. For the inverse he provides a formula where a singular integral is involved. Our aim is to approach the same transform from the theory developed in this report so far. As a result we will show that the singularity in the formule for the inverse can be avoided.

Moreover, an alternative transform is considered, for which a simple Parseval identity is valid. Note that in [S] no such identity is present.

Let $a \in S^{q-1}$ and define $B = \{b \in S^{q-1} \, |(a,b) = 0\}$. Let $n \in \mathbb{N}_0$. Construct the set V_n by

$$V_n = \{e_{n,b} : s \mapsto (a - \mathrm{i}b, s)^n \mid b \in B\}. \tag{12.63}$$

The elements $e_{n,b}$ are spherical harmonics of order n for all $n \in \mathbb{N}$ and $b \in B$. In Theorem 12.23 we will prove that V_n is total in $\mathcal{H}_{q,n}$. Define the function of positive type K_n by

$$\begin{aligned} K_n(b,b') &= (e_{n,b}, e_{n,b'})_q = \frac{\Gamma(\frac{1}{2}q)}{2^n\Gamma(n+\frac{1}{2}q)}\big(\overline{\tilde{e}_{n,b}(D)}\tilde{e}_{n,b'}\big)(0) \\ &= \ldots = \frac{\Gamma(\frac{1}{2}q)n!}{2^n\Gamma(n+\frac{1}{2}q)}(1+(b,b'))^n, \end{aligned} \tag{12.64}$$

for all $b, b' \in B$. By Theorem 2.3, the frame transform $W_n : \mathcal{H}_{q,n} \to \mathbb{C}^B_{K_n}$ defined by

$$\big(W_n f\big)(b) = (e_{n,b}, f)_q, \tag{12.65}$$

for all $f \in \mathcal{H}_{q,n}, b \in \mathcal{B}$ is unitary. Obviously, W_n maps spherical harmonics of order n onto polynomial on B with degree at most n. Moreover, since $dim(H_{q,n}) = dim(\mathrm{HomPol}(\mathbb{R}^{q-1}), n) + dim(\mathrm{HomPol}(\mathbb{R}^{q-1}, n-1)) = dim(\mathrm{Pol}(S^{q-2}, n))$, it follows that the space $\mathbb{C}^B_{K_n}$ consists of all polynomials on B with degree at most n. Note that $\mathbb{C}^B_{K_n}$ coincides with $\mathrm{Pol}(S^{q-2}, n)$ as a vector space, but the norm on the both spaces is different (although topological equivalent). They are related by a transform.

Define the constants $\alpha_{n,k,q}$ by

$$\alpha_{n,k,q} = \int_{-1}^{1} (1+t)^n Q_k^{\frac{q-3}{2}}(t)(1-t^2)^{q/2-2} \,\mathrm{d}t, \tag{12.66}$$

for all $0 \le k \le n$. Then, the linear map $\Phi : \mathbb{C}^B_{K_n} \to \mathrm{Pol}(B, n)$ defined by

$$\big(\Phi f\big)(b) = \Big(\sum_{k=0}^{n} \alpha_{n,k,q}^{-1} Q_k^{\frac{q-3}{2}}(b,\cdot), f\Big)_{\mathrm{Pol}(B,n)}, \tag{12.67}$$

for all $f \in \mathbb{C}^B_{K_n}$ maps the reproducing kernel K_n onto R_n. Therefore the inner product on $\mathbb{C}^B_{K_n}$ is given by

$$(f,g)_{\mathbb{C}^B_{K_n}} = (\Phi f, g)_{\mathrm{Pol}(B,n)}, \tag{12.68}$$

for all $f, g \in \mathbb{C}^B_{K_n}$.

Define $\tilde{e}_{n,s} \in \mathrm{Pol}(B,n)$ by $\tilde{e}_{n,s}(b) = \overline{e_{n,b}(s)}$, for all $n \in \mathbb{N}_0$, $b \in B$ and $S \in S^{q-1}$. As a result we obtain the following identities,

$$f_n(s) = (Q_n^{q/2-1}(s,\cdot), f_n)_{H_{q,n}} = (\tilde{e}_{n,s}, W_n f_n)_{\mathbb{C}^B_{K_n}} = (\Phi(\tilde{e}_{n,s}), W_n f_n)_{\mathrm{Pol}(B,n)}, \tag{12.69}$$

for all $f_n \in \mathcal{H}_{q,n}$, $s \in S^{q-1}$ and $n \in \mathbb{N}_0$. Moreover,

$$Q_n^{q/2-1}(s, s') = (\Phi(\tilde{e}_{n,s}), \tilde{e}_{n,s'})_{\mathrm{Pol}(B,n)}, \tag{12.70}$$

for all $s, s' \in S^{q-1}$, $f \in \mathcal{H}_{q,n}$ and $n \in \mathbb{N}_0$. The second identity is to be regarded as the analogon of key Lemma 3.9 as formulated in the article by Sherman. The singularity of the function $e^*_{n,b}$ introduced by Sherman, does not occur in $\Phi(\tilde{e}_{n,s})$.

Finally, define the reproducing kernel K by

$$K(b, n, b', n') = \delta_{nn'} K_n(b, b'), \tag{12.71}$$

for all $b, b' \in B$ and $n, n' \in \mathbb{N}$. Then the frame transform $W : \mathcal{H} \to \mathbb{C}_K^{B\times\mathbb{N}}$ defined by $\big(Wf\big)(b,n) = \big(W_n f_n\big)(b)$ where $f = \sum_{n=0}^{\infty} f_n$ and $f_k \in \mathcal{H}_k$ for all $k \in \mathbb{N}$, is unitary. Summarizing we get the following relations

$$\big(Wf\big)(b,n) = \frac{1}{\sigma_{q-1}} \int_{S^{q-1}} \overline{e_{n,b}(s)} f(s) \, \mathrm{d}\sigma_{q-1}(s), \tag{12.72}$$

for all $f \in \mathbb{L}_2(S^{q-1})$, $b \in B$ and $n \in \mathbb{N}_0$. Moreover,

$$f(s) = \mathrm{l.i.m.}_{N\to\infty} \frac{1}{\sigma_{q-2}^2} \sum_{n=0}^{N} \int_B \overline{\big(\Phi\tilde{e}_{n,s}\big)(b)} \big(Wf\big)(b,n) \, \mathrm{d}\sigma_{q-2}(b) \tag{12.73}$$

for $f \in \mathbb{L}_2(S^{q-1})$ and almost all $s \in S^{q-1}$. Moreover, we also find the Parseval identity

$$\|f\|^2_{\mathbb{L}_2(S^{q-1})} = \sum_{n=0}^{\infty} \|W_n f_n\|^2_{\mathbb{C}^B_{K_n}} \tag{12.74}$$

for all $f = \sum_{n=0}^{\infty} f_n$ in $\mathbb{L}_2(S^{q-1}$. Note that in [S] such a result is not mentioned. But as long as $\mathbb{C}^B_{K_n}$ is not characterized in a more tangible way, this identity remains unpractical.

12.3 An application of Theorem 9.2

In this section we consider the transform in an abstract fashion. It provides a nice illustration of Theorem 9.2, by Grossmann, Morlet and Paul.

For the sake of simplicity let $a = (1, 0, \ldots, 0)$. Define $J_{a,q} = \{A \in SO(q) \mid Aa = a\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \tilde{A} \end{pmatrix} \mid \tilde{A} \in SO(q-1) \right\}$. Then $J_{a,q}$ can be identified with $SO(q-1)$ and $S^{q-1} \simeq SO(q)/J_{a,q}$.

The representation $\mathcal{U}^q : SO(q) \to \mathcal{B}(\mathbb{L}_2(S^{q-1})) : A \mapsto \mathcal{U}_A^{(q)}$ defined by $\big(\mathcal{U}^q_A f\big)(x) = f(A^{-1}x)$ for all $f \in \mathbb{L}_2(S^{q-1})$, for all $A \in SO(q)$ and almost all $x \in S^{q-1}$, is unitary. Moreover, the restriction of $\mathcal{U}$ onto $\mathcal{H}_{q,n}$ is irreducible for all $n \in \mathbb{N}$. Note that

$$\big(\mathcal{U}^q_A e_{n,b}\big)(s) = e_{n,b}(A^{-1}s) = (a - \mathrm{i}b, A^{-1}s)^n = (a - iAb, s)^n = e_{n,Ab}(s), \tag{12.75}$$

for all $s \in S^{q-1}, b \in B, n \in \mathbb{N}_0$ and $A \in J_{a,q}$.

Fix $n \in \mathbb{N}$. The subrepresentation $\mathcal{U}^q|_{J_{a,q}}$ is no longer irreducible, but $\mathcal{H}_{q,n}$ decomposes in $\mathcal{H}_{q,n} = \oplus_{j=1}^n \mathcal{H}_{q,n}^j$ such that $\mathcal{U}^q|_{J_{a,q}}$ is irreducible if restricted to $\mathcal{H}_{q,n}^j$. Denote $\mathbb{P}_{q,n}^j$ as the projection operator from $\mathcal{H}_{q,n}$ into $\mathcal{H}_{q,n}^j$. For a detailed discussion of this space $\mathcal{H}_{q,n}^j$ see [Mu, §11].

Now fix $b \in B = \{(0, \xi_{q-1}) \mid \xi_{q-1} \in S^{q-2}\}$. Apparently, the representation $\mathcal{U}^q|_{J_{a,q}}$ is cyclic with respect to $\mathcal{H}_{q,n}$, since the span of

$$\tilde{V}_n = \{\mathcal{U}_A^q e_{n,b} \mid A \in J_{a,q}\} = \{e_{n,Ab} \mid A \in J_{a,q}\} = \{e_{n,b'} \mid b' \in B\} = V_n, \tag{12.76}$$

is dense in $\mathcal{H}_{q,n}$ and hence $e_{n,b}$ is a cyclic vector. Note that for all $A \in J_{a,q}$ with the property $Ab = b$ we have $\mathcal{U}_A^q e_{n,b} = e_{n,b}$. These elements form a subgroup of $J_{a,q}$ and will be denoted by $J_{a,b,q}$. Note that $J_{a,q}/J_{a,b,q}$ can be identified with S^{q-2}. We will replace the index set $J_{a,q}$ by S^{q-2} in a moment.

Define for $j = 1, \ldots, n$ the function $\tilde{K}_n^j$ of positive type by $K_n^j(A, A') = (\mathcal{U}_A e_{n,b}, \mathcal{U}_{A'} e_{n,b})_{\mathcal{H}_{q,n}^J}$ and the wavelet transform $\tilde{W}_n^j : \mathcal{H}_{q,n}^j \to \mathbb{C}_{\tilde{K}_n^j}^{J_{a,q}}$ by

$$\big(\tilde{W}_n^j f\big)(A) = (\mathcal{U}_A^q \mathbb{P}_n^j e_{n,b}, f)_{\mathcal{H}_{q,n}^j}, \tag{12.77}$$

for all $f \in \mathcal{H}_n^j$ and $A \in J_{a,q}$, which is unitary by Theorem 2.3. Since $\mathcal{U}^q|_{J_{a,q}}$ restricted to $\mathcal{H}_{q,n}^j$ is unitary irreducible and trivially square integrable, the functional Hilbert space $\mathbb{C}_{K_n^j}^{J_{a,q}}$ is a closed subspace of $\mathbb{L}_2(J_{a,q})$ by Theorem 9.2, (the inner products are equal up to a constant). Note that $\tilde{W}_n^j f$ is constant along the orbits of $J_{a,b,q}$ acting on $J_{a,q}$.

As mentioned earlier we replace $J_{a,q}$ by $J_{a,q}/J_{a,b,q}$, which we immediately identify by S^{q-2}. Define the unitary wavelet transform $W_n^j : \mathcal{H}_{q,n}^j \to \mathbb{C}_{K_n^j}^{S^{q-2}}$ by

$$\big(W_n^j f\big)(\xi_{q-1}) = (\mathbb{P}_n^j e_{n,(0,\xi_{q-1})}, f)_{\mathcal{H}_{q,n}^j}, \tag{12.78}$$

for all $f \in \mathcal{H}_n^j$ and $\xi_{q-1} \in S^{q-2}$. Since $\tilde{W}_n^j$ maps $\mathcal{H}_{g,n}^j$ unitary onto a closed subspace of $\mathbb{L}_2(J_{a,q})$ and $\tilde{W}_n^j f$ is constant along the orbits of $J_{a,b,q}$ acting on $J_{a,q}$ for all $f \in \mathcal{H}_{q,n}^j$, we obtain that the functional Hilbert space $\mathbb{C}_{K_n^j}^{S^{q-2}}$ is a closed subspace of $\mathbb{L}_2(S^{q-2})$ (up to a constant) .

An important observation now is that the map W_n^j intertwines the representation $\mathcal{U}^q|_{J_{a,q}} : J_{a,q} \to \mathcal{B}(\mathcal{H}_{q,n}^j) : A \mapsto \mathcal{U}_A^q|_{\mathcal{H}_{q,n}^j}$ and the representation $\mathcal{U}^{q-1} : SO(q-1) \to \mathcal{B}(\mathbb{C}_{K_n^j}^{S^{q-1}}) : \tilde{A} \mapsto \mathcal{U}_{\tilde{A}}^{q-1}|_{\mathbb{C}_{K_n^j}^{S^{q-1}}}$, i.e. $W_n \mathcal{U}_A^q|_{\mathcal{H}_{q,n}^j} = \mathcal{U}_{\tilde{A}}^{q-1} W_n$ for all $A = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{A} \end{pmatrix}$ and $\tilde{A} \in SO(q-1)$. Indeed,

$$\begin{aligned}
\big(W_n^j \mathcal{U}_A^q|_{\mathcal{H}_{q,n}^j} f\big)(\xi_{q-1}) &= (\mathbb{P}_n^j e_{n,(0,\xi_{q-1})}, \mathcal{U}_A^q|_{\mathcal{H}_{q,n}^j} f)_{\mathcal{H}_{q,n}^j} = (\mathcal{U}_{A^{-1}}^q|_{\mathcal{H}_{q,n}^j} \mathbb{P}_n^j e_{n,(0,\xi_{q-1})}, f)_{\mathcal{H}_{q,n}^j} \\
&= (\mathbb{P}_n^j \mathcal{U}_{A^{-1}}^q e_{n,(0,\xi_{q-1})}, f)_{\mathcal{H}_{q,n}^j} = (\mathbb{P}_n^j e_{n,A^{-1}(0,\xi_{q-1})}, f)_{\mathcal{H}_{q,n}^j} \\
&= (\mathbb{P}_n^j e_{n,(0,\tilde{A}^{-1}\xi_{q-1})}, f)_{\mathcal{H}_{q,n}^j} = \big(\mathcal{U}_{\tilde{A}}^{q-1} W_n f\big)(\xi_{q-1}),
\end{aligned} \tag{12.79}$$

for all $\xi_{q-1} \in S^{q-1}$, $f \in \mathcal{H}^j_{q,n}$ and for all $A = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{A} \end{pmatrix}$ and $\tilde{A} \in SO(q-1)$. Since the restriction of $\mathcal{U}^q|_{J_{a,q}} : J_{a,q} \to \mathcal{H}^j_{q,n}$ is irreducible and unitary and moreover W^j_n is unitary, the representation $\mathcal{U}^{q-1} : SO(q-1) \to \mathbb{C}^{S^{q-1}}_{K^j_n}$ is also irreducible and unitary. Therefore $\mathbb{C}^{S^{q-2}}_{K^j_n}$ is exactly the space $\mathcal{H}_{q-1,k}$ for some $k \in \mathbb{N}$. Since $\dim \mathcal{H}^j_{q,n} = \dim \mathcal{H}_{q-1,j}$ (see [Mu, §11]) and all the dimensions of $\mathcal{H}_{q-1,k}$ differ for $k \in \mathbb{N}$, it must be the case that $k = j$.

Summarizing, we can conclude that W^j_n is a unitary map (up to a constant) from $\mathcal{H}^j_{q,n}$ into $\mathcal{H}_{q-1,j}$. I.e. there exist a $c^j_{q,n} > 0$ such that

$$(f, g)_{\mathcal{H}^j_{q,n}} = c^j_{q,n} (W^j_n f, W^j_n g)_{\mathcal{H}_{q-1,j}}, \tag{12.80}$$

for all $f, g \in \mathcal{H}^j_{q,n}$. Naturally, the numbers $c^j_{q,n}$ are related to the coefficients $\alpha_{n,k,q}$ in (12.66).

From the transform W^j_n we can built the transform W_n on $\mathcal{H}_{q,n}$, and finally the transform W on $\mathbb{L}_2(S^{q-1})$.

12.4 An approach using a special orthonormal basis

First we give a summary of the most important result in §10 and §11 of [Mu]. Note that we use some different conventions. Let $q \geq 3$ and consider the coefficient $B^j_n(t)$ of the expansion

$$(t + \mathrm{i}s\sqrt{1-t^2})^n = \sum_{j=0}^{n} B^j_n(t) C_j^{\frac{q-3}{2}}(s), \tag{12.81}$$

for all $s, t \in (-1, 1)$. By the classical orthogonality relations, the formula of Rodriguez and integral representation for the Gegenbauer polynomials, one obtains

$$B^j_n(t) = a_{j,n,q}(1-t^2)^{j/2} C_{n-j}^{\frac{q+2j-3}{2}}(t), \tag{12.82}$$

for all $t \in [-1, 1]$ and some constant $a_{j,n,q}$. For fixed $j \in \mathbb{N}$ the functions $\{B^j_n(q;t) \mid n \geq j\}$ are orthogonal in the following way

$$\int_{-1}^{1} B^j_n(q;t) B^j_m(q;t)(1-t^2)^{\frac{q-3}{2}} \, \mathrm{d}t = 0, \tag{12.83}$$

if $n \neq m$. Let $A^j_n = \mathrm{i}^{-j} \beta_{n,j,q} B^j_n$, where $\beta_{n,j,q}$ is chosen such that

$$\int_{-1}^{1} A^j_n(q;t) A^j_m(q;t)(1-t^2)^{\frac{q-3}{2}} \, \mathrm{d}t = \delta_{nm}, \tag{12.84}$$

for $n, m \geq j$. The factors i^{-j} make the function A^j_n real-valued. The functions A^j_n are the building blocks for orthonormal basis for the space $\mathcal{H}_{q,n}$. Use the following coordinates for elements on the sphere S^{q-1}

$$\xi_q = (t; \sqrt{1-t^2}\xi_{q-1}), \tag{12.85}$$

where $\xi_{q-1} \in S^{q-2}$ and $t \in [-1, 1]$. The following is the key result of §11 in [Mu].

Theorem 12.22 *If for $m = 0, 1, \ldots, n$ orthonormal bases $\{Y_{m,j} \mid j = 1, \ldots, N(q-1, m)\}$ for $\mathcal{H}_{q-1,m}$ are given then*

$$\{(t; \sqrt{1-t^2}\xi_{q-1}) \mapsto A_n^m(q;t)Y_{m,j}(\xi_{q-1}) \mid j = 1, 2 \ldots N(q-1, m) \quad m = 0, 1, \ldots n\}, \tag{12.86}$$

is an orthonormal basis of $\mathcal{H}_{q,n}$.

The set $\{(x, y) \mapsto (x + \mathrm{i}y)^n, (x, y) \mapsto (x - \mathrm{i}y)^n\}$ is an orthonormal basis for $\mathcal{H}_{2,n}$. From this basis we can construct by induction a basis for $\mathcal{H}_{q,n}$ for all $q \geq 3$ and $n \in \mathbb{N}_0$.

Now we proceed by defining a wavelet transform on $\mathbb{L}_2(S^{q-1})$. The group $SO(q-2)$ has a natural action on S^{q-2} and therefore it has the natural action on S^{q-1} given by

$$A(t; \sqrt{1-t^2}\xi_{q-1}) = (t, \sqrt{1-t^2}A\xi_{q-1}), \tag{12.87}$$

for all $A \in SO(q-1)$, $\xi_{q-1} \in S^{q-2}$ and $t \in [-1, 1]$. Let $\mathcal{U}^{(q)} : SO(q-1) \to \mathcal{B}(\mathbb{L}_2(S^{q-1}))$ be the representation induced by this action. Define $\mathcal{U}^{(q,n)} : SO(q-1) \to \mathcal{H}_{q,n}$ by $\mathcal{U}_A^{(q,n)} f = \mathcal{U}_A^{(q)} f$, for all $A \in SO(q-1)$ and $f \in \mathcal{H}_{q,n}$. Next we search for cyclic wavelets for the representations $\mathcal{U}^{(q,n)}$. Let $\psi_n \in \mathbb{L}_2(S^{q-1})$ and write

$$\psi_n = \sum_{m=0}^{n} \sum_{j=0}^{N(q-1,m)} \alpha_{mj} A_n^m Y_{m,j}, \tag{12.88}$$

where the function $Y_{m,j}$ and A_n^m are defined as above. A useful criteria for the coefficients α_{mj} such that ψ_n is a cyclic wavelet is the following.

Theorem 12.23 *Let $n \in \mathbb{N}$. If there exists $\tilde{\alpha}_m$ and $b \in S^{q-2}$ such that*

$$\alpha_{mj} = \tilde{\alpha}_m \overline{Y_{m,j}(b)} \tag{12.89}$$

for all $j = 1, 2, \ldots, N(q-1, m)$ and $m = 0, 1 \ldots, n$ then ψ defined by (12.88) simplifies to

$$\psi_n(t; \sqrt{1-t^2}\xi_{q-1}) = \sum_{m=0}^{n} \tilde{\alpha}_m A_n^m(t) Q_m^{\frac{q-3}{2}}(b, \xi_{q-1}), \tag{12.90}$$

for all $(t; \sqrt{1-t^2}\xi_{q-1}) \in S^{q-1}$. Moreover, if $\tilde{\alpha}_m \neq 0$ for all $m = 0, 1, \ldots n$ then ψ is a cyclic wavelet for $\mathcal{U}^{(q,n)}$.

Proof:
First note that

$$\psi_n = \sum_{m=0}^{n} \sum_{j=0}^{N(q-1,m)} \alpha_{mj} A_n^m Y_{m,j} = \sum_{m=0}^{n} \tilde{\alpha}_m A_n^m \sum_{j=0}^{N(q-1,m)} \overline{Y_{m,j}(b)} Y_{m,j} = \sum_{m=0}^{n} \tilde{\alpha}_m A_n^m Q_m^{\frac{q-3}{2}}(b, \cdot).$$

Moreover, $\big(\mathcal{U}_A^{(q,n)}\psi_n\big)(t;\sqrt{1-t^2}\xi_{q-1}) = \tilde{\alpha}_m A_n^m(t) Q_m^{\frac{q-3}{2}}(Ab,\xi_{q-1})$ for all $(t;\sqrt{1-t^2}\xi_{q-1}) \in S^{q-1}$ and $A \in SO(q-1)$. Let $f \in \mathcal{H}_{q,n}$ and write

$$f = \sum_{m=0}^{n} \sum_{j=0}^{N(q-1,m)} \beta_{mj} A_n^m Y_{m,j}.$$

Then

$$(\mathcal{U}_A^{(q,n)}\psi_n, f) = \sum_{m=0}^{n} \sum_{j=0}^{N(q-1,m)} \overline{\tilde{\alpha}_m} \beta_{mj} Y_{m,j}(Ab),$$

for all $A \in SO(q-2)$. Assume now that $f \in \{\mathcal{U}_A^{(q,n)}\psi \mid A \in SO(q-1)\}^{\perp}$. Since $\{Y_{m,j} \mid j = 1,2\dots N(q-1,m) \quad m = 0,1,\dots n\}$ is an orthonormal basis for $\mathrm{Pol}(S^{q-2},n)$ and by the transitivity of the action of $SO(q-1)$ on S^{q-2}, it now follows that $\tilde{\alpha}_m\beta_{mj} = 0$ for all $j = 0,1,\dots,N(q-1,m)$ and $m = 0,1,\dots n$. Therefore $\beta_{mj} = 0$ for all $j = 0,1,\dots,N(q-1,m)$ and $m = 0,1,\dots n$ and hence $f = 0$. □

Note that the procedure of the Section 12.3 holds for any wavelet ψ which satisfies all the condition of Lemma 12.23. Therefore their associated transforms all isometrically (up to a constant) map $\mathcal{H}_{q,n}^j$ onto $\mathcal{H}_{q-1,n}$. These constants however can differ.

Lemma 12.23 proves that the set $\{e_{n,b} \mid b \in B\}$ introduced by Sherman, has a dense span in $\mathcal{H}_{q,n}$ for all $n \in \mathbb{N}_0$. Moreover, the function K of positive type ie represented by

$$K_n(b,b) = \sum_{k=0}^{n} |\beta_{k,n,q}|^{-2} Q_k^{\frac{q-3}{2}}(b,b'), \tag{12.91}$$

for all $b, b' \in B$. Hence we see that $|\beta_{n,k,q}|^{-2} = \alpha_{n,k,q}$.

Next we introduce a different transform based on a cyclic wavelet which satisfies the condition of Lemma 12.23 in a trivial way. Let $n \in \mathbb{N}_0$. Let $\phi_{n,b} \in \mathcal{H}_{n,q}$ be defined by

$$\phi_{b,n}(t;\sqrt{1-t^2}\xi_{q-1}) = \sum_{m=0}^{n} A_n^m(t) Q_m^{\frac{q-3}{2}}((b,\xi_{q-1})), \tag{12.92}$$

for all $t \in [-1,1]$, $\xi_{q-1} \in S^{q-2}$ and $b \in S^{q-2}$. By Theorem 12.23 the set $\{\phi_{n,b} \mid b \in S^{q-2}\}$ has dense span in $\mathcal{H}_{q,n}$. Since

$$(\phi_{b,n}, \phi_{b',n})_{\mathcal{H}_{n,q}} = \sum_{m=0}^{n} Q_m^{\frac{q-3}{2}}((b,b')) = R(b,b'), \tag{12.93}$$

for all $b, b' \in S^{q-1}$ and by Theorem 2.3, it follows that the frame transform defined by

$$\big(\tilde{W}_n f\big)(b) = (\phi_{b,n}, f)_{\mathcal{H}_{n,q}} \tag{12.94}$$

for all $b \in S^{q-2}$ and $f \in \mathcal{H}_{q,n}$ is a unitary map from $\mathcal{H}_{q,n}$ onto $\mathrm{Pol}(S^{q-2}, b)$. Now define $\tilde{\phi}_{n,s}$ by $\tilde{\phi}_{n,s}(b) = \big(\tilde{W}_n Q_{n,s}^{\frac{q-1}{3}}\big)(b) = \overline{\phi_{n,b}(s)}$. Then

$$\big(W_n^{-1} F\big)(s) = (\tilde{\phi}_{n,s}, F)_{\mathrm{Pol}(S^{q-2},n)}, \tag{12.95}$$

for all $s \in S^{q-1}$ and $F \in \mathrm{Pol}(S^{q-2}, n)$.

Finally, define the function K of positive type on $S^{q-2} \times \mathbb{N}_0$ by $K(b, n, b', n') = K_n(b, b')\delta_{nn'}$ for all $n, n' \in \mathbb{N}_0$ and $b, b' \in S^{q-1}$.

Theorem 12.24 *The transform* $\tilde{W} : \mathbb{L}_2(S^{q-1}) \to \mathbb{C}_K^{S^{q-2} \times \mathbb{N}_0}$ *defined by*

$$\big(\tilde{W} f\big)(b, n) = (\tilde{\phi}_{n,b}, f)_{\mathbb{L}_2(S^{q-1})} \tag{12.96}$$

is unitary. The inverse is defined by

$$f(s) = \mathrm{l.i.m.}_{N\to\infty} \sum_{n=0}^{N} (\tilde{\phi}_{n,s}, \tilde{W}_n f)_{\mathrm{Pol}(S^{q-2},n)}, \tag{12.97}$$

for almost all $s \in S^{q-1}$ *and all* $f \in \mathbb{L}_2(S^{q-1})$ *where* f_n *is the projection of* f *on* $\mathcal{H}_{q,n}$ *for all* $n \in \mathbb{N}_0$. *Moreover,*

$$\|f\|^2_{\mathbb{L}_2(S^{q-1})} = \sum_{n=0}^{\infty} \|W_n f_n\|^2_{\mathrm{Pol}(S^{q-2},n)}, \tag{12.98}$$

for all $f \in \mathbb{L}_2(S^{q-1})$, *where* f_n *is the projection of* f *on* $\mathcal{H}_{q,n}$.

Question: Does there exist a more tangible expression for the function $\phi_{n,b}$?

A Schur's lemma

Schur's lemma is mostly known for the special case of irreducible representations $\mathcal{U}$ on finite dimensional spaces or irreducible representations of compact groups. In these cases the proof is straightforward. The main idea is that if A has an eigen-value, then the eigen space is invariant under $\mathcal{U}_g$, which follows by the assumption $\mathcal{U}_g A = A\mathcal{U}_g$, and by irreducibility of $\mathcal{U}$ it then follows that $\overline{E_\lambda} = E_\lambda = \mathcal{H}$. Nevertheless, Schur's lemma has serious consequences such as the orthogonality relations by Weyl for compact groups. We will give a generalization of this theorem which is applied in Theorem 9.2 and which is formulated as an exercise in [D, vol.V, pp.21].

Theorem A.1 (Schur's Lemma) *Let G be a group and let $g \mapsto \mathcal{U}_g$ be a unitary irreducible representation of G in a Hilbert space $\mathcal{H}$. If A is a closed operator on $\mathcal{H}$ such that*

$$\mathcal{U}_g A f = A\mathcal{U}_g f \qquad \text{for all } g \in G \text{ , } f \in \mathcal{D}(A) \text{ ,}$$

then $A = cI$ for some $c \in \mathbb{C}$.

Proof:
First we will prove the theorem for a self-adjoint bounded operator A. Note that $\mathcal{D}(A) = \mathcal{H}$. It follows from the spectral theorem for self-adjoint operators that A is in the norm closure of the linear span V of all orthogonal projections P commuting with all the bounded operators commuting with A. In particular $\mathcal{U}_g$ is a bounded operator commuting with A and therefore every $P \in V$ commutes with $\mathcal{U}_g$. Therefore the space on which P projects (which is closed since it equals the $\mathcal{N}(I-P)$) is invariant under $\mathcal{U}_g$. But $\mathcal{U}$ was supposed to be irreducible and therefore this space equals $\mathcal{H}$ or $\{0\}$, i.e. $P = 0$ or $P = I$. Since A is within the span of such P, we have that $A = cI$, for some constant $c \in \mathbb{C}$.

Assume now that $A \in \mathcal{B}(\mathcal{H})$. By unitarity of $\mathcal{U}_g$

$$(f_1, A^*\mathcal{U}_g f_2)_\mathcal{H} = (\mathcal{U}_{g^{-1}} A f_1, f_2)_\mathcal{H} = (A\mathcal{U}_{g^{-1}} f_1, f_2)_\mathcal{H} = (f_1, \mathcal{U}_g A^* f_2)_\mathcal{H},$$

for all $g \in G$ and $f_1, f_2 \in \mathcal{H}$. Therefore, $\mathcal{U}_g$ commutes with A^* and thus with $A + A^*$ and $\mathrm{i}(A - A^*)$ (which are both self-adjoint) for all $g \in G$. Hence there exists $c_1, c_2 \in \mathbb{C}$ such that

$$A = \frac{A + A^*}{2} + \frac{A - A^*}{2} = \frac{A + A^*}{2} + \frac{\mathrm{i}(A - A^*)}{2\mathrm{i}} = c_1 I + \frac{c_2}{\mathrm{i}} I$$

and thus the result follows for any bounded operator A on a Hilbert space A.

Finally, let A be any closed operator commuting with $\mathcal{U}$. The domain $\mathcal{D}(A)$ is invariant under $\mathcal{U}$, therefore by the irreducibility of $\mathcal{U}$ it follows that $\mathcal{D}(A)$

is dense in $\mathcal{H}$. Since A is closed and densely defined, the adjoint A^* is well-defined. Moreover, $A^*\mathcal{U}_g = \mathcal{U}_g A^*$ for all $g \in G$ by a same argument as in the case $A \in \mathcal{B}(\mathcal{H})$. By closedness of A, the domain $\mathcal{D}(A)$ is a Hilbert space (say $\mathcal{D}_A$) equipped with inner product

$$(f_1, f_2)_A = (f_1, f_2)_{\mathcal{H}} + (Af_1, Af_2)_{\mathcal{H}} = (f_1, (I + A^*A)f_2)_{\mathcal{H}},$$

for all $f_1, f_2 \in \mathcal{D}_A$.

Define the representation $\tilde{\mathcal{U}} : g \mapsto \tilde{\mathcal{U}}_g$ of G in $\mathcal{D}_A$ by $\tilde{\mathcal{U}}_g f = \mathcal{U}_g f$, for all $f \in \mathcal{D}_A$ and $g \in G$. Since $\mathcal{U}$ is unitary and $\mathcal{U}_g$ commutes with A for all $g \in G$, it follows that $\tilde{\mathcal{U}}$ is unitary. Moreover, $\tilde{\mathcal{U}}$ is irreducible. Indeed, if V is a non-trivial closed subspace of $\mathcal{D}_A$ and $f_1 \in V$ with $f_1 \neq 0$. Let $f_2 \in V^\perp$, then

$$0 = (f_2, \tilde{\mathcal{U}}_g f_1)_A = (f_2, (1 + A^*A)\mathcal{U}_g f_1)_{\mathcal{H}} = (f_2, \mathcal{U}_g (1 + A^*A) f_1)_{\mathcal{H}},$$

for all $g \in G$. Hence $f_2 = 0$ by the irreducibility of $\mathcal{U}$. Thus $V = \mathcal{D}_A$.

Obviously, the operator $\tilde{A} : \mathcal{D}_A \to \mathcal{H}$ given by $\tilde{A}f = Af$ is a bounded operator and satisfies $\tilde{A}\tilde{\mathcal{U}}_g = \mathcal{U}_g \tilde{A}$ for all $g \in G$. Moreover, $\tilde{A}^*\mathcal{U}_g = \tilde{\mathcal{U}}_g \tilde{A}^*$ for all $g \in G$ by a same argument as in the case $A \in \mathcal{B}(\mathcal{H})$. As a result the operator $\tilde{A}^*\tilde{A} : \mathcal{D}_A \to \mathcal{D}_A$ is a bounded operator on the Hilbert space $\mathcal{D}_A$ commuting with $\tilde{\mathcal{U}}_g$ for all $g \in G$. As a result we have by the preceding that $\tilde{A}^*\tilde{A} = dI$, but then we have $(\tilde{A}f, \tilde{A}f)_{\mathcal{H}} = d(f, f)_A$ and therefore

$$\frac{1}{d}(Af, Af)_{\mathcal{H}} = (f, f)_{\mathcal{H}} + (Af, Af)_{\mathcal{H}} \Leftrightarrow (Af, Af)_{\mathcal{H}} = |c|^2 (f, f)_{\mathcal{H}},$$

for all $f \in \mathcal{D}_A$, with $|c|^2 = d/(1 - d)$. Now A is a closed operator, $\mathcal{D}(A)$ is dense in $\mathcal{H}$ and A is bounded. Hence $\mathcal{D}(A) = \mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$. As a result A is is equal to cI by the previous part of the proof for some $c \in \mathbb{C}$, with $|c|^2 = d/(1 - d)$. □

See [Ta, Prop. 0.4.5] for a more general version of the Schur's Lemma.

B Gelfand triples

Let $\mathcal{H}$ be a complex Hilbert space and R an unbounded, positive and self-adjoint operator on $\mathcal{H}$, for which the inverse $R^{-1} \in \mathcal{B}(\mathcal{H})$. Note that the boundedness of R^{-1} implies that $D(R^{-1}) = \mathcal{H}$ and hence $D(R) = R^{-1}(\mathcal{H})$.

Define the space $\mathcal{H}_I$ as the linear space $D(R)$ equipped with the inner product $(f,g)_I = (Rf, Rg)_{\mathcal{H}}$ for all $f, g \in \mathcal{H}$. Since R is closed and R^{-1} is bounded, $\mathcal{H}_I$ is a Hilbert space. Next define the Hilbert space $\mathcal{H}_{-I}$ as the completion of $\mathcal{H}$ equipped with the norm $(f,g)_{-I} = (R^{-1}f, R^{-1}g)_{\mathcal{H}}$.

The operator R on $\mathcal{H}$ induces the map $\tilde{R} : \mathcal{H}_I \to \mathcal{H}$ by $\tilde{R}f = Rf$ for all $f \in \mathcal{H}_I = D(R)$. Since $\|\tilde{R}f\|_{\mathcal{H}} = \|f\|_I$ for all $f \in \mathcal{H}_I$, the map $\tilde{R}$ is on isometry. By boundedness of R^{-1}, it follows that $\tilde{R}$ is also surjective and hence a unitary map.

Define $\check{R} : D(R) \to \mathcal{H}_{-I}$ by $\check{R}f = Rf$ for all $f \in D(R)$. Since $\|\check{R}f\|_{-I} = \|f\|_{\mathcal{H}}$ for all $f \in D(R)$, the map $\check{R}$ is closable and its extension is an isometry. Since $R(D(R)) = \mathcal{H}$ and $\mathcal{H}$ is dense in $\mathcal{H}_{-I}$ the closure is also surjective, hence a unitary map. Write $\tilde{\tilde{R}}$ for the closure of $\check{R}$.

Hence the following triple is obtained

$$\mathcal{H}^I \overset{\tilde{R}}{\hookrightarrow} \mathcal{H} \overset{\tilde{\tilde{R}}}{\hookrightarrow} \mathcal{H}^{-I}. \tag{B.1}$$

A triple of this type is called a Gelfand triple.

It follows by the Riesz representation theorem and the unitarity of $\tilde{R}$ and $\tilde{\tilde{R}}$ that the space $\mathcal{H}_{-I}$ is naturally isomorphic to the anti-dual space of $\mathcal{H}_I$ under the pairing

$$\langle F, f \rangle = (\tilde{\tilde{R}}^{-1} F, \tilde{R}f)_{\mathcal{H}} \tag{B.2}$$

for all $F \in \mathcal{H}_{-I}$ and $f \in \mathcal{H}_I$. Note that by the selfadjoint-ness of R

$$\langle F, f \rangle = (F, f)_{\mathcal{H}} \tag{B.3}$$

if $F \in \mathcal{H}$ for all $f \in \mathcal{H}^I$. In this paper R, $\tilde{R}$ and $\tilde{\tilde{R}}$ are all denoted by the same symbol R. From the context it is clear which operator is meant by this symbol.

Example: Let $k \in \mathbb{N}$. Then it is well-known that the operator $D_k = 1 + |\triangle|^k$ with domain $H^{2k,2}(\mathbb{R}^2)$ is an unbounded, positive and self-adjoint operator on $\mathbb{L}_2(\mathbb{R}^2)$ with bounded inverse. Define the Gelfand triple

$$H^{2k,2}(\mathbb{R}^2) \hookrightarrow \mathbb{L}_2(\mathbb{R}^2) \hookrightarrow H^{-2k,2}(\mathbb{R}^2). \tag{B.4}$$

For a detailed discussion of these spaces, see [Y, §I.10, pp.56].

C An open problem

C.1 Introduction

Let Ω be an open connected subset of $\mathbb{C}$ and denote the space of all analytic functions on Ω by $\mathcal{A}(\Omega)$. Let $Z = \{a_n \mid n \in \mathbb{N}\}$ be a countable set of points in Ω and $w : Z \to (0, \infty)$

a positive valued function on $\mathbb{N}$. Finally, let $\mathcal{H} \subset \mathcal{A}(\Omega)$ consist of all $f \in \mathcal{A}(\Omega)$ with

$$\sum_{n\in\mathbb{N}} |f(a_n)|^2 w(a_n) < \infty. \tag{C.1}$$

Consider the questions: Under what conditions

1. is $\mathcal{H}$ a pre-Hilbert space under the inner-product $(\cdot,\cdot) : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ defined by

$$(f,g)_{\mathcal{H}} = \sum_{n\in\mathbb{N}} \overline{f(a_n)} g(a_n) w(n), \tag{C.2}$$

   for all $f, g \in \mathcal{H}$?

2. is $\mathcal{H}$ a Hilbert space, i.e. is $\mathcal{H}$ complete?

3. is $\mathcal{H}$ a functional Hilbert space?

If $\mathcal{H}$ is a pre-Hilbert space, but not a Hilbert space,

4. can we characterize the completion of $\mathcal{H}$?

This is still a challenging open problem. We will make some remarks about it.

The space $\mathcal{H}$ is a pre-Hilbert space if and only $f(a_n) = 0$ for all $n \in \mathbb{N}$ implies $f = 0$. A sufficient condition is that the set $Z = \{a_n \mid n \in \mathbb{N}\}$ has an accumulation point in Ω, since the set of zeros of an analytic function has no accumulation points. In case $\Omega = \mathbb{C}$, another sufficient condition is that

$$\sum_{n=1}^{\infty} |a_n|^{-1-h} = \infty, \tag{C.3}$$

for some $h \in \mathbb{N}$, but then the space $\mathcal{A}(\Omega)$ must at least be restricted to the subspace of entire function of order $\lambda < h + 1$.

The choice of Ω is troublesome. For example, replace Ω by a subset $\Omega_1 \subset \Omega$ such that $\{a_n \mid n \in \mathbb{N}\}$ is still contained in Ω_1. How are the two spaces $\mathcal{A}(\Omega)$ and $\mathcal{A}(\Omega_1)$ related?

Note that if the space $\mathcal{H}$ is indeed a Hilbert space, then it straightforwardly follows that point-evaluation on elements of Z is continuous. Hence there exist $K_{a_n} \in \mathcal{H}$ such that

$$f(a_n) = (K_{a_n}, f)_{\mathcal{H}}, \tag{C.4}$$

for all $f \in \mathcal{H}$. Moreover, the set $\{K_{a_n} \mid n \in \mathbb{N}\}$ is total in $\mathcal{H}$.

C.2 q-functions

A challenging example concerns the special q-functions, as mentioned in Section 6.3. Consider the space $\mathcal{A}(\mathbb{C})$ consisting of analytic functions on $\mathbb{C}$ and let $0 < q < 1$. Define $Z = \{q^n \mid n \in \mathbb{N}_0\}$ and $w : Z \to (0, \infty)$ by $w(q^n) = q^n$. Condition (C.1) is now given by

$$\sum_{n=0}^{\infty} |f(q^n)|^2 q^n < \infty, \tag{C.5}$$

and $\mathcal{H}$ is the set of all element of $\mathcal{A}(\mathbb{C})$ which satisfy (C.5). Since Z has accumulation point it is clear that $\mathcal{H}$ now is pre-Hilbert space.

This example is a nice illustration that the set Ω (which we have chosen to be $\mathbb{C}$), must contain the point 0. If for example Ω was chosen to be the open unit disk around $z = 1$, denoted by $D_{1,1}$. Then $g : D_{1,1} \to \mathbb{C}$ given by

$$g(z) = \sin \pi \frac{\log z}{\log q}, \tag{C.6}$$

for all $z \in D_{1,1}$ defines an analytic function on $D_{1,1}$, which is zero on all elements of Z. Therefore it satisfies condition (C.5) and belongs to $\mathcal{H}$, but has norm equal to zero. Hence $\mathcal{H}$ fails to be a pre-Hilbert space.

Although the space $\mathcal{H}$ is a pre-Hilbert space, it is not a Hilbert space. For example, consider the sequence $\{f_n \in \mathcal{H}\}_{n\in\mathbb{N}}$ defined by

$$f_n(z) = \prod_{k=1}^{n} \frac{z - q^k}{1 - q^k}, \tag{C.7}$$

for all $n \in \mathbb{N}$ and $z \in \mathbb{C}$. Since there exists a $N \in \mathbb{N}$ such that $|\frac{z-q^k}{1-q^k}| \leq \frac{\min(q^k, 1-q^k)}{1-q^k} \leq 1$ for all $k \geq N$ and $z \in (0, 1)$, we obtain $|f_n(z)| \leq |f_N(z)| \leq \max_{z\in[0,1]} |f_N(z)| =: A$ for all $z \in (0, 1)$ and $n \geq N$. Moreover, $f_n(q^k) = 0$ for all $1 \leq k \leq n$ and $f_n(1) = 1$. Now

$$\|f_n - f_m\|^2_{\mathcal{H}} = \sum_{k=\min(n,m)+1}^{\infty} |f_n(q^k) - f_m(q^k)|^2 q^k \leq 4A^2 \sum_{k=\min(n,m)+1}^{\infty} q^k = 4A^2 \frac{q^{\min(n,m)+1}}{1-q}, \tag{C.8}$$

for all $n, m \geq N$. Hence $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. But it does not have a limit in $\mathcal{H}$. Indeed, if $f \in \mathcal{H}$ is the limit of $\{f_n\}_{n\in\mathbb{N}}$, then $f(q^k) = 0$ for all $k \in \mathbb{N}$ and $f(1) = 1$. Therefore f is non-zero and the set of zeros has an accumulation point. Hence f is not analytic.

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References

[Ab] L.D. Abreu, *A q-sampling theorem related to the q-hankel transform*, (Pré-Publicacoes do Departemento de Matemática, Universidade de Coimbra, Pre-print number 03 28)

[Ah] L.V. Ahlfors, *Complex analysis*, (McGraw-Hill, second edition 1966)

[An] T. Ando, *Reproducing Kernel Spaces and Quadratic Inequalities* (Sapporo, Hokkaido University, 1987)

[Ar] N. Aronszajn, Theory of reproducing kernels, *Trans. A.M.S.*,**68** (1950), pp 337-404.

[AS] M. Abramowitz and I.A. Stegun, *Handbook of mathematical funtions* , (Dover Publ. New York, 1970).

[B] V. Bargmann, On A Hilbert Space of Analytic Functions and an associated Integral Transform (I), *Communcations on pure and applied mathematics XIV* (1961) pp. 187-214

[C] C.V.L. Charlier, *Les applications de la théorie des probabilit'es aux sciences math'ematiques et aux sciences physiques; Application de la théorie des probabiltiés à l'astronomie* (Paris, 1930, pp. 52)

[D] J. Dieudonné, *Treatise on Analysis* (Academic Press New York, vol II & V, 1976).

[EG1] S.J.L. van Eijndhoven and J. de Graaf, A mathematical interpretation of Dirac's formalism part a-b-c, *Reports on Mathematical Physics* (22, no. 2,1985)

[EG2] S.J.L. Eijndhoven and J. de Graaf, A measure theoretical Sobolev Lemma, *Journal of Functional Analysis* (63(1),1985,pp. 74-85)

[Fa] R.C. Fabec, *Fundamentals of infinite dimensional representation theory*, (Chapman & Hall/CRC, 2000).

[FH] J. Faraut and K. Harzallah. *Deux cours d' analyse harmonique: Ecole d'Été d'Analyse Harmonique* (Birkhauser Basel,1987)

[FKNP] M. Fashandi, R.A. Kamyabi Gol, A. Nikman, and M. A. Pourabdollah, Continuous wavelet trnasform on a special homogeneous space, *Jour. of Math. Phys. 44(9)* (2003), pp 4260-4266.

[FM] Hartmut Führ, Matthias Mayer, Continuous Wavelet Transforms from Cyclic Representations: A General Approach Using Plancherel Measure, *Journal of Fourier Analysis and Applications* 8 (2002), pp. 375-398.

[Fo] Folland, G.B., *A course in abstract Harmonic analysis*, (CRC Press, Boca Raton, FL, 1995).

[GMP] A. Grosmann, J. Morlet and T. Paul, Transform associated to square integrable group representations, *Jour of Math. Phys. 23(10)* (1985), pp 2473-2479.

[H] J.R. Higgins, *An interpolation series associated with the Bessel-Hankel transform*, (J. London Math. Soc. (2), 5 (1972), 707-714)

[KHV] S. N. Kalitzin, B.M. ter Haar Romeny and M.A. Viergever, Invertible apertured orientation filters in image analysis, (31(2/3): 145-158, 1999)

[Ma] F.J.L. Martens, *Spaces of analytic functions on inductive/projective limits of Hilbert Spaces*, (1988). [1]

[MOS] W. Magnus, F. Oberhettinger and R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, (Springer, 1996)

[Mu] C. Müller, *Analysis of spherical symmetries in Euclidean spaces*, (Springer Berline, 1998)

[O] F.W.J. Olver, *Asymptotics and special functions*, (Academic Press, New York 1974)

[RS] H. Reiter and J.D.Stegeman, *Classical harmonic analysis and locally compact groups* (Oxford at the Clarendon press, 2000).

[Ta] M.E.Taylor, *Noncommutative Harmonic Analysis*, (American Mathematical Society , 1985)

[Te] N.M. Temme, *Special functions: an introduction to the classical functions of mathematical physics*, (Chichester: Wiley-Interscience, 1996)

[S] T.O. Sherman *Fourier analysis on the sphere* (Trans AMS, 1975, Vol. 209)

[Y] K. Yosida, *Functional Analysis* (Springer Verlag , Edition 1980).

[1] This PHD thesis is available on the webpages of the Technische Universtiteit Eindhoven. Webpage in 2004 : http://alexandria.tue.nl/extra3/proefschrift/PRF6A/8810117.pdf