MASTER

2-D flow evolution in bounded domains

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MASTER’S THESIS

2-D Flow Evolution in Bounded Domains

by

Chun Dong Chau

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Eindhoven, January 2005
Abstract

The aim of this work is to analyze the evolution of an array of vortices. From laboratory experiments one knows that some flow patterns last much longer than others. The present study considers two different patterns: a vorticity field with a rectangular flow pattern and a vorticity field with a triangular flow pattern.

By assuming periodic boundary conditions one may consider only one cell from the flow field with a specific geometry (rectangular or triangular).

First, we consider stationary potential flow. Although in this case the stream function and the velocity potential are governed by the relative simple Laplace equation, the nature of the boundary conditions and the shape of the domain may complicate the mathematical problem. We show that the use of complex function theory is very useful here. Especially the technique of conformal mapping will play an important role in the analysis of potential flow.

Second, the dynamics of vortex decay will be considered, which is governed by the vorticity equation. This nonlinear partial differential equation is not easy to solve. Therefore, in the first instance the nonlinear term is neglected, which physically means that the viscosity is considered as the only reason for the decay of the vortex. This yields a diffusion equation for the vorticity. Since the vorticity is related to the stream function by the Poisson equation, it is important to find the eigenstructure of the corresponding domain. Having the eigenfunctions one can solve the diffusion equation and also the Green’s function can be constructed for solving the Poisson equation. The decay of the flow due to viscosity turns out to be fully determined by the eigenvalues of the system.

Last but not least, the nonlinear convection term will be taken into account. It turns out that for our bounded domains the convection term does not have any contribution. So perhaps it should be: "Last and the least,...".
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Eindhoven, The Netherlands
Januari, 2005

Chun Dong Chau
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Chapter 1

Introduction

Lots of research have been done on the dynamics of vortices, especially of two-dimensional vortices. These vortices play an important role in nature. Flows in the atmosphere and oceans on Earth can be considered as being approximately two-dimensional. This is due to the fact that both atmosphere and oceans are thin layers of only a few kilometers depth, whereas the horizontal scales of the flow inside these layers are typically of the order of hundreds or even thousands of kilometers.

The dynamics of two-dimensional flows is most of the time hard to calculate. The vortices are also often located in a certain background flow field. See, e.g., Trieling [19] for two-dimensional vortices in strain and shear flows. Therefore, the dynamics of two-dimensional flows is often simulated by numerical methods. Contour dynamics is a well-known and widely used method in the field of vortex dynamics, see e.g., Schoemaker [17] and Vosbeek [20].

Another common phenomenon is merging of vortices. Small vortices in a two-dimensional flow tend to organize spontaneously in large scale coherent structures, which is called self-organization, see, e.g., Maassen [8]. Bokhoven [3] did experiments with an array of rotating permanent magnets in a container with fluid. Using different configurations of the magnets, he was able to create different vorticity fields, see Figure [1.1].
It turns out that in Bokhoven’s experiment, some patterns have longer life times than others, if the driving magnets are taken away.

This report presents investigations on the time evolution of vortices ordered in different patterns. To make an analytical approach tractable we assume that the patterns extend to infinity, so that boundary effects from the container walls can be neglected. Then, we can apply periodic boundary conditions. This implies that we may focus on the dynamics within one unit cell. The effect of the other cells is incorporated via the boundary conditions. Since we want to apply analytical methods, we do not present an exhaustive list of results for all type of patterns. Instead, we restrict ourselves to square and triangular basic cells, see Figure 1.2.

This report is organized as follows: some theory of fluid dynamics and the derivation of the vorticity equation are presented in chapter 2. In chapter 3 the flow field of the potential flow is found from conformal mapping theory. Also the kinetic energies in these domains are presented in this chapter. In chapter 4 the vorticity equation is solved without the convection term. In chapter 5 the contribution of the convection term is included. This report ends with a conclusion and summary of the investigations on the time evolution of regular patterns of vortices in 2-D.
Figure 1.2: One cell
All fluid mechanics is based on the conservation laws of mass, momentum and energy. The derivation of the conservation laws can be found in, amongst others, Kundu and Cohen [5]. This section shows how the vorticity equation is derived from these conservation laws.

2.1 Conservation laws

The integral formulation of the conservation law of mass reads as

$$\frac{d}{dt} \iiint_{V} \rho \, dV + \iint_{A} \rho (n \cdot v) \, dA = 0,$$

(2.1.1)

where $V$ and $A$ are the inside and surface of an arbitrary volume, $\rho$ represents the density of the fluid, $t$ the time and $v = (u, v, w)$ the local velocity. The normal $n$ on $A$ points outward. The conservation of mass equation expresses, that the rate of accumulation of mass inside the control volume and the net rate of outflow of mass across the control surface balance each other. By using the theorem of Gauss (see A.0.1), the integral formulation of the conservation law of mass can be written as

$$\iiint_{V} \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) \right] \, dV = 0,$$

(2.1.2)

which must hold for arbitrary control volume $V$. This implies that the integrand must be zero. This leads to the so-called differential form of the conservation law of mass (also
called the continuity equation)
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \]  
(2.1.3)

which is equal to
\[ \frac{D \rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0, \]  
(2.1.4)

with the material derivative \( \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla. \)

The integral formulation of the conservation law of momentum is
\[ \frac{d}{dt} \iiint_{V} \rho \mathbf{v} \, dV + \iiint_{A} \rho \mathbf{v} (\mathbf{n} \cdot \mathbf{v}) \, dA = \iiint_{V} \rho \mathbf{f} \, dV + \iiint_{A} \mathbf{t} \, dA, \]  
(2.1.5)

where \( \mathbf{t} \) can be rewritten into
\[ \mathbf{t} = \mathbf{S} \cdot \mathbf{n}, \]  
(2.1.6)

with \( \mathbf{S} \) the stress tensor. The force \( \mathbf{f} \) (which could be gravity) acts on the mass inside the control volume \( V \). The conservation of momentum equation expresses, that the total change of momentum within \( V \) is equal to the net force on the control volume \( V \) and the surface \( A \). Like the conservation of mass case, one can find the differential form of the conservation law of momentum by using the theorem of Gauss,
\[ \frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = \rho \mathbf{f} + \nabla \mathbf{S}. \]  
(2.1.7)

The left hand side of the equation equals
\[ \frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = \mathbf{v} \frac{\partial \rho}{\partial t} + \rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \nabla \cdot (\rho \mathbf{v}) + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = \rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v}, \]

where the latter equality holds due to the continuity equation. This yields for the differential form of the conservation of momentum
\[ \rho \frac{D \mathbf{v}}{Dt} = \rho \mathbf{f} + \nabla \mathbf{S}. \]  
(2.1.8)

The stress tensor \( \mathbf{S} \) can be decomposed into
\[ \mathbf{S} = -p \mathbf{I} + \mathbf{\sigma}, \]  
(2.1.9)
or using the index notation

\[ S_{ij} = -p\delta_{ij} + \sigma_{ij}. \]  

(2.1.10)

Here \( p \) is the hydrostatic pressure, \( \delta_{ij} \) the Kronecker Delta and \( \sigma_{ij} \) the shear stress which depends on the strain rate tensor \( D \) and \( \nabla \cdot v \) in the following way:

\[ \sigma = 2\mu[D - \frac{1}{3}(\nabla \cdot v)I] + \zeta(\nabla \cdot v)I, \]  

where \( \mu \) is the shear viscosity and \( \zeta \) the bulk viscosity. Equation (2.1.8) can be rewritten into

\[ \rho \frac{Dv}{Dt} = \rho f - \nabla p + \nabla \cdot (2\mu D) + \nabla [(\zeta - \frac{2}{3}\mu)(\nabla \cdot v)]. \]  

(2.1.11)

From the fact that

\[ D_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad \text{and} \quad \frac{\partial D_{ij}}{\partial x_i} = \frac{1}{2} \frac{\partial}{\partial x_i} (\nabla \cdot v) + \frac{1}{2} \nabla^2 v_j, \]

equation (2.1.12) turns into

\[ \rho \frac{Dv}{Dt} = \rho f - \nabla p + \mu \nabla^2 v + (\zeta - \frac{1}{3}\mu)\nabla(\nabla \cdot v). \]  

(2.1.12)

(2.1.13)

After assuming incompressibility of the fluid, thus

\[ \nabla \cdot v = 0, \]  

(2.1.14)

one obtains the well-known Navier Stokes equation

\[ \rho \frac{Dv}{Dt} = \rho f - \nabla p + \mu \nabla^2 v. \]  

(2.1.15)

If the viscous effects can be neglected, so if \( \mu = 0 \), the Navier Stokes equation turns into the Euler equation.

The vorticity \( \omega \) is defined as

\[ \omega = \nabla \times v. \]  

(2.1.16)

By taking the curl of both sides of the Navier Stokes equation (2.1.15) and making use of the vector identity

\[ (v \cdot \nabla)v = (\nabla \times v) \times v + \frac{1}{2} \nabla(v \cdot v) = \omega \times v + \frac{1}{2} \nabla(v \cdot v), \]  

(2.1.17)
one finds
\[ \frac{\partial \omega}{\partial t} + \nabla \times (\omega \times v) = \frac{\nabla \rho \times \nabla p}{\rho^2} + \nu \nabla^2 \omega. \] (2.1.18)

The second term on the left hand side of the equation can be written as
\[ \nabla \times (\omega \times v) = \omega \nabla \cdot v + (v \cdot \nabla)\omega - \omega \cdot \nabla v - v \nabla \cdot \omega. \]

The first term equals zero for incompressible flows and the last term is equal to zero due to the fact that \( \nabla \cdot \omega = \nabla \cdot \nabla \times v = 0 \). The obtained vorticity equation is
\[ \frac{D\omega}{Dt} = \omega \cdot \nabla v + \frac{\nabla \rho \times \nabla p}{\rho^2} + \nu \nabla^2 \omega + \nabla \times f, \] (2.1.19)
where \( \nu = \frac{\mu}{\rho} \) is the kinematic viscosity. No external forces will be considered in this report, so \( f = 0 \). The flow will also be considered as barotropic, meaning that the pressure and density contours are everywhere exactly parallel, i.e. \( \nabla \rho \) and \( \nabla p \) are parallel and thus
\[ \frac{\nabla \rho \times \nabla p}{\rho^2} = 0. \]

The reduced vorticity equation to be studied in this report reads as
\[ \frac{D\omega}{Dt} = \omega \cdot \nabla v + \nu \nabla^2 \omega. \] (2.1.20)

### 2.2 Two-dimensional flow

In two-dimensional flows the velocity has two components, \( v = (u, v) \) in the plane of the flow and therefore the vorticity has only one component
\[ \omega = \nabla \times v = \omega e_z, \] (2.2.1)
where \( e_z \) is the unit vector pointing in the \( z \)-direction (perpendicular to the plane of the flow).

For a two-dimensional incompressible flow a stream function \( \psi(x, y, t) \) can be defined as
\[ v = \left( \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right). \] (2.2.2)

The vorticity \( \omega \) and the stream function \( \psi \) are related through the Poisson equation
\[ \omega = -\nabla^2 \psi. \] (2.2.3)
In the case of two-dimensional motion, the vorticity is everywhere normal to the plane of flow and thus
\[ \boldsymbol{\omega} \cdot \nabla \mathbf{v} = 0. \] (2.2.4)

This yields for the vorticity equation (2.1.20)
\[ \frac{D\boldsymbol{\omega}}{Dt} = \frac{\partial \boldsymbol{\omega}}{\partial t} + J(\boldsymbol{\omega}, \psi) = \nu \nabla^2 \boldsymbol{\omega}, \] (2.2.5)
where the Jacobian operator \( J \) is defined as
\[ J(\boldsymbol{\omega}, \psi) = \frac{\partial \boldsymbol{\omega}}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial \boldsymbol{\omega}}{\partial y} \frac{\partial \psi}{\partial x}, \] (2.2.6)
which will be called the ”convection term”.

An important integral quantity characterizing a flow is the circulation \( \Gamma \) defined as
\[ \Gamma = \oint_C \mathbf{v} \cdot ds, \] (2.2.7)
where \( ds \) is a line element of the closed contour \( C \). One can use Stokes’ theorem (see [A.0.3]) to write the circulation in terms of the vorticity
\[ \Gamma = \oint_C \mathbf{v} \cdot ds = \iint_A (\nabla \times \mathbf{v}) \cdot \mathbf{n} \, dA = \iint_A \boldsymbol{\omega} \cdot \mathbf{n} \, dA = \iint_A \boldsymbol{\omega} \, dx \, dy, \] (2.2.8)
where \( A \) is any surface of which the contour \( C \) forms the boundary.

Another quantity is the kinetic energy \( E \), which is given by
\[ E = \frac{1}{2} \iint_A \rho \mathbf{v}^2 \, dA = \frac{1}{2} \iint_A \rho (u^2 + v^2) \, dx \, dy, \] (2.2.9)
taken over the surface \( A \) occupied by the 2-D fluid.
Chapter 3

Potential Flow

In this chapter the vorticity \( \omega \) will be taken vanishing, which means that the flow is irrotational. From (2.2.3) we find that the stream function \( \psi \) is harmonic: \( \nabla^2 \psi = 0 \). It is convenient to introduce a velocity potential \( \phi \) by

\[
\nabla \phi = v,
\]

since \( \omega = \nabla \times v = \nabla \times \nabla \phi = 0 \). So for a two-dimensional irrotational flow with velocity \( v = (u,v) \) we have

\[
u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y}.
\] (3.0.2)

The incompressibility of the flow gives

\[
\nabla^2 \phi = 0,
\] (3.0.3)

so \( \phi \) is also harmonic. Combining (2.2.2) and (3.0.2) we get the relation

\[
\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.
\] (3.0.4)

This strongly suggests to introduce \( \phi \) and \( \psi \) as the real and the imaginary part of a complex potential. Thus the complex velocity potential \( w(z) \) is defined as

\[
w(z) = \phi + i\psi,
\] (3.0.5)

where \( z \) is a complex number. If one knows \( w(z) \) on a certain domain, then the complex velocity potential on some other domain can be found by using the technique of conformal mapping. In this chapter the complex velocity potential of a vortex in a square and an equilateral triangular domain will be obtained, starting from the knowledge of the complex velocity potential of a vortex in a half plane. In view of this we first present some theory of conformal mappings.
3.1 Conformal mapping

The theory of conformal mapping is described in most books on hydrodynamics, like Lamb [6] and Milne-Thomson [13]. The applications in these books mostly concern flows in unbounded regions, like flow near acute and obtuse corners or flow past a plate or flow at an open pipe end. Here we deal with bounded domains, namely the square and the equilateral triangle. For these domains the same theory holds. Good overviews of the theory of conformal mappings are given by Boersma [2] and Derrick [4].

Consider a mapping from one complex plane to another given by

\[ \zeta = F(z). \]  \hspace{1cm} (3.1.1)

This mapping, from a point in the \( z \)-plane to a point in the \( \zeta \)-plane, is a conformal mapping if the function \( F \) is analytic. The technique of conformal mapping is a very useful method by which complex flow patterns (in geometric difficult spaces) can be transformed into simple ones and vice versa. The reason is that the Laplace operator \( \nabla^2 \) is preserved under a conformal mapping. The complex flow potential is governed by the Laplace equation. So if this potential is found for a simple domain and one knows how to map this domain on a complex domain, the potential is also directly known on the latter domain. The result of the Riemann Mapping Theorem guarantees us that any two simply connected regions except the whole \( \mathbb{R}^2 \) (the Euclidean plane) can be mapped conformally onto each other.

**Theorem 3.1.1. Riemann Mapping Theorem**

Let \( z_0 \) be a point in a simply connected region \( \mathbb{K} \neq \mathbb{C} \). Then there is a unique analytic function \( \zeta = F(z) \) mapping \( \mathbb{K} \) one-to-one onto the disk \( |\zeta| \leq 1 \) such that \( F(z_0) = 0 \) and \( F'(z_0) \geq 0 \).

The Riemann Mapping Theorem says something about the mapping from one interior region to another interior region. The following theorem says something about the mapping of the points at the boundary.

**Theorem 3.1.2.**

Let \( G_z \) and \( G_\zeta \) be areas with boundaries \( \Gamma_z \) and \( \Gamma_\zeta \), which are Jordan curves. Let \( \zeta = F(z) \) be a conformal mapping from \( G_z \) to \( G_\zeta \). Then for each \( z_0 \in \Gamma_z \) exists

\[ \lim_{z \to z_0} F(z) =: \zeta_0 \quad \text{and} \quad \zeta_0 \in \Gamma_\zeta. \]

Define now \( F(z_0) = \zeta_0 \) for \( z_0 \in \Gamma_z \). The extended function \( F(z) \) is continuous on \( (G_z \cup \Gamma_z) \) and the function \( F(z) = \zeta \) maps \( (G_z \cup \Gamma_z) \) one-to-one onto \( (G_\zeta \cup \Gamma_\zeta) \).
The proofs of both theorems can be found in Boersma [2]. The theorems guarantee that any two simply connected regions except the whole $\mathbb{C}$ can be mapped conformally onto each other.

The Joukowski transformation and the Schwarz-Christoffel transformation are very useful conformal mappings in hydrodynamics. The Joukowski transformation maps a circle onto an ellipse. For applications one might think of a flow around a circle which can be mapped onto an inclined flat plate. A more interesting Joukowski transformation is the mapping of a circle to a domain with aerofoil shape. These aerofoil shapes are called the Joukowski aerofoils, and play an important role in the aerodynamics of wings. More details about the Joukowski transformation and some aerofoil theory can be found in Milne-Thomson [13]. The Schwarz-Christoffel transformation is a very useful mapping for polygons. This transformation will be applied to the square and the triangular domain in this report. Therefore, the theory and details of this transformation will be considered in the following subsection.

### 3.1.1 Schwarz-Christoffel Mapping

A closed plane figure with $n$ straight sides is called an $n$-sided polygon. We are looking for a conformal mapping from an $n$-sided polygon in the $z$-plane to the upper-half plane in the $\zeta$-plane, i.e. $\Im(\zeta) > 0$. According to the Riemann Mapping Theorem there exists a function $z = H(\zeta)$ which maps $\Im(\zeta) > 0$ onto the $n$-sided polygon $G_z$. Using theorem [3.1.2] this function can be continuously extended to the boundary $\Im(\zeta) = 0$, such that the function maps $\Im(\zeta) = 0$ onto the boundaries (n sides) of the $n$-sided polygon. Denote the interior angles of a simple closed $n$-sided polygon at each vertex $z_k$ by $\theta_k$, for $k = 1, ..., n$ and define the corresponding mapped points of these vertices in the $\zeta$-plane by $\zeta_k$, for $k = 1, ..., n$, which are on the real axis. The following formula is called the Schwarz-Christoffel formula

\[
\frac{dz}{d\zeta} = C \prod_{k=1}^{n} (\zeta - \zeta_k)^{\theta_k/2^{k-1}}, \tag{3.1.2}
\]

or

\[
z = A + C \int_{\zeta_0}^{\zeta} \prod_{k=1}^{n} (t - \zeta_k)^{\theta_k/2^{k-1}} dt, \tag{3.1.3}
\]

where $A$ and $C$ are the unknown constants. For the derivation of the Schwarz-Christoffel formula, the reader is referred to Boersma [2] and Nehari [15].

When the angles of an $n$-sided polygon are given, then the shape of the polygon is determined only if $n = 3$. This means that if $n = 3$ then $\theta_1, \theta_2$ and $\theta_3$ determine the shape and
\(\zeta_1, \zeta_2\) and \(\zeta_3\) can be chosen arbitrary. But for \(n > 3\), \(\theta_1, ..., \theta_n\) do not determine the shape and thus not all \(\zeta_1, ..., \zeta_n\) can be chosen arbitrary. This means that three \(\zeta_k\) points on the real axis which correspond with three vertices \(z_k\) of a given polygon can be chosen arbitrary. The remaining \(n - 3\) points must be arranged such that the polygon is of the right shape. In the limit \(\zeta_k \to \pm \infty\), the corresponding term in the Schwarz-Christoffel formula can be omitted by a proper choice of \(C\). Choose \(C = C_1(1 - \zeta_k)^{\frac{\theta_k}{\pi}} - 1\) then

\[
\frac{dz}{d\zeta} = C_1(\zeta - \zeta_1)^{\frac{\theta_1}{\pi}} - 1 ... (\frac{1}{-\zeta_k})^{\frac{\theta_k}{\pi}} - 1 ... (\zeta - \zeta_n)^{\frac{\theta_n}{\pi}} - 1. \quad (3.1.4)
\]

For

\[
\zeta_k \to \pm \infty, \quad \left(\frac{1}{-\zeta_k}\right)^{\frac{\theta_k}{\pi}} - 1 ... (\zeta - \zeta_k)^{\frac{\theta_k}{\pi}} - 1 = \left(\frac{\zeta - \zeta_k}{-\zeta_k}\right)^{\frac{\theta_k}{\pi}} - 1 \to 1, \quad (3.1.5)
\]

thus the Schwarz-Christoffel formula becomes

\[
\frac{dz}{d\zeta} = C_1(\zeta - \zeta_1)^{\frac{\theta_1}{\pi}} - 1 ... (\zeta - \zeta_k-1)^{\frac{\theta_k-1}{\pi}} - 1 ... (\zeta - \zeta_{k+1})^{\frac{\theta_{k+1}}{\pi}} - 1 ... (\zeta - \zeta_n)^{\frac{\theta_n}{\pi}} - 1. \quad (3.1.6)
\]

The Schwarz-Christoffel formula can be used to find a mapping from an n-sided polygon to the upper-half plane. In the following section, the flow field of a potential vortex in a square and an equilateral triangle will be found, by making use of this formula.

### 3.2 Mapping to a Square and an Equilateral Triangle

As stated in the previous section the flow field of a certain domain is preserved under a conformal mapping. Using the Schwarz-Christoffel formula one can find a mapping from an n-sided polygon in the \(z\)-plane to the upper-half plane in the \(\zeta\)-plane. In this section the Schwarz-Christoffel formula will be applied to the square and the equilateral triangle. Using this formulae, one can find the complex velocity potential \(w(z) = \phi + i\psi\) for the different geometries, since the complex velocity potential for the flow due to a vortex in a half plane is known.

#### 3.2.1 Rectangular domain

Consider a rectangle in the \(z\)-plane with vertices \(\{z_A, z_B, z_C, z_D\}\). All interior angles are equal to \(\frac{\pi}{2}\). In general there is no simple analytic determination of the corresponding points \(\{\zeta_A, \zeta_B, \zeta_C, \zeta_D\}\) for an n-sided polygon with \(n > 3\). But in this case the symmetry allows us to find an explicit solution. Consider the following coordinates of the vertices in the complex planes:
vertex | coordinates in the $z$-plane | coordinates in the $\zeta$-plane
--- | --- | ---
$A$ | $z_A = -K + iK'$ | $\zeta_A = m^{-\frac{1}{2}}$
$B$ | $z_B = -K$ | $\zeta_B = -1$
$C$ | $z_C = K$ | $\zeta_C = 1$
$D$ | $z_D = K + iK'$ | $\zeta_D = m^{-\frac{1}{2}}$

where $K'$ and $m$ are real numbers. A vortex is placed in the center of mass $z_1$ of the rectangle. The vortex at $z_1$ in the rectangle will be mapped onto $\zeta_1$ in the half plane.

Using the Schwarz-Christoffel equation (3.1.2) one finds

$$g'(\zeta) := \frac{dz}{d\zeta} = C'[(\zeta-1)(\zeta+1)(\zeta - m^{-\frac{1}{2}})(\zeta + m^{-\frac{1}{2}})]^{-\frac{1}{2}} = \frac{C}{\sqrt{(\zeta^2 - 1)(\zeta^2 - m^{-1})}}$$

(3.2.1)

Integrating this equation yields

$$z = g(\zeta) = A + C' \int_0^\zeta \frac{dt}{\sqrt{(t^2 - 1)(t^2 - m^{-1})}}.$$

(3.2.2)
The image of infinity in the \( \zeta \)-plane turns out to be the point \( iK' \) in the \( z \)-plane. The image of 0 in the \( \zeta \)-plane is also 0 in the \( z \)-plane, which means that the integration constant \( A \) equals zero. The mapping is thus given by an elliptic integral of the first kind

\[
z = g(\zeta) = C \int_0^\zeta \frac{dt}{\sqrt{(t^2 - 1)(t^2 - m^{-1})}} = C m \int_0^{\sin^{-1} \zeta} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}, \tag{3.2.3}
\]

and this gives for \( \zeta \)

\[
\zeta = \text{sn}(C_m^{-1} z|m), \tag{3.2.4}
\]

where \( \text{sn}(z|m) \) is the Jacobi elliptic sine with parameter \( m \). If the rectangle is scaled such that \( K = g(1) \) then \( C_m = 1 \). More precisely \( g(1) \) is the elliptic integral of the first kind with parameter \( m \). In this report the parameter \( m = (3 - 2\sqrt{2})^2 \), so that the rectangle becomes a square. For more details and properties of elliptic integrals the reader is referred to Abramowitz & Stegun [1] and Magnus & Oberhettinger [9].

For a vortex placed in the center of mass, one has the following coordinates:

<table>
<thead>
<tr>
<th>vortex coordinates in the ( z )-plane</th>
<th>coordinates in the ( \zeta )-plane</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z_1 = i \frac{K'}{2} )</td>
<td>( \zeta_1 = \text{sn}(i \frac{K'}{2}</td>
</tr>
</tbody>
</table>

The velocity field induced by a potential vortex in an unbounded domain (the whole complex plane) with strength \( \gamma \) is purely azimuthal. The velocity components in polar coordinates are given by

\[
v_r = 0, \quad v_\theta = \frac{\gamma}{2\pi r},
\]

and in Cartesian coordinates

\[
u = -v_\theta \sin(\theta), \quad v = v_\theta \cos(\theta).
\]

From the relation

\[
\frac{dw}{dz} = u - iv,
\tag{3.2.5}
\]

one obtains the complex velocity potential \( w(z) \)

\[
w(z) = -\frac{i\gamma}{2\pi} \ln(z).
\tag{3.2.6}
\]

When the vortex is located in the point \( z = z_0 \), the complex velocity potential \( w(z) \) becomes

\[
w(z) = -\frac{i\gamma}{2\pi} \ln(z - z_0).
\tag{3.2.7}
Note that the velocity is scaled with a factor $2\pi$ such that the circulation of a potential vortex equals the strength of the vortex

$$\Gamma = \oint_C v \, ds = \oint_0^{2\pi} \frac{\gamma}{2\pi r} \, r \, d\theta = \gamma. \quad (3.2.8)$$

The flow field of a vortex in a half plane can be determined by using the image principle. One places a vortex with opposite sign at $\bar{\zeta}_1$ the complex conjugate of the position $\zeta_1$ of the original vortex. The complex velocity potential $\hat{w}(\zeta)$ for a half plane becomes

$$\hat{w}(\zeta) = -\frac{i\gamma}{2\pi} \ln(\zeta - \zeta_1) + \frac{i\gamma}{2\pi} \ln(\zeta - \bar{\zeta}_1). \quad (3.2.9)$$

The flow field in the $z$-plane is easily found by using (3.2.4). So the complex velocity potential $w_\diamond(z)$ for the square becomes

$$w_\diamond(z) = -\frac{i\gamma}{2\pi} \ln(\text{sn}(z|m) - \zeta_1) + \frac{i\gamma}{2\pi} \ln(\text{sn}(z|m) - \bar{\zeta}_1). \quad (3.2.10)$$

By the definition of the complex velocity potential, one can write $w_\diamond(z) = \phi_\diamond(x,y) + i\psi_\diamond(x,y)$. Consequently the stream function for the square is found by taking the imaginary part of the complex velocity potential, i.e. $\Im(w_\diamond(z))$. See Figure 3.2 for the streamlines. Analogously the velocity potential is found by taking the real part of the complex velocity potential, i.e. $\Re(w_\diamond(z))$. See Figure 3.3 for the potential level lines and the streamlines.

![Streamlines of a vortex in a half plane](image.png)

![Streamlines of a vortex in a square](image.png)

**Figure 3.2:** Conformal mapping of the streamlines from a half plane to a square
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2-D Flow Evolution in Bounded Domains

(a) Streamlines and potential level lines of a vortex in a half plane

(b) Streamlines and potential level lines of a vortex square

Figure 3.3: Conformal mapping of the potential level lines (dashed lines) and the streamlines (solid lines) from a half plane to a square

3.2.2 Triangular domain

Consider an equilateral triangle in the $z$-plane with vertices \{ $z_A$, $z_B$, $z_C$ \} with interior angles $\frac{\pi}{3}$ and side $a$. We take the following coordinates of the vertices in the complex plane:

<table>
<thead>
<tr>
<th>vertex</th>
<th>$z$-plane coordinates</th>
<th>$\zeta$-plane coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$z_A = -\frac{a}{2}$</td>
<td>$\zeta_A = 0$</td>
</tr>
<tr>
<td>$B$</td>
<td>$z_B = \frac{a}{2}$</td>
<td>$\zeta_B = 1$</td>
</tr>
<tr>
<td>$C$</td>
<td>$z_C = \frac{a}{2} \sqrt{3}$</td>
<td>$\zeta_C = \infty$</td>
</tr>
</tbody>
</table>

A vortex is placed in the center of mass $z_1$ of the triangle to induce the flow. The vortex will be mapped onto $\zeta_1$ in the half plane.
Using the Schwarz-Christoffel equation one finds

\[ g'(\zeta) := \frac{dz}{d\zeta} = C(\zeta(\zeta - 1))^{-\frac{2}{3}} = \frac{C}{(\zeta(\zeta - 1))^{2/3}} \]  \hspace{1cm} (3.2.11)

Integrating this equation yields

\[ z = g(\zeta) = A + C \int_0^\zeta \frac{dt}{(t(t - 1))^{2/3}}, \]  \hspace{1cm} (3.2.12)

The image of \( \frac{1}{2} \) in the \( \zeta \)-plane turns out to be the point 0 in the \( z \)-plane. The integration constant \( A \) is obtained by using \( g(0) = -\frac{a}{2} \), thus \( A = -\frac{a}{2} \). The constant \( C \) is determined by making use of \( g(1) = \frac{a}{2} \) and thus \( C \) is

\[ C = \frac{a}{\int_0^1 (t(t - 1))^{-2/3} dt}. \]

The mapping from a half plane to an equilateral triangle with side length \( a \) becomes

\[ z = g(\zeta) = a \frac{\int_0^\zeta (t(t - 1))^{-2/3} dt}{\int_0^1 (t(t - 1))^{-2/3} dt} - \frac{a}{2} = a \frac{B_{\zeta}(1/3, 1)}{B(1/3, 1/3)} - \frac{a}{2} = a I_{\zeta}(1/3, 1/3) - \frac{a}{2}, \]  \hspace{1cm} (3.2.13)
where $B(p, q)$ is the beta function, $B_\zeta(p, q)$ is the incomplete beta function and $I_\zeta(p, q)$ is the regularized beta function, all with parameters $p$ and $q$. This gives for $\zeta$

$$\zeta = I_\zeta^{-1}(1/3, 1/3),$$

(3.2.14)

where $I_\zeta^{-1}(p, q)$ is the inverse of the regularized beta function. For more details and properties of beta functions, the reader is referred to Abramowitz & Stegun [1] and Magnus & Oberhettinger [9].

For a vortex placed in the center of mass of the triangle, one have the following coordinates:

<table>
<thead>
<tr>
<th>vortex</th>
<th>coordinates in the $z$-plane</th>
<th>coordinates in the $\zeta$-plane</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$z_1 = \frac{ia}{2\sqrt{3}}$</td>
<td>$\zeta_1 = I_\zeta^{-1}(1/3, 1/3)$</td>
</tr>
</tbody>
</table>

The complex velocity potential $\hat{w}(\zeta)$ for a half plane equals is given by (3.2.9) So the complex velocity potential $w_\Delta(z)$ for the equilateral triangle becomes

$$w_\Delta(z) = -\frac{i\gamma}{2\pi} \ln(I_\zeta^{-1}(1/3, 1/3) - \zeta_1) + \frac{i\gamma}{2\pi} \ln(I_\zeta^{-1}(1/3, 1/3) - \bar{\zeta}_1).$$

(3.2.15)

Again, the stream function and the velocity potential are found by taking the imaginary part and respectively the real part of $w_\Delta(z)$, i.e. $\Im(w_\Delta(z))$ and $\Re(w_\Delta(z))$. See Figures 3.5 and 3.6.

**Figure 3.5:** Conformal mapping of the streamlines from a half plane to an equilateral triangle
(a) Streamlines and potential level lines of a vortex in a half plane
(b) Streamlines and potential level lines of a vortex in an equilateral triangle

Figure 3.6: Conformal mapping of the potential level lines (dashed lines) and the streamlines (solid lines) from a half plane to an equilateral triangle

3.3 Kinetic Energy

The kinetic energy is given in equation (2.2.9). In this section the density is assumed to be uniform and equals \( \rho = 1 \). Since this chapter deals with potential flows, one can write for the kinetic energy \( E \)

\[
2E = \iint_{A} \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 \, dx \, dy = \iint_{A} \nabla \phi \cdot \nabla \phi \, dx \, dy.
\] (3.3.1)

Applying Green’s Theorem (see equation A-3 in the appendix) and \( \nabla^2 \phi = 0 \) yields

\[
2E = \oint_{\Gamma} \phi \nabla \phi \cdot \mathbf{n} \, ds = \oint_{\Gamma} \phi \, d\psi,
\] (3.3.2)

where the latter equality is a result from the substitution of \( \nabla \phi \cdot \mathbf{n} = \frac{\partial \psi}{\partial s} \). The integration contour \( \Gamma \) is a simple closed curve, a connected curve that does not cross itself and ends at the same point where it begins, which encloses the area \( A \). Note that one cannot apply Green’s Theorem on areas with singularities.
In the following sections the kinetic energies of the flow induced by a vortex in a square and an equilateral triangle are obtained.

### 3.3.1 Kinetic Energy in a Square

Consider a square in the $z$-plane with a vortex placed at the center of mass. The region $\mathcal{A}$ in equation (3.3.1) is the inner part of the square which is enclosed by the boundary of the square. However one cannot take the boundary of the square for applying equation (3.3.2) since the area enclosed by this boundary has a singularity at the place of the vortex. It is thus not possible to use equation (3.3.2) directly. But consider now the following region enclosed by the contour $\mathcal{C} = C_1 + C_2 + C_3 + C_4 + C_5 + C_6$, see Figure 3.7.

![Figure 3.7: Simple closed contour](image)

The area enclosed by curve $\mathcal{C}$ does not have singular points, therefore equation (3.3.2) can be used to obtain the kinetic energy $\tilde{E}_\circ$ inside this curve. The kinetic energy inside curve $\mathcal{C}$ is

$$2\tilde{E}_\circ = \oint_{\mathcal{C}} \phi_\circ \, d\psi_\circ = \int_{C_1+C_2+C_3+C_4+C_5+C_6} \phi_\circ \, d\psi_\circ,$$

(3.3.3)
where the integrals along $C_1, C_2, C_3, C_5$ are zero since $\psi_0$ is constant along these (stream)lines. The lines $C_4$ and $C_6$ are orthogonal to the streamlines, which means that the velocity potential at the lines $C_4$ and $C_6$ are constant, say $\phi_{o4}$ and $\phi_{o6}$ respectively. So this yields for the kinetic energy inside curve $C$

$$2\tilde{E}_o = \oint_C \phi_0 \, d\psi_0$$

$$= \int_{C_4} \phi_{o4} \, d\psi_0 + \int_{C_6} \phi_{o6} \, d\psi_0$$

$$= (\psi_0(b) - \psi_0(a))\phi_{o4} + (\psi_0(d) - \psi_0(c))\phi_{o6}$$

$$= \psi_0(b)\phi_{o4} - \psi_0(c)\phi_{o6}, \quad (3.3.4)$$

where the latter equality holds since the value of the streamline at the boundary is equal to zero. By symmetry the kinetic energy of almost the whole square (the square without the small area around the vortex) $E_\circ$ is equal to

$$E_\circ = \psi_0(b)\phi_{o4} - \psi_0(c)\phi_{o6}. \quad (3.3.5)$$

The values of the potential level lines $\phi_{o4}$ and $\phi_{o6}$ need to be calculated. For simplicity of the calculation, the complex velocity potential in the $\zeta$-plane is used. The potential level line $\phi_{o6}$ in the $z$-plane has the same value as the potential level line which runs from $\zeta = 0$ to the place of the vortex $\zeta_1$ along a part of the imaginary axes in the $\zeta$-plane, called $D_6$ as indicated in Figure 3.8

![Figure 3.8: Path $D_6$ in the $\zeta$-plane](image)
The complex velocity potential \( \hat{w}(\zeta) \) for the half plane is given by

\[
\hat{w}(\zeta) = -\frac{i\gamma}{2\pi} \ln(\zeta - \zeta_1) + \frac{i\gamma}{2\pi} \ln(\zeta - \bar{\zeta}_1) \\
= \frac{i\gamma}{2\pi} \left[ \ln|\zeta - \bar{\zeta}_1| + i \arg(\zeta - \bar{\zeta}_1) - \ln|\zeta - \zeta_1| - i \arg(\zeta - \zeta_1) \right],
\]

(3.3.6)

where \(|z|\) denotes the absolute value of \(z\) and \(\arg(z)\) its argument. A branch cut discontinuity in the complex \(\zeta\)-plane along the negative real axis is placed to make the logarithm single valued. Since \(\hat{w}(\zeta) = \phi + i\psi\), one needs to take the real part of \(\hat{w}(\zeta)\) for the velocity potential, which is

\[
\Re(\hat{w}(\zeta)) = \frac{\gamma}{2\pi} \left[ \arg(\zeta - \zeta_1) - \arg(\zeta - \bar{\zeta}_1) \right].
\]

(3.3.7)

The points \(\zeta\) along \(D_6\) are purely imaginary. Since \(\zeta_1\) is also a purely imaginary number and \(\zeta < \zeta_1\) for all \(\zeta\) on \(D_6\), \(\zeta - \zeta_1\) will be on the negative imaginary axes. The argument of \(\zeta - \zeta_1\) is thus \(-\frac{\pi}{2}\). Since \(\zeta_1 = -\bar{\zeta}_1\), \(\zeta - \bar{\zeta}_1\) is on the positive imaginary axes, hence \(\Re(\hat{w}(\zeta))\) becomes

\[
\Re(\hat{w}(\zeta)) = \frac{\gamma}{2\pi} \left[ -\frac{\pi}{2} - \frac{\pi}{2} \right] = -\frac{\gamma}{2}.
\]

(3.3.8)

In a similar way \(\phi_{\diamond}\) can be obtained. The value \(\phi_{\diamond}\) turns out to be zero. The kinetic energy for the square becomes

\[
E_\diamond = \frac{\gamma}{2} \psi_\diamond(c).
\]

(3.3.9)

It looks as if the strength of the vortex can cause the kinetic energy to be negative. However, \(\psi = \Im(w)\) is given by

\[
\Im(\hat{w}(\zeta)) = \frac{\gamma}{2\pi} \ln \left( \frac{|\zeta - \bar{\zeta}_1|}{|\zeta - \zeta_1|} \right),
\]

(3.3.10)

so \(E_\diamond\) is proportional to \(\gamma^2\). Also the logarithm, which runs from 0 at the boundary to \(+\infty\) close to the vortex \(\zeta_1\), is positive.

The kinetic energy depends on how close the contour \(C_5\) is taken to the vortex. If the radius of this contour is taken smaller and smaller, the kinetic energy approaches \(+\infty\). This is a direct consequence of introducing a vortex with a velocity singularity in the center. So calculation of the kinetic energy for such a vortex is in itself not useful. However, we want to compare the same vortex in a square and a triangle. It will turn out in section 3.3.3 that this comparison still yields interesting information.
3.3.2 Kinetic Energy in an Equilateral Triangle

The kinetic energy of a flow induced by a vortex in an equilateral triangle is found in a similar way as for the square. Consider the following region enclosed by the contour $C = C_1 + C_2 + C_3 + C_4 + C_5$, see Figure (3.7).

![Figure 3.9: Simple closed contour](image)

The area enclosed by curve $C$ does not have singular points, therefore equation (3.3.2) can be used to obtain the kinetic energy $\tilde{E}_\Delta$ inside this curve. The kinetic energy inside curve $C$ is

$$2\tilde{E}_\Delta = \oint_C \phi_\Delta d\psi_\Delta = \int_{C_1+C_2+C_3+C_4+C_5} \phi_\Delta d\psi_\Delta,$$  \hspace{1cm} (3.3.11)

where the integrals along $C_1, C_2, C_4$ are zero since $\psi_\Delta$ is constant along these (stream)lines. The lines $C_3$ and $C_5$ are orthogonal to the streamlines, which means that the velocity potential at the lines are constant there, say $\phi_\Delta C_3$ and $\phi_\Delta C_5$, respectively. So this yields for the kinetic energy inside curve $C$

$$2\tilde{E}_\Delta = \oint_C \phi_\Delta d\psi_\Delta$$

$$= \int_{C_3} \phi_\Delta C_3 d\psi_\Delta + \int_{C_5} \phi_\Delta C_5 d\psi_\Delta$$

$$= (\psi_\Delta(b) - \psi_\Delta(a))\phi_\Delta C_3 + (\psi_\Delta(d) - \psi_\Delta(c))\phi_\Delta C_5$$

$$= \psi_\Delta(b)\phi_\Delta C_3 - \psi_\Delta(c)\phi_\Delta C_5.$$  \hspace{1cm} (3.3.12)
where the latter equality holds since the streamline at the boundary is equal to zero. By
symmetry the kinetic energy of almost the whole triangle (the triangle without the small
area around the vortex) \( E_\triangle \) is equal to
\[
E_\triangle = \psi_\triangle(b) \phi_\triangle(b) - \psi_\triangle(c) \phi_\triangle(c).
\] (3.3.13)
The values \( \phi_\triangle\beta \) and \( \phi_\triangle\gamma \) can be obtained in a similar as in the square case. The kinetic
energy of the equilateral triangle is given by
\[
E_\triangle = \frac{\gamma}{2} \psi_\triangle(c).
\] (3.3.14)

### 3.3.3 Kinetic Energy in a Square vs. Kinetic Energy in an in an Equi-

lateral Triangle

It is interesting to compare the kinetic energies of the flows induced by the same vortex
in different domains. It is clear that the kinetic energies tend to infinity, since there is a
singularity at the place of the vortex. In this subsection the kinetic energies in the square
and the equilateral triangle are compared in the limit that the point \( c \), see Figure [3.7 and
[3.9] approaches the vortex center.

Call the distance to the vortex center \( \varepsilon \). The difference of the kinetic energy for the square
and the triangle is
\[
E_\square - E_\triangle \equiv \frac{\gamma}{2} [\psi_\square(c) - \psi_\triangle(c)] = \frac{\gamma}{2} [\psi_\square(z_\circ + i\varepsilon) - \psi_\triangle(z_\circ + i\varepsilon)],
\] (3.3.15)
where \( z_\circ \) and \( z_\triangle \) are the positions of the vortex in the square and the triangle, respectively.
The equation inside the square bracket at the right hand side can be rewritten as
\[
\Im \left[ w_\circ(\xi) - w_\triangle(\eta) \right] = \Im \left[ \frac{i\gamma}{2\pi} \left( \ln(\text{sn}(\xi|m) - \zeta_\circ) - \ln(\text{sn}(\xi|m) - \zeta_\triangle) \right) + \ln(\frac{\Gamma^{-1}_{\frac{3}{2}}(1/3, 1/3) - \zeta_\circ}{\Gamma^{-1}_{\frac{3}{2}}(1/3, 1/3) - \zeta_\triangle}) \right] \nonumber \nonumber
= \Im \left[ \frac{i\gamma}{2\pi} \left( \frac{\text{sn}(\xi|m) - \zeta_\circ}{\text{sn}(\xi|m) - \zeta_\triangle} + \ln(\frac{\Gamma^{-1}_{\frac{3}{2}}(1/3, 1/3) - \zeta_\circ}{\Gamma^{-1}_{\frac{3}{2}}(1/3, 1/3) - \zeta_\triangle}) \right) \right],
\] (3.3.16)
where \( w_\circ(\xi) \) and \( w_\triangle(\eta) \) are the complex velocity potentials of the corresponding geome-

tries given by [3.2.10] and [3.2.15] and \( \xi = z_\circ + i\varepsilon \) and \( \eta = z_\triangle + i\varepsilon \).
By use of Taylor expansion, $sn(\xi|m)$ can be written as,

$$sn(\xi|m) = sn(z_0) + i\varepsilon \frac{\partial}{\partial z} sn(z|m)|_{z=z_0} + O(\varepsilon^2).$$  \hspace{1cm} (3.3.17)

Before rewriting $I^{-1}_{1/2}(\frac{1}{3}, \frac{1}{3})$ as a Taylor series, we define

$$w = F(\zeta) = a \int_0^\zeta \left( t(t-1) \right)^{-2/3} dt - \frac{a}{2} = a I_\zeta(\frac{1}{3}, \frac{1}{3}) - \frac{a}{2},$$ \hspace{1cm} (3.3.18)

and thus

$$I_\zeta(\frac{1}{3}, \frac{1}{3}) = \frac{F(\zeta)}{a} + \frac{1}{2} = \frac{w}{a} + \frac{1}{2}. \hspace{1cm} (3.3.19)$$

Taking the inverse of the previous equations yields

$$\zeta = I^{-1}_{1/2}(\frac{1}{3}, \frac{1}{3}) = F^{-1}(w). \hspace{1cm} (3.3.20)$$

The expansion for $I^{-1}_{1/2}(\frac{1}{3}, \frac{1}{3})$ is given by

$$I^{-1}_{1/2}(\frac{1}{3}, \frac{1}{3}) = F^{-1}(\eta) = F^{-1}(z_\delta) + i\varepsilon \frac{\partial}{\partial w} F^{-1}(w)|_{w=z_\delta} + O(\varepsilon^2). \hspace{1cm} (3.3.21)$$

So we need to calculate the derivatives of the functions $sn(z|m)$ and $F^{-1}(w)$. By the inverse function theorem we have

$$\frac{d}{dw} F^{-1}(w) = \frac{d\zeta}{d\omega} = \left( \frac{dw}{d\zeta} \right)^{-1} = \frac{1}{F''(\zeta)}. \hspace{1cm} (3.3.22)$$

and from the definition of $F(\zeta)$ in (3.3.18)

$$\frac{d}{dw} F^{-1}(w) \big|_{w=z_\delta} = \frac{\int_0^{\zeta} \left( t(t-1) \right)^{-2/3} dt}{a(\zeta(\zeta-1))^{-2/3}} \big|_{w=z_\delta, \zeta=F^{-1}(z_\delta)} = \frac{(\zeta_\delta(\zeta_\delta-1))^{2/3}}{a} \int_0^1 \left( t(t-1) \right)^{-2/3} dt. \hspace{1cm} (3.3.23)$$

For the derivative of the Jacobi elliptic sine we have

$$\frac{d}{dz} sn(z|m) \big|_{z=z_0} = cn(z|m)dn(z|m) \big|_{z=z_0} = \sqrt{1-\zeta_\delta^2} \sqrt{1-m\zeta_\delta^2}, \hspace{1cm} (3.3.24)$$
where \( cn(z|m) \) is the Jacobi elliptic cosine and \( dn(z|m) \) is the Jacobi elliptic delta function. By neglecting the higher order terms and leaving out \( \frac{\gamma}{2} \) for the time being, one finds the following expression for the difference between the kinetic energy in the square and the triangle

\[
\Im \left[ w_\circ(\xi) - w_\triangle(\eta) \right] = \Im \left[ \frac{i}{2\pi} \ln \left( \sqrt{1 - \zeta_\circ^2} \right) \frac{\zeta_\circ - \zeta_\circ}{a(\zeta_\circ - \zeta_\circ)} \right],
\]

\[
= \Im \left[ \frac{i}{2\pi} \ln \left( \frac{\zeta_\circ - \zeta_\circ}{a(\zeta_\circ - \zeta_\circ)} \right) \right].
\]

(3.3.25)

There is still one variable in the equation, namely the length \( a \) of the side of the equilateral triangle. To compare the different geometries, \( a \) is chosen such that both geometries have the area. For \( m = (3 - 2\sqrt{2})^2 \) one obtains a square with sides \( 2K = K' = g(1) \), where \( g \) is the function defined in 3.2.3 with \( C_m = 1 \). Then, the difference \( \Im \left[ w_\circ(\xi) - w_\triangle(\eta) \right] \), with the vortex located in the center of mass, has been calculated for different values of the distance \( \epsilon \). The results are given in Figure 3.10. It turns out to converge to 0.01313. This corresponds with the difference found by using the Taylor expansion.

---

**Figure 3.10:** Plot for \( \epsilon \to 0 \) in \( \Im(w_\circ(\xi) - w_\triangle(\eta)) \), with \( \xi = z_\circ + i\epsilon \) and \( \eta = z_\triangle + i\epsilon \).
Chapter 4

Viscous decay of a vortex in a bounded domain

Recall that in a two dimensional area the vorticity \( \omega \) and the stream function \( \psi \) are related through the Poisson equation
\[
\omega = -\nabla^2 \psi. \tag{4.0.1}
\]

Consider the vorticity equation
\[
\frac{D\omega}{Dt} = \frac{\partial \omega}{\partial t} + J(\omega, \psi) = \nu \nabla^2 \omega, \tag{4.0.2}
\]
with Jacobian operator \( J \)
\[
J(\omega, \psi) = \frac{\partial \omega}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial \omega}{\partial y} \frac{\partial \psi}{\partial x}. \tag{4.0.3}
\]

Due to the fact that the velocity must vanish at the boundary, we have \( \psi = 0 \) at the boundaries. Since the potential vortex is located in the center of mass of the domain, the motion is close to azimuthal near the vortex. Therefore, close to the boundary and close to the vortex one might neglect the Jacobian operator \( J \) in the vorticity equation. In this approximation, the vorticity equation becomes the well-known diffusion equation,
\[
\frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega. \tag{4.0.4}
\]

By solving this equation with the conditions that the vorticity is zero at the boundary and some initial distribution of the vorticity over the domain, we can calculate the vorticity \( \omega(x, y, t) \). The stream function can then be obtained from equation (4.0.1). In this chapter we study the reduced model embodied in (4.0.4). Later on, we shall show that the obtained results are also very useful in the study of the full model equation (4.0.2).
4.1 Initial and Boundary Conditions

The Poisson equation (4.0.1) needs boundary conditions and the diffusion equation (4.0.4) boundary conditions and also an initial condition to be uniquely solved. Assume that initially the vortex is concentrated at one point, say \((x_0, y_0)\) in a region \(\Omega\) with boundary \(\partial \Omega\). The initial vorticity distribution then reads as

\[
\omega(x, y, 0) = \gamma \delta(x - x_0, y - y_0),
\]

(4.1.1)

with \(\delta(x - x_0, y - y_0)\) the Dirac delta distribution, also written as a product of the one-dimensional counterparts: \(\delta(x-x_0)\delta(y-y_0)\). This distribution has the following properties,

\[
\int_{y_0-\varepsilon}^{y_0+\varepsilon} \int_{x_0-\varepsilon}^{x_0+\varepsilon} \delta(x - x_0, y - y_0) \, dx \, dy = 1, \quad \text{for all } \varepsilon > 0, \quad (4.1.2a)
\]

\[
\int_{y_0-\varepsilon}^{y_0+\varepsilon} \int_{x_0-\varepsilon}^{x_0+\varepsilon} f(x, y) \delta(x - x_0, y - y_0) \, dx \, dy = f(x_0, y_0), \quad \text{for all test functions } f. \quad (4.1.2b)
\]

Note that the circulation \(\Gamma = \gamma\), since

\[
\Gamma = \oint_C \mathbf{v} \cdot ds = \iint_A \omega \cdot \mathbf{n} \, dA = \gamma, \quad (4.1.3)
\]

which is consistent with the potential case (see equation (3.2.8)). The boundary condition comes from the fact that the vorticity vanishes at the boundary, thus

\[
\omega(x, y, t) = 0, \quad (x, y) \in \partial \Omega. \quad (4.1.4)
\]

The boundary condition for the Poisson equation (4.0.1) follows from the fact that the velocity \(\mathbf{v} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right)\) vanishes at the boundary:

\[
\psi(x, y, t) = 0, \quad (x, y) \in \partial \Omega. \quad (4.1.5)
\]

This gives two second order boundary value problems,

\[
\begin{align*}
\frac{\partial \omega(x,y,t)}{\partial t} &= \nu \nabla^2 \omega, \\
\omega(x, y, t) &= 0, \quad (x, y) \in \partial \Omega, \\
\omega(x, y, 0) &= \gamma \delta(x - x_0, y - y_0), \quad (x_0, y_0) \in \Omega,
\end{align*}
\]

(4.1.6)
and
\begin{equation}
\begin{aligned}
\omega(x, y, t) &= -\nabla^2 \psi, \\
\psi(x, y, t) &= 0, \quad (x, y) \in \partial \Omega,
\end{aligned}
\end{equation}
which will be solved by making use of Sturm-Liouville theory and Green’s functions.

### 4.1.1 Sturm Liouville theory; Green’s functions

The diffusion equation will be solved by using the method of separation of variables. This leads to simple Sturm-Liouville problems. A Sturm-Liouville problem consist of finding a real function $u \in C^2[a, b]$ that satisfies
\begin{equation}
\begin{aligned}
\frac{d}{dx} \left[p(x) \frac{du}{dx}\right] - \left[\lambda r(x) + q(x)\right]u(x) &= L[u] - \lambda r(x)u = 0, \\
\alpha_1 u(a) + \alpha_2 u'(a) &= 0, \\
\beta_1 u(b) + \beta_2 u'(b) &= 0,
\end{aligned}
\end{equation}
where $\frac{dp(x)}{dx}$, $q(x)$ and $r(x)$ are given real, continuous functions. $\alpha_1, \alpha_2, \beta_1$ and $\beta_2$ are given real numbers. The differential operator $L$ is the so called Sturm-Liouville operator. The problem is called regular when the functions $p(x)$ and $r(x)$ are positive for $a \leq x \leq b$. The non-trivial solutions $\varphi_n$, are called the eigenfunctions and the corresponding $\lambda_n$ the eigenvalues, for $n = 1, 2, \ldots$. The following theorem shows why the Sturm-Liouville problem is so convenient.

**Theorem 4.1.1.** For a regular Sturm-Liouville problem the following holds

i. The Sturm-Liouville operator is self-adjoint.

ii. The set of eigenvalues $\lambda_1, \lambda_2, \ldots$ is countable and the eigenvalues are real.

iii. The associated eigenfunctions in $\varphi_1, \varphi_2, \ldots$ are orthogonal to each other with respect to the inner product with weight function $r(x)$
\begin{equation}
(\varphi_n, \varphi_m) = \int_a^b r(x) \varphi_n \varphi_m \, dx = 0.
\end{equation}

iv. The eigenfunction $\varphi_n$ has exactly $n$ zero points in the interval $a < x < b$ and is uniquely determined up to a constant factor.
v. The orthonormal sequences of eigenfunctions of a regular Sturm-Liouville system is complete in the set $C[a, b]$.

The proof of this well-known theorem can be found in most books on elementary boundary value problems, e.g. van Duijn [14]. Another kind of problem associated with the Sturm-Liouville operator is the inhomogeneous differential equation

$$L[u] = f(x), \quad (4.1.10)$$

with the same boundary problems as in (4.1.8). The solution of this problem can be expressed in an integral, which contains the Greens’s function. The Green’s function is defined as follows.

**Definition** A function $G(x|x')$ for $x, x' \in \Omega$ is called the Green’s function of the Dirichlet problem in $\Omega$ if $G(x|x')$ satisfies the following equations

$$L_x[G](x|x') = \delta(x - x'), \quad (4.1.11a)$$

$$G(x|x') = 0 \quad \text{at the boundary} \partial\Omega, \quad (4.1.11b)$$

where $L_x$ is a linear differential operator. If $\Omega \subset \mathbb{R}^2$, the Dirac delta function $\delta(x - x')$ is given by $\delta(x - x'_1)\delta(x - x'_2)$ and similar for $\Omega \subset \mathbb{R}^n$.

After multiplication of (4.1.11a) by $f(x')$ and integration over $\Omega$ one obtains

$$\int_{\Omega} L_x[G](x|x') f(x') \, d\Omega = \int_{\Omega} \delta(x - x') f(x') \, d\Omega = f(x). \quad (4.1.12)$$

Integration and differentiation may be interchanged since $L_x$ is a linear operator;

$$L_x \left[ \int_{\Omega} G(x|x') f(x') \, d\Omega \right] = f(x). \quad (4.1.13)$$

Comparing to (4.1.10) we conclude that

$$u = \int_{\Omega} G(x|x') f(x') \, d\Omega, \quad (4.1.14)$$

for $\Omega$ being one dimensional and $L_x = L$. In this report the stream function is obtained by solving the Poisson equation, so $L = \nabla^2$, $u = \psi$ and $f = \omega$,

$$\begin{cases}
\nabla^2[G](x|x') = \delta(x - x'), & x \in \Omega, \\
G(x|x') = 0, & x \in \partial\Omega.
\end{cases} \quad (4.1.15)$$
In the one-dimensional case we have the Sturm-Liouville problem with \( r(x) = 1 \). According to Sturm-Liouville’s theorem the set of eigenfunctions is complete. That’s why \( G \) can be written as a linear combination of the eigenfunctions in the one-dimensional case. But what about higher dimensions? For domains that are Cartesian products of orthogonal coordinate systems, completeness of the eigenfunctions formed from products of one-dimensional counterparts has been established, see Strauss [18]. The differential operator \( L_\mathbf{x} \) must also be separable into the different components of \( \mathbf{x} = \{x_1, x_2, \ldots, x_n\} \) i.e. \( L_\mathbf{x} = L_{x_1} + L_{x_2} + \ldots + L_{x_n} \). The construction of the Green’s function for the two-dimensional case is shown below.

Consider the Cartesian coordinates \( \{x, y\} \) in two-dimensions. The equation \( L_\mathbf{x}[G](\mathbf{x}|\mathbf{x'}) = \delta(\mathbf{x} - \mathbf{x'}) \) needs to be solved. Since \( L_\mathbf{x} = L_x + L_y \) one can write

\[
L_\mathbf{x}[G](\mathbf{x}|\mathbf{x'}) = L_x G + L_y G = \delta(x - x') \delta(y - y').
\]  

(4.1.16)

If one writes \( \delta(x - x') \delta(y - y') = \lambda G = (\lambda_x + \lambda_y) G \), then one has two (one-dimensional) Sturm-Liouville problems

\[
L_x G = \lambda_x G, \quad \text{with solutions } \varphi_{n_x} \text{ and } \lambda_{n_x} \text{ for } n_x = 1, 2, \ldots,
\]

(4.1.17a)

\[
L_y G = \lambda_y G, \quad \text{with solutions } \varphi_{n_y} \text{ and } \lambda_{n_y} \text{ for } n_y = 1, 2, \ldots.
\]

(4.1.17b)

Since the eigenfunctions \( \varphi_{n_x}(x) \) and \( \varphi_{n_y}(y) \) are complete in their one-dimensional counterparts, the products of them form also a complete set in the two-dimensional region. Therefore, the Green’s function can be written as a linear combination of the products,

\[
G(x, y|x', y') = \sum_{n_x} \sum_{n_y} a_{n_x n_y} \varphi_{n_x}(x) \varphi_{n_y}(y).
\]

(4.1.18)

The delta distribution can also be written as a linear combination of \( \varphi_{n_x} \varphi_{n_y} \),

\[
\delta(x - x') \delta(y - y') = \sum_{n_x} \sum_{n_y} c_{n_x n_y} \varphi_{n_x}(x) \varphi_{n_y}(y),
\]

(4.1.19)

where \( c_{n_x n_y} \) can be obtained by multiplying both sides of equation (4.1.19) with \( \varphi_{n_x} \varphi_{n_y} \) and integrating over the domain. Due to the orthogonality of \( \varphi_{n_x} \varphi_{n_y} \) for different \( n_x, n_y \) and due to the property of the delta distribution in a integral, this yields for \( c_{n_x n_y} \)

\[
c_{n_x n_y} = \frac{\varphi_{n_x}(x') \varphi_{n_y}(y')}{(\varphi_{n_x} \varphi_{n_y}, \varphi_{n_x} \varphi_{n_y})},
\]

(4.1.20)

where the inner product \( (\varphi_{n_x} \varphi_{n_y}, \varphi_{n_x} \varphi_{n_y}) \) is defined as

\[
(\varphi_{n_x} \varphi_{n_y}, \varphi_{n_x} \varphi_{n_y}) = \iint_{\Omega} \varphi_{n_x}(x) \varphi_{n_y}(y) \varphi_{n_x}(x) \varphi_{n_y}(y) \, dx \, dy = \| \varphi_{n_x} \varphi_{n_y} \|^2.
\]

(4.1.21)
From $L_x[G] - \lambda G = 0$ and $\delta(x - x', y - y') = \lambda G$ one obtains

$$
\lambda G = \sum_{n_x} \sum_{n_y} (\lambda_{n_x} + \lambda_{n_y}) a_{n_x n_y} \varphi_{n_x}(x) \varphi_{n_y}(y)
$$

$$
= \sum_{n_x} \sum_{n_y} \varphi_{n_x}(x') \varphi_{n_y}(y') \varphi_{n_x}(x) \varphi_{n_y}(y)
$$

$$
\Rightarrow a_{n_x n_y} = \frac{1}{\lambda_{n_x} + \lambda_{n_y} (\varphi_{n_x} \varphi_{n_y}, \varphi_{n_x} \varphi_{n_y})} \varphi_{n_x}(x') \varphi_{n_y}(y')
$$

(4.1.22)

and thus

$$
G(x, y| x', y') = \sum_{n_x} \sum_{n_y} \frac{1}{\lambda_{n_x} + \lambda_{n_y} (\varphi_{n_x} \varphi_{n_y}, \varphi_{n_x} \varphi_{n_y})} \varphi_{n_x}(x') \varphi_{n_y}(y') \varphi_{n_x}(x) \varphi_{n_y}(y).
$$

(4.1.23)

The diffusion problem thus reduces to the problem of finding eigenfunctions of the Sturm-Liouville problem after the separation of variables method has been applied. For solving the Poisson equation one needs to find the Green’s function for the corresponding domain. To solve the Poisson equation the eigenstructure, i.e. the eigenfunctions and eigenvalues, of the Laplacian $\nabla^2$ for the square and the equilateral triangle needs to be found. First, a nice result will be shown by combining the diffusion and the Poisson equation. Recall that the solution of the diffusion equation is used for the input of the Poisson equation.

### 4.1.2 The Diffusion and the Poisson equation combined

The diffusion problem $\frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega$ needs to be solved. After obtaining the vorticity $\omega$, it can be put as a known function in the Poisson equation $\omega = -\nabla^2 \psi$. As mentioned before, the diffusion equation becomes a Sturm-Liouville problem after separation of variables. So we try the separation

$$
\omega(x, y, t) = X(x)Y(y)T(t).
$$

After substitution in the diffusion equation one obtains two Sturm-Liouville problems and a first order ordinary differential equation. For the $x$-coordinate the Sturm-Liouville operator reads as $L_x = \frac{d^2}{dx^2}$, the functions $p(x), r(x)$ equal 1 and $q(x) = 0$. The solution of this problem consist of the eigenfunctions $\varphi_{n_x}$ with its corresponding eigenvalues $\lambda_{n_x}^2$. Similarly, for the $y$-coordinate we obtain eigenfunctions $\varphi_{n_y}$ with corresponding eigenvalues $\lambda_{n_y}^2$. For $T(x)$ we find the ODE

$$
T' + \nu \lambda^2 T = 0, \quad t > 0,
$$

(4.1.24)
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The full solution of the diffusion equation is thus

$$
\omega(x, y, t) = X(x)Y(y)T(t) = \sum_{n_x} \sum_{n_y} c_{n_x n_y} e^{-t \lambda^2_{n_x n_y}} \phi_{n_x}(x) \phi_{n_y}(y),
$$

(4.1.25)

where $\lambda^2_{n_x n_y} = \lambda^2_{n_x} + \lambda^2_{n_y}$. Substituting $t = 0$ we thus have

$$
\omega(x, y, 0) = \sum_{n_x} \sum_{n_y} c_{n_x n_y} \phi_{n_x}(x) \phi_{n_y}(y) = \gamma \delta(x - x_0, y - y_0).
$$

(4.1.26)

Multiplication with $\phi_{n_x}(x) \phi_{n_y}(y)$ and integrating over the domain we obtain for $c_{n_x n_y}$

$$
c_{n_x n_y} = \frac{\gamma \phi_{n_x}(x_0) \phi_{n_y}(y_0)}{\| \phi_{n_x} \phi_{n_y} \|^2}.
$$

(4.1.27)

The full solution of the diffusion equation is thus

$$
\omega(x, y, t) = X(x)Y(y)T(t) = \sum_{n_x} \sum_{n_y} e^{-t \lambda^2_{n_x n_y}} \frac{\gamma \phi_{n_x}(x_0) \phi_{n_y}(y_0)}{\| \phi_{n_x} \phi_{n_y} \|^2} \phi_{n_x}(x) \phi_{n_y}(y).
$$

(4.1.28)

The solution $\omega(x, y, t)$ is used as input to solve the Poisson equation. This boils down to finding the Green’s function for the different domains. Consider domains which are products of orthogonal coordinate systems. The Green’s functions $G(x, y|x', y')$ of these domains are given by equation (4.1.23). For the stream function $\psi(x, y, t)$ we find,

$$
\psi(x, y, t) = \int_\Omega G(x, y|x', y')w(x', y', t) \, dx' \, dy'
$$

$$
= \int_\Omega \left\{ \left( \sum_{n_x} \sum_{n_y} \frac{1}{\lambda^2_{n_x n_y}} \phi_{n_x}(x) \phi_{n_y}(y) \phi_{n_x}(x') \phi_{n_y}(y') \right) \right. \\
\left. \left( \sum_{n_x} \sum_{n_y} \frac{\gamma \phi_{n_x}(x_0) \phi_{n_y}(y_0)}{\| \phi_{n_x} \phi_{n_y} \|^2} e^{-t \lambda^2_{n_x n_y}} \phi_{n_x}(x') \phi_{n_y}(y') \right) \right\} \, dx' \, dy'
$$

$$
= \int_\Omega \left\{ \sum_{n_x} \sum_{n_y} \lambda^2_{n_x n_y} \frac{\gamma}{\| \phi_{n_x} \phi_{n_y} \|^2} \phi_{n_x}(x) \phi_{n_y}(y) \phi_{n_x}(x_0) \phi_{n_y}(y_0) e^{-t \lambda^2_{n_x n_y}} \phi_{n_x}(x') \phi_{n_y}(y') \right. \\
\left. \left. \cdot \phi_{n_x}(x') \phi_{n_y}(y') \phi_{n_x}(x') \phi_{n_y}(y') \right) \right\} \, dx' \, dy'
$$

$$
= \sum_{n_x} \sum_{n_y} \frac{\gamma}{\lambda^2_{n_x n_y}} \sigma_{n_x n_y} \phi_{n_x}(x) \phi_{n_y}(y) \phi_{n_x}(x_0) \phi_{n_y}(y_0) e^{-t \lambda^2_{n_x n_y}}
$$

$$
= \sum_{n_x} \sum_{n_y} \frac{\gamma}{\lambda^2_{n_x n_y}} \phi_{n_x}(x) \phi_{n_y}(y) \phi_{n_x}(x_0) \phi_{n_y}(y_0) e^{-t \lambda^2_{n_x n_y}},
$$

(4.1.29)
where the latter equation holds due to the orthogonality of the eigenfunctions and the fact that $\sigma_{n_xn_y}$ is defined as

\[
\sigma_{n_xn_y} = \int_{\Omega} \varphi_{n_x}(x')\varphi_{n_y}(y')\varphi_{n_x}(x')\varphi_{n_y}(y')dx'dy' = \|\varphi_{n_x}\varphi_{n_y}\|^2. \tag{4.1.30}
\]

Thanks to the fact that the Poisson and diffusion equation are defined on the same domain and have similar boundary conditions, we can use the same eigenfunctions and Green’s functions in both cases. This greatly simplifies their simultaneous treatment.

### 4.2 The eigenstructure of the different domains

As shown in the previous section, solving the PDE’s is mainly about finding eigenfunctions that satisfy the boundary conditions of a certain geometry. The first domain to be studied will be the rectangular domain. The eigenvalue problem in this domain is well-known. Therefore, the reader may skip the derivation and only take a look at the resulting expression for the stream function. The eigenvalue problem for the equilateral triangle is not as common as for the rectangle. The eigenfunctions for the equilateral triangle were already found in the 19th century, but it is still unknown how this set of eigenfunctions has been constructed. In the last section the obtained expressions for the stream functions are analysed and compared to the potential flow case.

#### 4.2.1 Rectangular domain

Consider the following problem in a rectangle with sides of length $a$ and $b$,

\[
\begin{aligned}
\frac{\partial \omega}{\partial t} &= \nu \left( \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right), \quad 0 < x < a, \quad 0 < y < b, \quad t > 0, \\
\omega(x, y, 0) &= \omega_0(x, y), \quad 0 < x < a, \quad 0 < y < b, \\
\omega(0, y, t) &= \omega(a, y, t) = 0, \quad 0 < y < b, \quad t > 0, \\
\omega(x, 0, t) &= \omega(x, b, t) = 0, \quad 0 < x < a, \quad t > 0,
\end{aligned} \tag{4.2.1}
\]

Due to the separate boundary conditions it is possible to use the separation of variables method. Write

\[
\omega(x, y, t) = X(x)Y(y)T(t).
\]
After substitution in the differential equation we find
\[ \frac{1}{\nu} T' T = \frac{X''}{X} + \frac{Y''}{Y} = -\kappa^2, \]
(4.2.2)
for some constant \( \kappa \). This turns into the following set of ordinary differential equations for some constant \( \mu \),
\[
\begin{cases}
X'' + \mu^2 X = 0, & 0 < x < a, \\
X(0) = X(a) = 0,
\end{cases}
\]
(4.2.3)
and
\[
\begin{cases}
Y'' + (\kappa^2 - \mu^2) Y = 0, & 0 < y < b, \\
Y(0) = Y(b) = 0,
\end{cases}
\]
(4.2.4)
and
\[
\begin{cases}
T' + \nu \kappa^2 T = 0, & t > 0, \\
T(0) = 1.
\end{cases}
\]
(4.2.5)
Problem (4.2.3) leads to eigenvalues
\[ \mu^2 = \mu_n^2 = \left( \frac{n\pi}{a} \right)^2, \quad n = 1, 2, \ldots, \]
(4.2.6)
with the orthogonal eigenfunctions
\[ X = X_n(x) = \sin\left( \frac{n\pi}{a} x \right), \quad 0 \leq x \leq a, \quad n = 1, 2, \ldots. \]
(4.2.7)
For problem (4.2.4) we similarly find
\[ \kappa^2 - \mu^2 = \left( \frac{m\pi}{b} \right)^2, \quad m = 1, 2, \ldots, \]
(4.2.8)
with orthogonal eigenfunctions
\[ Y = Y_m(x) = \sin\left( \frac{m\pi}{b} y \right), \quad 0 \leq y \leq b, \quad m = 1, 2, \ldots. \]
(4.2.9)
Problem (4.2.5) leads to
\[ T = T_0 e^{-\nu \kappa^2 t}. \]
(4.2.10)
The yet unknown $\kappa^2$ is determined from

$$\kappa^2 = \left(\frac{m\pi}{b}\right)^2 + \mu^2 = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}, \quad n, m = 1, 2, \ldots, \quad (4.2.11)$$

which we denote as $\lambda^2_{mn}$. Thus the family of product solutions of this problem consist of the constant multiples of

$$\omega_{mn}(x, y, t) = e^{-t\nu\lambda_{mn}} \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right), \quad (4.2.12)$$

and due to the linearity of the differential equation with its homogeneous boundary condition this yields

$$\omega(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} e^{-t\nu\lambda_{mn}} \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right). \quad (4.2.13)$$

One can find $c_{mn}$ by making use of the initial condition $w_0(x, y)$,

$$\omega(x, y, 0) = \omega_0(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right), \quad (4.2.14)$$

with

$$\omega_0(x, y) = \gamma \delta(x - x_0) \delta(y - y_0), \quad (4.2.15)$$

where $x_0 = \frac{a}{2}, y_0 = \frac{b}{2}$ and $\gamma$ the strength of the vortex. This equation can be rewritten into a linear combination of the orthogonal functions $\sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right)$ by using the double fourier series

$$\delta(x - x_0) \delta(y - y_0) = \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a} x_0\right) \sin\left(\frac{m\pi}{b} y_0\right) \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right). \quad (4.2.16)$$

Which gives

$$c_{mn} = \gamma \frac{4}{ab} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{m\pi}{2}\right), \quad (4.2.17)$$

and the solution $\omega_0$ of the diffusion equation in a rectangular domain becomes

$$\omega_0(x, y, t) = \gamma \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{m\pi}{2}\right) \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right) e^{-t\nu\lambda_{mn}}. \quad (4.2.18)$$

The found $\omega_0(x, y, t)$ is used to obtain the stream function.

Before solving the Poisson equation, the Green’s function for the Laplace operator in the
rectangle needs to be found. Finding this Green’s function \( g(x, y|x', y') \) requires solving the following problem

\[
\begin{aligned}
\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} &= -\delta(x - x', y - y'), \\
g(0, y|x', y') &= g(a, y|x', y') = 0, \\
g(x, 0|x', y') &= g(x, b|x', y') = 0,
\end{aligned}
\]  
\hspace{1cm} 0 < x, x' < a, \quad 0 < y, y' < b, \quad (4.2.19)

One approach for finding the Green’s function is to expand it in terms of eigenfunctions \( \varphi(x, y) \) which are solutions of the following problem:

\[
\begin{aligned}
\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} &= -\lambda \varphi, \\
\varphi(0, y) &= \varphi(a, y) = 0, \\
\varphi(x, 0) &= \varphi(x, b) = 0,
\end{aligned}
\]  
\hspace{1cm} 0 < x < a, \quad 0 < y < b, \quad (4.2.20)

The eigenvalues and the corresponding eigenfunctions are

\[
\varphi_{mn}(x, y) = \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right), \quad \lambda_{mn} = \frac{n^2 \pi^2}{a^2} + \frac{m^2 \pi^2}{b^2}, \quad n, m = 1, 2, \ldots \quad (4.2.21)
\]

Therefore the Green’s function is given by

\[
g(x, y|x', y') = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right), \quad (4.2.22)
\]

where \( A_{mn} \) depends on \( x' \) and \( y' \) only. From (4.2.16) one has

\[
\left(\frac{n^2 \pi^2}{a^2} + \frac{m^2 \pi^2}{b^2}\right) A_{mn} = \frac{4}{ab} \sin\left(\frac{n\pi}{a} x'\right) \sin\left(\frac{m\pi}{b} y'\right). \quad (4.2.23)
\]

Thus the Green’s function of the Laplace operator for a rectangular domain is given by

\[
g(x, y|x', y') = \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{a} x'\right) \sin\left(\frac{m\pi}{b} y'\right) \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right)}{n^2 \pi^2 / a^2 + m^2 \pi^2 / b^2}. \quad (4.2.24)
\]
Figure 4.1: Green’s function of the rectangle with the peak at \((x', y')\)

From \(\nabla \psi = -\omega\) one obtains the stream function \(\psi\),

\[
\psi_0(x, y, t) = \int_0^a \int_0^b g(x, y|x', y')\omega(x', y', t) \, dx' \, dy'
\]

\[
= \gamma \left(\frac{4}{ab}\right)^2 \int_0^a \int_0^b \left\{ \left( \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\sin \left(\frac{n\pi}{2} x'\right) \sin \left(\frac{m\pi}{2} y'\right)}{n^2\pi^2/a^2 + m^2\pi^2/b^2} \right) \right.
\]

\[
\left. \left( \sum_{k=1}^\infty \sum_{l=1}^\infty \sin \left(\frac{k\pi}{2} \right) \sin \left(\frac{l\pi}{2} \right) \sin \left(\frac{k\pi}{a} x'\right) \sin \left(\frac{l\pi}{b} y'\right) e^{-\nu t\lambda kl} \right) \right\} \, dx' \, dy'.
\]

\[
= \gamma \left(\frac{4}{ab}\right)^2 \int_0^a \int_0^b \sum_{m=1}^\infty \sum_{n=1}^\infty \sigma_{mn} \frac{\sin \left(\frac{m\pi}{2} \right) \sin \left(\frac{n\pi}{2} \right) \sin \left(\frac{n\pi}{a} x'\right) \sin \left(\frac{n\pi}{b} y'\right)}{n^2\pi^2/a^2 + m^2\pi^2/b^2} \, dx' \, dy' \cdot e^{-\nu t\lambda mn},
\]

where the latter equations holds due to the fact that for \(n \neq k\) or \(m \neq l\) the pairs

\(\sin \left(\frac{n\pi}{a} x'\right) \sin \left(\frac{m\pi}{b} y'\right)\) and \(\sin \left(\frac{k\pi}{a} x'\right) \sin \left(\frac{l\pi}{b} y'\right)\) are orthogonal and \(\sigma_{mn}\) equals

\[
\sigma_{mn} = \int_0^a \int_0^b \sin \left(\frac{n\pi}{a} x'\right) \sin \left(\frac{m\pi}{b} y'\right) \sin \left(\frac{n\pi}{a} x'\right) \sin \left(\frac{n\pi}{b} y'\right) \, dx' \, dy',
\]

\[
= \| \sin \left(\frac{n\pi}{a} x'\right) \sin \left(\frac{m\pi}{b} y'\right) \|^2,
\]

\[
= \frac{ab}{4}.
\]
The stream function thus is found as

$$\psi_\diamond(x, y, t) = \gamma \frac{4}{ab} \sum_{n,m=1}^{\infty} \frac{\sin\left(\frac{m\pi}{2}\right) \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right)}{n^2 \pi^2 / a^2 + m^2 \pi^2 / b^2} e^{-t\nu(n^2 \pi^2 / a^2 + m^2 \pi^2 / b^2)}. \quad (4.2.27)$$

Note that after a long time the fluid should be in rest, which is indeed the case since $\psi(x, y, t) \to 0$ for $t \to \infty$.

### 4.2.2 Triangular domain

The Dirichlet eigenvalue problem for the equilateral triangle were first presented by Lamé [7] and then further explored by Pockels [16]. However, Lamé only stated his formulas and showed that they satisfied the boundary conditions but McCartin [11] provided a derivation of Lamé’s formulas from first principles. McCartin also presented a corresponding treatment of the Neumann and Robin problems, see McCartin [10][12]. The proofs for the theorems in this section can be found in the work of McCartin [11]. Lamé considered the following eigenvalue problem

$$\begin{cases} \nabla^2 T(x, y) + k^2 T(x, y) = 0, & (x, y) \in \tau, \\ T(x, y) = 0, & (x, y) \in \partial\tau, \end{cases} \quad (4.2.28)$$

where $\tau$ is the equilateral triangle and $\partial\tau$ is the boundary of the triangle. Lamé was able to show that the eigenfunctions of problem 4.2.2 could be expressed in terms of combinations of sines and cosines. Before showing this representation, the following theorem observed by Lamé must be presented.

**Theorem 4.2.1.** Suppose that $T(x, y)$ can be represented by the trigonometric series

$$T(x, y) = \sum_i A_i \sin(\lambda_i x + \mu_i y + \alpha_i) + B_i \cos(\lambda_i x + \mu_i y + \beta_i) \quad (4.2.29)$$

with $\lambda_i^2 + \mu_i^2 = k^2$. Then

i. $T(x, y)$ is antisymmetric about any line on which it vanishes.

ii. $T(x, y)$ is symmetric about any line along which its normal derivative vanishes.

Theorem 4.2.1 has the following consequences.

**Corollary 4.2.2.** With $T(x, y)$ written as (4.2.29),
i. if \( T(x, y) = 0 \) along the boundary of a polygon, then \( T(x, y) = 0 \) along the boundaries of the family of congruent and symmetrically placed polygons obtained by reflection about its sides.

ii. if the normal derivative of \( T(x, y) \) is zero along the boundary of a polygon, then the normal derivative of \( T(x, y) \) is zero along the boundaries of the family of congruent and symmetrically placed polygons obtained by reflection about its sides.

Consider Figure 4.2 where the antisymmetry (denoted by \( \pm \)) is produced by repeatedly antireflecting about its edges.

\[
\begin{align*}
\text{Figure 4.2: The antisymmetric, equilateral, triangular lattice}
\end{align*}
\]

The lattice of the equilateral triangles and Theorem 4.2.1 suggest that the solution of the eigenvalue problem in this domain might be written as (4.2.29). It might be written in trigonometric series, since the representation in trigonometric series implies the vanishing properties at the boundaries and not vice versa. However, Lamé constructed a family of eigenfunctions which solves the eigenvalue problem in an equilateral triangle and where the eigenfunctions are indeed trigonometric functions.

Consider an equilateral triangle of side \( a \) in Cartesian coordinates \((x, y)\). Define the triangular coordinate system \((u, v, w)\) by

\[
\begin{align*}
  u &= r - y, \\
  v &= \frac{\sqrt{3}}{2} \left( x - \frac{a}{2} \right) + \frac{1}{2} (y - r), \\
  w &= \frac{\sqrt{3}}{2} \left( \frac{a}{2} - x \right) + \frac{1}{2} (y - r),
\end{align*}
\]
where \( r = a/(2\sqrt{3}) \) is the inradius of the triangle. The triangular coordinates satisfy the relation

\[
u + v + w = 0.
\]

The center of the triangle has coordinates \((0, 0, 0)\) and the three sides of the triangle are given by \( u = r, v = r \) and \( w = r \), see Figure 4.3

By defining the orthogonal coordinates \((\xi, \eta)\) as

\[
\xi = u, \quad \text{and} \quad \eta = v - w,
\]

the eigenvalue problem changes into

\[
\begin{cases}
\frac{\partial^2 T}{\partial \xi^2} + 3 \frac{\partial^2 T}{\partial \eta^2} + k^2 T = 0, \\
T(u, v, w) = 0, \quad \text{for} \quad u \in \{r, -2r\} \text{ or } v \in \{r, -2r\} \text{ or } w \in \{r, -2r\}.
\end{cases}
\]

Due to the separated boundary conditions one can seek a solution of the form \( T = f(\xi)g(\eta) \), which yields the following set of equations

\[
\begin{align*}
f'' + \alpha^2 f &= 0, \\
g'' + \beta^2 g &= 0, \\
\alpha^2 + 3\beta^2 &= k^2,
\end{align*}
\]
where \( k \) is constant. Here the sought eigenfunction \( T(u, v, w) = f(u)g(v - w) \) will be decomposed into a symmetric part \( T_s(u, v, w) \) and an antisymmetric part \( T_a(u, v, w) \) about the altitude \( v = w \),

\[
T(u, v, w) = T_s(u, v, w) + T_a(u, v, w),
\]

(4.2.35)

where

\[
T_s(u, v, w) = \frac{T(u, v, w) + T(u, w, v)}{2} \quad \text{and} \quad T_a(u, v, w) = \frac{T(u, v, w) - T(u, w, v)}{2}.
\]

(4.2.36a)

(4.2.36b)

The trigonometric series for \( T_s \) and \( T_a \) are obtained separately. Both \( T_s \) and \( T_a \) must vanish along all three sides of the triangle, while \( T_s \) being even as a function of \( v - w \) and \( T_a \) being odd as a function of \( v - w \). One can try for \( T_s \) a function of the form

\[
\sin \left[ \frac{\pi l}{3r} (u + 2r) \right] \cos[\beta_1(v - w)],
\]

(4.2.37)

where \( l \) is an integer and \( \left[ \frac{\pi l}{3r} \right]^2 + 3\beta_1^2 = k^2 \). Note that when the Dirichlet boundary condition is satisfied for \( v = r \) then, by symmetry, it would be automatically satisfied along \( w = r \). McCartin [11] showed that the form in equation 4.2.37 does not fit the boundary condition at \( v = r \). He also showed that a sum of the form

\[
\sin \left[ \frac{\pi l}{3r} (u + 2r) \right] \cos[\beta_1(v - w)] + \sin \left[ \frac{\pi m}{3r} (u + 2r) \right] \cos[\beta_2(v - w)],
\]

(4.2.38)

with \( \left[ \frac{\pi l}{3r} \right]^2 + 3\beta_1^2 = \left[ \frac{\pi m}{3r} \right]^2 + 3\beta_2^2 = k^2 \) and \( l, m \) integers, also does not fit the boundary condition \( (v = r) \). McCartin showed that a sum of three terms does satisfy the boundary conditions. The sum of three terms, which was also found by Lamé, looks like

\[
T_s = \sin \left[ \frac{\pi l}{3r} (u + 2r) \right] \cos[\beta_1(v - w)] + \\
\sin \left[ \frac{\pi m}{3r} (u + 2r) \right] \cos[\beta_2(v - w)] + \\
\sin \left[ \frac{\pi n}{3r} (u + 2r) \right] \cos[\beta_3(v - w)],
\]

(4.2.39)

with \( \left[ \frac{\pi l}{3r} \right]^2 + 3\beta_1^2 = \left[ \frac{\pi m}{3r} \right]^2 + 3\beta_2^2 = \left[ \frac{\pi n}{3r} \right]^2 + 3\beta_3^2 = k^2 \) and \( l, m, n \) integers. To satisfy the boundary condition, one has

\[
\beta_1 = \frac{\pi (m - n)}{9r}, \quad \beta_2 = \frac{\pi (n - l)}{9r}, \quad \beta_3 = \frac{\pi (l - m)}{9r},
\]

(4.2.40)
and also the relation
\[ l + m + n = 0. \] (4.2.41)

The eigenvalue \( k^2 \) is now given by
\[ k^2 = \frac{2}{27} \left( \frac{\pi}{r} \right)^2 [l^2 + m^2 + n^2] = \frac{4}{27} \left( \frac{\pi}{r} \right)^2 [m^2 + mn + n^2]. \] (4.2.42)

The antisymmetric part \( T_a(u, v, w) \) is obtained in the same way. Now the oddness of \( T_a \) as a function of \( v - w \) plays an important role. One has
\[ T_a = \sin \left[ \frac{\pi l}{3r} (u + 2r) \right] \sin[\alpha_1(v - w)] + \sin \left[ \frac{\pi m}{3r} (u + 2r) \right] \sin[\alpha_2(v - w)] + \sin \left[ \frac{\pi n}{3r} (u + 2r) \right] \sin[\alpha_3(v - w)], \] (4.2.43)

where \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) turn out to be equal to \( \beta_1, \beta_2 \) and \( \beta_3 \), respectively. Also relation (4.2.41) holds and the eigenvalue is the same as in the symmetric mode.

McCartin showed that only the set \( \{T_{mn}^s(n \leq m), T_{mn}^a(n < m); 0 < m \leq n\} \) needs to be considered, where \( T_{mn}^s \) equals
\[ T_{mn}^s = \sin \left[ \frac{\pi l}{3r} (u + 2r) \right] \cos \left[ \frac{\pi (m - n)}{9r} (v - w) \right] + \sin \left[ \frac{\pi m}{3r} (u + 2r) \right] \cos \left[ \frac{\pi (n - l)}{9r} (v - w) \right] + \sin \left[ \frac{\pi n}{3r} (u + 2r) \right] \cos \left[ \frac{\pi (l - m)}{9r} (v - w) \right], \] (4.2.44)
and \( T_{mn}^a \) equals
\[ T_{mn}^a = \sin \left[ \frac{\pi l}{3r} (u + 2r) \right] \sin \left[ \frac{\pi (m - n)}{9r} (v - w) \right] + \sin \left[ \frac{\pi m}{3r} (u + 2r) \right] \sin \left[ \frac{\pi (n - l)}{9r} (v - w) \right] + \sin \left[ \frac{\pi n}{3r} (u + 2r) \right] \sin \left[ \frac{\pi (l - m)}{9r} (v - w) \right]. \] (4.2.45)

He also showed that the collection of eigenfunctions \( \{T_{mn}^s, T_{mn}^a\} \) is complete. Also the orthogonality of the eigenfunctions holds and the norms are
\[ \|T_{mn}^s\|^2 = \|T_{mn}^a\|^2 = \frac{9r^2 \sqrt{3}}{4} \quad (m \neq n) \quad \text{and} \quad \|T_{mn}^s\|^2 = \frac{9r^2 \sqrt{3}}{2}. \] (4.2.46)
Using equation (4.1.23) the Green’s function \( g(x, y|x', y') \) is given by

\[
g(x, y|x', y') = \sum_{m=1}^{\infty} \frac{1}{\lambda_{mn}} \frac{T_{s}^{mn}(x', y')T_{s}^{mn}(x, y)}{\|T_{s}^{mn}\|^2} + \sum_{n=m+1}^{\infty} \frac{1}{\lambda_{mn}} \left[ \frac{T_{s}^{mn}(x', y')T_{s}^{mn}(x, y)}{\|T_{s}^{mn}\|^2} + \frac{T_{a}^{mn}(x', y')T_{a}^{mn}(x, y)}{\|T_{a}^{mn}\|^2} \right],
\]

(4.2.47)

where \( \lambda_{mn} = \frac{1}{27}(\frac{\pi}{r})^2[m^2 + mn + n^2] \). The stream function is given by

\[
\psi_{\Delta}(x, y, t) = \int_{\Omega} g(x, y|x', y') \omega_{\Delta}(x', y', t) \, dx' \, dy',
\]

(4.2.48)

where \( \omega_{\Delta}(x', y', t) \) is given by (see equation (4.1.28)),

\[
\omega_{\Delta}(x', y', t) = \sum_{k=1}^{\infty} \gamma \left\{ e^{-t\nu_{kk}} \frac{T_{s}^{kk}(x_0, y_0)T_{s}^{kk}(x', y')}{\|T_{s}^{kk}\|^2} + \sum_{l=k+1}^{\infty} e^{-t\nu_{kl}} \left[ \frac{T_{s}^{kl}(x_0, y_0)T_{s}^{kl}(x', y')}{\|T_{s}^{kl}\|^2} + \frac{T_{a}^{kl}(x_0, y_0)T_{a}^{kl}(x', y')}{\|T_{a}^{kl}\|^2} \right] \right\},
\]

(4.2.49)

and thus

\[
\psi_{\Delta}(x, y, t) = \int_{\Omega} \left\{ \sum_{m=1}^{\infty} \frac{e^{-t\lambda_{mm}} T_{s}^{mn}(x_0, y_0) T_{s}^{mn}(x, y) T_{s}^{mn}(x', y') T_{s}^{mn}(x', y')}{\lambda_{mn} \|T_{s}^{mn}\|^2} + \sum_{n=m+1}^{\infty} \frac{e^{-t\lambda_{mn}} T_{s}^{mn}(x_0, y_0) T_{s}^{mn}(x, y) T_{s}^{mn}(x', y') T_{s}^{mn}(x', y')}{\lambda_{mn} \|T_{s}^{mn}\|^2} + \frac{T_{a}^{mn}(x_0, y_0) T_{a}^{mn}(x, y) T_{a}^{mn}(x', y') T_{a}^{mn}(x', y')}{\|T_{a}^{mn}\|^2} \right\} \, dx' \, dy',
\]

(4.2.50)
Figure 4.4: Green’s function of the triangle with the peak at \((x', y')\)

In the extended lattice of Figure 4.2, one can clearly see the antisymmetric structure: see Figure 4.5

Figure 4.5: Different views of the antisymmetry of the equilateral lattice
4.3 Results

From the stream function one can directly find the streamlines on different times. It has already been noticed that the fluid comes to rest after a long time. But what about the initial situation? In the previous chapter the stationary solution for the complex velocity potential has been found for the potential flow. In this chapter, this potential flow is taken as initial condition. So the stream function obtained from the complex velocity potential is considered to be equal to the stream function at $t = 0$, for the decaying system described by the combination of the diffusion and the Poisson equation. The decay of the flow field will be calculated for both the square and the triangle, so that a comparison is possible.

4.3.1 The initial cases

The stream function for the potential flow in a square is obtained by taking the imaginary part of equation (3.2.10). This leads to

$$
\Im(w_\diamond(z)) = \Im\left(\frac{i\gamma}{2\pi} \left(\ln(sn(z - K|m) - sn(ib/2 - K|m)) - \ln(sn(z - K|m) - sn(ib/2 - K|m))\right)\right),
$$

(4.3.1)

where $z = x + iy$, $K = \frac{a}{2}$ and $b = a$ for making the rectangle a square with sides $a$. On the other hand, by setting $t = 0$ in equation (4.2.27) we obtain an alternative expression for the stream function, namely

$$
\psi_\diamond(x, y, 0) = \gamma \frac{4}{ab} \sum_{n,m=1}^{\infty} \frac{\sin\left(\frac{m\pi}{2}\right) \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right)}{n^2\pi^2/a^2 + m^2\pi^2/b^2},
$$

(4.3.2)

Since both expressions are solutions of the same equations with the same boundary conditions, they represent the same profile, although this is far from straightforward to see from their functional forms. To check this, their level plots are given in Figure 4.6.
The shapes of the streamlines look indeed the same in both figures. Due to the complexity of the functions, different points in the square are calculated numerically and compared to each other.

Define the relative error $\epsilon_\diamond$ for the square as

$$
\epsilon_\diamond = \frac{\Im(w_\diamond(z)) - \psi_\diamond(x, y, 0)}{\Im(w_\diamond(z)) + \psi_\diamond(x, y, 0)}.
$$

In Figure 4.7 the relative errors are plotted as a function of position in the square. The points in Figure 4.7(a) are running from the bottom left corner to the vortex in the center of mass. In Figure 4.7(b) 500 random points in the square are taken.
From the figures one can conclude that the relative error is constant. This means that $\Im(w_\diamond(z))$ and $\psi_\diamond(x, y, 0)$ are related by a multiplicative factor. This multiplicative factor, $C_{\epsilon_\diamond}$ say, equals

$$C_{\epsilon_\diamond} = \frac{1 - \epsilon_\diamond}{\epsilon_\diamond + 1} \rightarrow \Im(w_\diamond(z)) = C_{\epsilon_\diamond} \psi_\diamond(x, y, 0).$$

To understand why the multiplicative factor do not equal unity, one should realize that the governing equation is linear, so that the solution is determined up to a constant factor. The boundary conditions do not fix this constant, since they are of the Neumann type.

Similarly, for the equilateral triangular case, the following expressions must be the same up to a multiplicative constant:

$$\Im(w_\Delta(z)) = \Im\left(\frac{i\gamma}{2\pi} \left(\ln\left(1^{1/3} + \frac{1}{2}\right) - \ln\left(1^{1/3} - 1^{1/3}\right)\right) - \ln\left(1^{1/3} + \frac{1}{2}\right) - \ln\left(1^{1/3} - 1^{1/3}\right)\right),$$

and

$$\psi_\Delta(x, y, 0) = \sum_{m=1}^{\infty} \left\{ \frac{\gamma}{\lambda_{mn}} \frac{T_{s_{mn}}^m(x_0, y_0)T_{s_{mn}}^m(x, y)}{\|T_{s_{mn}}^m\|^2} + \sum_{n=m+1}^{\infty} \frac{\gamma}{\lambda_{mn}} \frac{T_{a_{mn}}^m(x_0, y_0)T_{a_{mn}}^m(x, y)}{\|T_{a_{mn}}^m\|^2}\right\}. $$

Figure 4.7: Relative errors in the square
where \( z = x + iy \) and \( a \) is the side of the equilateral equation. The streamlines of both stream functions are given in Figure 4.8.

![Streamlines in an equilateral triangle](image)

**(a)** Streamlines constructed from \( \Im(w_\Delta(z)) \)

**(b)** Streamlines constructed from \( \psi_\Delta(x, y, 0) \)

**Figure 4.8:** Streamlines in an equilateral triangle

Define for the equilateral triangle the relative error \( \epsilon_\Delta = (\Im(w_\Delta(z)) - \psi_\Delta(x, y, 0))/\psi_\Delta(x, y, 0) + \Im(w_\Delta(z)) \). In Figure 4.9 the relative errors are plotted. The points in Figure 4.9(a) are running from the bottom left corner to the vortex in the center of mass. In Figure 4.9(b) 500 random points in the equilateral triangle are taken.

![Relative errors in the equilateral triangle](image)

**(a)** Points from bottom left corner to vortex

**(b)** Random 500 points

**Figure 4.9:** Relative errors in the equilateral triangle
From Figure 4.9(a) one can conclude that the relative error is constant. The points in Figure 4.9(b) do not all converge to the same value due to numerical inaccuracies. To calculate the relative error we used a finite sum for equation (4.3.4) which causes the deviations. This means that \( \Im(w_\Delta(z)) \) and \( \psi_\Delta(x, y, 0) \) are related by a multiplicative factor. This multiplicative factor, \( C_{\epsilon_\Delta} \) say, equals
\[
C_{\epsilon_\Delta} = \frac{1 - \epsilon_\Delta}{\epsilon_\Delta + 1} \rightarrow \Im(w_\Delta(z)) = C_{\epsilon_\Delta} \psi_\Delta(x, y, 0).
\]

### 4.3.2 Evolution of a Vortex in a Square vs. an Equilateral Triangle

The viscous decay of the vortex is caused by the \( e^{-t\nu\lambda_{mn}} \) term in \((4.2.27)\) and \((4.2.50)\), which will be called the decay factor of mode\((mn)\). To see in which domain the fluid will be first in rest one should investigate the different eigenvalues. According to \((4.2.21)\), the eigenvalue for the rectangular domain is
\[
\lambda^\diamond_{mn} = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}, \tag{4.3.5}
\]
where \( a \) and \( b \) are the lengths of the sides of the rectangle. According to \((4.2.42)\), the eigenvalue of the equilateral triangle equals
\[
\lambda^\Delta_{mn} = \frac{4}{27} \left( \frac{\pi}{r} \right)^2 [m^2 + mn + n^2], \tag{4.3.6}
\]
where \( r \) is the inradius of the triangle which equals \( s/(2\sqrt{3}) \) if \( s \) is the length of the side of the triangle. Below some figures of \( e^{-t\lambda_{mn}} \) for different modes are plotted as functions of time \( t \). The side \( s \) of the triangle is chosen such that the two domains have equal areas and the sides of the rectangle is chosen such that it is a square, i.e. \( a = b \).

![Figure 4.10: Decay factors of different modes for square (left) and triangle (right)](image)
When only looking at the decay factors one can conclude that the vortex placed in the triangular domain decays a little bit faster. To show this, the decay factors of the square and the triangle are simultaneously plotted in Figure 4.11.

![Figure 4.11: Modes of the triangle](image)

The first mode clearly has the longest life time. The eigenvalues of the \((1, 1)\)-mode equal

\[
\lambda_{1,1}^\circ = \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}, \quad \text{and} \quad \lambda_{1,1}^\Delta = \frac{4 \pi^2}{9 r^2},
\]

with ratio

\[
\frac{\lambda_{1,1}^\circ}{\lambda_{1,1}^\Delta} = \frac{s^2}{8a^2} = \frac{1}{2\sqrt{3}},
\]

where \(s = \frac{2a}{\sqrt{3}}\), such that the areas are the same. In general the ratio of the eigenvalues of corresponding modes of the rectangle and the equilateral triangle equals

\[
\frac{\lambda_{mn}^\circ}{\lambda_{mn}^\Delta} = \frac{3s^2}{16} \frac{b^2n^2 + m^2a^2}{a^2b^2(m^2 + mn + n^2)},
\]

where \(s\) is the side of the triangle and \(a, b\) are the sides of the rectangle.
Chapter 5

Convection term \( J(\omega, \psi) \)

In the previous chapter, all calculations were done for the reduced model in which the convection term

\[
J(\omega, \psi) = \frac{\partial \omega}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial \omega}{\partial y} \frac{\partial \psi}{\partial x}
\]

(5.0.1)

was omitted in the vorticity equation

\[
\frac{D\omega}{Dt} = \frac{\partial \omega}{\partial t} + J(\omega, \psi) = \nu \nabla^2 \omega.
\]

(5.0.2)

In this chapter the contribution of the convection term is investigated. The two-dimensional domains in this chapter are assumed to be Cartesian products of orthogonal one-dimensional coordinate systems. In such domains the eigenfunctions formed from products of one-dimensional counterparts are complete. This will play an important role in this chapter.

5.1 Expansion in Eigenfunctions

From the relation

\[
\omega = -\nabla^2 \psi,
\]

(5.1.1)

one can write the vorticity equation (5.0.2) as

\[
\frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega - \frac{\partial \omega}{\partial x} \frac{\partial}{\partial y} \int \int G(x, y|x', y') \omega(x', y', t) dx'dy' + \frac{\partial \omega}{\partial y} \frac{\partial}{\partial x} \int \int G(x, y|x', y') \omega(x', y', t) dx'dy' \]

(5.1.2)
where \( G(x, y|x', y') \) is the Green’s function of the domain under consideration. Denote the orthogonal eigenfunctions formed from products of one-dimensional counterparts as \( \{ \phi^n(x)\phi^m(y)|n, m = 0, 1, 2, \ldots \} \). By completeness one can write \( \omega \) as

\[
\omega(x, y, t) = \sum_n \sum_m c_{mn}^n(t)\phi^n(x)\phi^m(y),
\]

(5.1.3)

and the Green’s function (see equation (4.1.23)) can be written as

\[
G(x, y|x', y') = \sum_n \sum_m \frac{1}{\lambda^2_m + \lambda^2_n} \phi^n(x)\phi^m(y) \| \phi^n \phi^m \|^2 \phi^n(x')\phi^m(y').
\]

(5.1.4)

This yields for the integral in (5.1.2)

\[
\int \int G(x, y|x', y')\omega(x', y', t) dx'dy' = \int \int \left\{ \sum_n \sum_m \frac{1}{\lambda^2_m + \lambda^2_n} \phi^n(x)\phi^m(y) \| \phi^n \phi^m \|^2 \phi^n(x')\phi^m(y') \right\} dx'dy'
= \sum_n \sum_m \left\{ \sum_k \sum_l \frac{c_{mn}^n(t)}{\lambda^2_m + \lambda^2_n} \| \phi^n \phi^m \|^2 \int \phi_n(x')\phi_m(y')dx'dy' \right\}
= \sum_n \sum_m \sum_k \sum_l c_{mn}^n(t) \phi^n(x)\phi^m(y).
\]

(5.1.5)

So after expanding into eigenfunctions, the vorticity equation (5.1.2) becomes

\[
\sum_n \sum_m c_{tn}^n t \phi^n(x)\phi^m(y) = \nu \left( \sum_n \sum_m c_{mn}^n \phi^n_{xx}(x)\phi^m(y) + \sum_n \sum_m c_{mn}^n t \phi^n(x)\phi^m_{yy}(y) \right)
- \sum_n \sum_m c_{mn}^n t \phi^n_x(x)\phi^m(y) \sum_k \sum_l \frac{c_{kl}^n(t)}{\lambda^2_k + \lambda^2_l} \phi^n_{kl}(x)\phi^m_{kl}(y)
+ \sum_n \sum_m c_{mn}^n t \phi^n(x)\phi^m_{yy}(y) \sum_k \sum_l \frac{c_{kl}^n(t)}{\lambda^2_k + \lambda^2_l} \phi^n_{kl}(x)\phi^m_{kl}(y),
\]

(5.1.6)

where \( \frac{\partial \phi^n}{\partial y} \equiv c_{mn}^n \), \( \frac{\partial \phi^n_{xx}}{\partial x^2} \equiv \phi^n_{xx} \) and \( \frac{\partial \phi^n_{yy}}{\partial x^2} \equiv \phi^n_{yy} \). Multiplication of equation (5.1.6) by \( \phi^n(x)\phi^m(y) \) and integration over the domain yields

\[
c_{tn}^n(t) \| \phi^n(y)\phi^n(x) \|^2 = -\nu(\lambda^2_m + \lambda^2_n)c_{mn}^n(t)\| \phi^n \phi^n \|^2 \int \phi^n_x(x)\phi^n(y) \phi^n_{xx}(y) dx dy
- \sum_n \sum_m \sum_k \sum_l \frac{c_{mn}^n c_{kl}^n(t)}{\lambda^2_k + \lambda^2_l} \int \phi^n_x(x)\phi^n_{kl}(x)(\phi^n_{kl}(y))^2 \phi^n_{y}(y) dx dy
+ \sum_n \sum_m \sum_k \sum_l \frac{c_{mn}^n c_{kl}^n(t)}{\lambda^2_k + \lambda^2_l} \int (\phi^n(x))^2 \phi^n_{xx}(x)\phi^n_{yy}(y) \phi^n_{y}(y) \phi^n_{y}(y) dx dy
\]

(5.1.7)
One can write the first integral of equation (5.1.7) as

\[
\int \int \phi^n_x(x)\phi^n_x(x)\phi^k_x(x)(\phi^m_y(y))^2\phi^l_y(y) \, dx \, dy
\]

\[= \int \phi^n_x(x)\phi^n_x(x)\phi^k_x(x) \, dx \int (\phi^m_y(y))^2\phi^l_y(y) \, dy \quad (5.1.8)
\]

The first integral on the right hand side of the equation equals

\[
\int \phi^n_x(x)\phi^n_x(x)\phi^k_x(x) \, dx
\]

\[= \phi^n(x)\phi^n(x)\phi^k(x) - \int (\phi^n_x(x)\phi^n_x(x)\phi^k(x) + \phi^n(x)\phi^n(x)\phi^k_x(x)) \, dx
\]

\[= -\frac{1}{2} \int \phi^n(x)\phi^n(x)\phi^k_x(x) \, dx, \quad (5.1.9)
\]

where the latter equation holds due to the homogeneous boundary conditions. The second integral in equation (5.1.7) can also be rewritten

\[
\int \int (\phi^n_x(x))^2\phi^k_x(x)\phi^m_y(y)\phi^m_y(y)\phi^l_y(y) \, dx \, dy
\]

\[= \int (\phi^n_x(x))^2\phi^k_x(x) \, dx \int \phi^m_y(y)\phi^m_y(y)\phi^l_y(y) \, dy. \quad (5.1.10)
\]

The second second integral on the right hand side of this equation equals

\[
\int \phi^m_y(y)\phi^m_y(y)\phi^l_y(y) \, dy
\]

\[= \phi^m(y)\phi^m(y)\phi^l(y) - \int (\phi^m_y(y)\phi^m_y(y)\phi^l(y) + \phi^m(y)\phi^m(y)\phi^l_y(y)) \, dy
\]

\[= -\frac{1}{2} \phi^m(y)\phi^m(y)\phi^l_y(y) \, dy. \quad (5.1.11)
\]

This shows that the integrals in equation (5.1.7) are the same. This leads to the remarkable observations that there is no contribution of the convection term in these domains.

In the following subsection the vorticity equation is solved by using the expanding in eigenfunctions method for the rectangle. One will see that the convection does indeed not have a contribution and that the solution equals the solution found in the previous chapter. The equilateral triangle domain can be done in a similar way by using the orthogonal coordinate system of Lamé, see (4.2.30).
5.1.1 Rectangular domain

For the rectangular domain, one has the following eigenfunctions

\[ \phi_n(x) = \sin\left(\frac{n\pi}{a} x\right), \quad \phi_m(y) = \sin\left(\frac{m\pi}{b} y\right), \]  

(5.1.12)

where \( a \) and \( b \) are the length of the sides of the rectangle. The first integral of equation (5.1.7) now reads as

\[
\int_0^b \int_0^a \phi^n(x)\phi^n(x)\phi^k(x)(\phi^m(y))^2\phi^l(y)\,dx\,dy
\]

\[
= \int_0^a \phi^n(x)\phi^n(x)\phi^k(x)\,dx \times \int_0^b (\phi^m(y))^2\phi^l(y)\,dy
\]

\[
= \frac{n\pi}{a} \int_0^a \cos\left(\frac{n\pi}{a} x\right) \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{k\pi}{a} x\right)\,dx \times \frac{l\pi}{b} \int_0^b (\sin\left(\frac{m\pi}{b} y\right))^2 \cos\left(\frac{l\pi}{b} y\right)\,dy
\]

\[
= \frac{n\pi}{2a} \int_0^a \sin\left(\frac{2n\pi}{a} x\right) \sin\left(\frac{k\pi}{a} x\right)\,dx \times \frac{l\pi}{2b} \int_0^b (1 - \cos\left(\frac{2m\pi}{b} y\right)) \cos\left(\frac{l\pi}{b} y\right)\,dy,
\]

(5.1.13)

where the first integral is equal to zero except for \( k = 2n \neq 0 \). For \( k = 2n \neq 0 \) one has

\[
\frac{n\pi}{2a} \int_0^a \left(\sin\left(\frac{2n\pi}{a} x\right)\right)^2\,dx = \frac{n\pi}{4}.
\]

(5.1.14)

The second part of equation (5.1.13) equals

\[
\frac{l\pi}{2b} \int_0^b (1 - \cos\left(\frac{2m\pi}{b} y\right)) \cos\left(\frac{l\pi}{b} y\right)\,dy =
\]

\[
\frac{l\pi}{2b} \int_0^b \cos\left(\frac{l\pi}{b} y\right)\,dy - \frac{l\pi}{2b} \int_0^b \cos\left(\frac{2m\pi}{b} y\right) \cos\left(\frac{l\pi}{b} y\right)\,dy,
\]

(5.1.15)

where the first integral on the righthand side of equation (5.1.15) is zero for \( l \neq 0 \) and for \( l = 0 \) the constant factor \( \frac{l\pi}{2b} \) is 0. So one only needs to consider the second integral on the righthand side, which only contributes when \( l = 2m \neq 0 \)

\[
\frac{2m\pi}{2b} \int_0^b (\cos\left(\frac{2m\pi}{b} y\right))^2\,dy = \frac{m\pi}{2}.
\]

(5.1.16)
The second integral in equation (5.1.7) can be calculated analogously:

\[
\int \int (\phi^n(x))^2 \phi^k_x(x) \phi^m_y(y) \phi^l(y) \, dx \, dy \\
= \int_0^a (\phi^n(x))^2 \phi^k_x(x) \, dx \int_0^b \phi^m_y(y) \phi^l(y) \, dy \\
= \frac{k\pi}{a} \int_0^a (\sin(n\pi x/a))^2 \cos(k\pi x/a) \, dx \times \frac{m\pi}{b} \int_0^b \cos(m\pi y/b) \sin(m\pi y/b) \sin(l\pi y/b) \, dy \\
= \frac{\pi}{2} \times \frac{m\pi}{4},
\]

(5.1.17)

for \( k = 2n \neq 0 \) and \( l = 2m \neq 0 \).

This yields for \( c^{mn}(t) \),

\[
c^{mn}_t(t) \\
= -\nu(\lambda^2_n + \lambda^2_m) c^{mn}(t) - \sum_m \sum_n c^{mn}(t) c^{2n,2m}(t) \left( \frac{n\pi}{4} \frac{m\pi}{4} - \frac{m\pi}{4} \frac{n\pi}{4} \right) \\
= -\nu(\lambda^2_n + \lambda^2_m) c^{mn}(t),
\]

(5.1.18)

which is a simple ODE. It is not hard to see that the expression for the vorticity will become the same as in the previous chapter, where the convection term was neglected.
Chapter 6

Conclusions

The investigations in this research have been inspired by the experimental observation that different vorticity patterns may have quite different life times. In this report we studied the decay of two types of vorticity patterns. The origin of the decay is viscosity, i.e. internal friction. One pattern consists of a plan filling array of identical squares, and the second consists of identical equilateral triangles. Since the patterns extend to infinity, we may apply periodic boundary conditions. This implies that we can concentrate on the dynamics of one unit cell, since the influence of the environment is automatically included via the boundary conditions.

To calculate the velocity profiles of potential flows explicitly, we applied the technique of conformal mapping. The dynamics of the patterns is described by an equation in which the Laplace operator plays a central role. The functional form of this operator is preserved under conformal mapping. That is why this technique provides a powerful tool to obtain analytical solutions. An important conformal mapping is the Schwarz-Christoffel mapping, which is a mapping from a half plane to an n-sided polygon. Using this mapping the flow fields of the potential flow in a square and an equilateral triangle have been obtained.

This study focuses on the comparison of square and triangular patterns. In principle, both patterns have unbounded kinetic energies if one introduces potential vortices. However, we have shown that a comparison of the kinetic energies of both patterns still can be done and yields interesting information. The difference of the kinetic energies between the same vortex in a square and a triangle tends to a constant value.

To analyze the decay of the vortex, we solved the reduced vorticity equation, the vorticity equation without the convection term. Due to the separated boundary conditions it was possible to use the method of separation of variables. This method gives simple Sturm-Liouville problems. The products of the eigenfunctions with its corresponding eigenvalues
obtained from the Sturm-Liouville problems are all solutions of the reduced vorticity equation (diffusion equation).

To calculate the stream function, we solved the Poisson equation with the solution of the diffusion equation as input. For solving the Poisson equation the Green’s function is needed. Due to the same geometry and boundary conditions the eigenfunctions and corresponding eigenvalues of the Greens function are the same as the eigenfunctions and eigenvalues of the diffusion equation. This resulted in lots of cancelation in the integral formulation of the stream function.

The eigenstructure of the Laplacian for the square is obtained by solving the Sturm-Liouville problems. The eigenstructure of the Laplacian for the equilateral triangle was found by Lamé. He stated the eigenfunctions and showed that they satisfied the boundary conditions. Using his stated eigenfunctions, the diffusion equation for the triangle has been solved and the Green’s function has been found.

The viscous decay of the flow field is given by the stream function. The only time dependent term in the stream function is $e^{-t\nu\lambda_{mn}}$. For $t=0$ the stream function indeed gives the flow field of a potential flow. The first modes in the stream function are the most important. For the equilateral triangle with the same area as the square one found that the flow in the triangle decays faster than in the square.

In the last chapter it has been shown that if the domain has separated homogeneous boundary conditions and orthogonal eigenfunctions which are formed as a product from the one dimensional counterparts, then the convection term does not have any contribution in the vorticity equation. This is indeed the case for the square and the equilateral triangle. The explanation lies in the highly symmetrical structure of these domains.
Appendix A

Integral Theorems

**Theorem A.0.1. Gauss’s Theorem**

Let $V \subset \mathbb{R}^3$ be a compact region with a smooth boundary $A$, then

$$\iiint_V \nabla \cdot H \, dV = \iint_A (H \cdot n) \, dA,$$

(A-1)

where $H$ is a vector field whose component functions have continuous partial derivatives.

**Theorem A.0.2. Green’s Theorem**

Let $C$ be a positively oriented, simply closed and piecewise-smooth curve in $\mathbb{R}^2$ and let $A$ be the region bounded by $C$. Let $P$ and $Q$ be functions having continuous partial derivatives on an open region containing $A$ then

$$\int_C P \, dx + Q \, dy = \iint_A \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy.$$

(A-2)

The following result follows directly from the Green’s Theorem

$$\iint_A \left( h \nabla^2 g + \nabla h \cdot \nabla g \right) \, dA = \oint_C h \nabla g \cdot n \, ds,$$

(A-3)

where $A$ is the region enclosed by the simply closed curve $C$, $ds$ is the arc length line element of $C$, and $n$ is the outward normal to the curve at $(x(s), y(s))$. 
Theorem A.0.3. *Stokes’ Theorem*
Let $C$ be a positively oriented, simply closed and piecewise-smooth curve in $\mathbb{R}^2$ and let $A$ be the region bounded by $C$, then

$$\int\int_{A} (\nabla \times \mathbf{H}) \cdot \mathbf{n} \, dA = \oint_{C} \mathbf{H} \cdot \mathbf{ds}, \quad (A-4)$$

where $\mathbf{H}$ is a vector field whose component functions have continuous partial derivatives.
Bibliography


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