MASTER

Controllability analysis of industrial processes

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CONTROLLABILITY ANALYSIS OF INDUSTRIAL PROCESSES

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Report of the M.Sc. project
Performed for Setpoint IPCOS (Best, Holland),
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Setpoint IPCOS coordinator: ir. J. Ludlage
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Summary

The physical and chemical process industry want increased performance for their plants. A way to improve performance on existing plants is to apply multi input multi output model based control on unit processes that appear to be bottle necks.

When a model has been derived, the controller design is not a trivial task. To perform it as efficiently as possible, realistic specifications have to be derived in advance. By studying the model, information can be obtained about the extent to which process outputs can be manipulated and what performance is the maximum achievable. The goal of this M.Sc. project was to derive a method that arrives at the wanted information: the controllability of the process.

To simplify the problem the process outputs are assumed to be ordered according to their priority. Output 1 is most important. The controllability of the process is studied by the controllability of its outputs because the performance is expressed in terms of the process outputs. The controllability of an output, given its position in the ranking, is the freedom that is left to manipulate the output without diminishing the control of more important outputs. The freedom is determined by restrictions; the amplitude of the control signals is limited and stability must be guaranteed for a specific upper bound on the model uncertainty. Another restriction is introduced by demanding a minimum nominal and robust performance which is determined by the deviation of the setpoint input vector from the process output vector. Expressions are derived for the restrictions and for the ideal controller that relate the restrictions and the controller to the process as directly as possible. Because of physical restrictions, the ideal controller exists as a concept only. By a decomposition of the ideal controller the effort to decouple all outputs is uncovered and by applying a special filter matrix the ideal controller is adapted to a realizable one. For each output the filter restricts the related entries of the ideal controller matrix as much as is necessary. The performance of the controlled system with and without model uncertainty, that is given by the deviation of setpoint input and process output, can be related relatively easily to the filter. From this the wanted information can be derived. The method is applied to a distillation column with three inputs and two outputs.

The method that is derived can be applied to stable processes without zeros in the complex right half frequency plane. Robustness of stability and performance are studied for diagonal uncertainty at the outputs only. Further research has to be performed on how to extend the applicability of the method to processes with diagonal uncertainty at the process inputs and on instable processes and processes with RHP zeros.

Contents

1 Introduction

In the industrial process control world, at least two parties are involved; the site that exploits the process and the control engineers. The site is confronted with a very dynamic and hardly predictable marketplace. The quality of the products has to meet tight specifications, the product quantities become smaller and the product deliveries have to take place 'just in time'. The control engineers are asked to design a control system that achieves the wanted flexibility for the production process.

Most industrial processes generate more than one product at the same time. All of them have to meet tight specifications. The wishes of the site are seen as outputs of the process by the control engineers. To influence the outputs the engineers use process inputs which are carefully chosen. They build a controller that generates control signals that are fed to the process inputs (the manipulated variables). Setpoint values that represent the specifications for the outputs, are the inputs for the controller. The function of the control system is to realize the setpoint values at the process outputs and to suppress disturbances that affect the outputs.

The possibility to manipulate all process outputs independently results in large flexibility. For a process with more than one input and output there will always be interaction as a function of frequency. Decoupling over the whole frequency band is impossible in general but, since in the setpoint values some outputs will always depend on others this need not be necessary. It can be more important for the site to be sure of a specific response characteristic with for example minimum settling time. However, the more decoupling is possible, the more flexibility. That is why it should be built into the controller. Other relevant properties of the controlled system are: disturbance suppression and control accuracy of specific outputs.

The reason why decoupling is impossible for some frequencies is that the process has characteristics that cause full dependency of the outputs. No controller will exist that realizes different values at the outputs at those frequencies. In order to perform the controller design phase as efficient as possible, the control engineer should be able to tell to what extent the process can be decoupled and what process characteristics limit. A feasibility report could contain the following information (suppose the process has three outputs, the first output representing the client's most important wish): changes in the setpoint inputs with frequencies up to 0.1 Hz. are followed perfectly by all outputs of the process. For frequencies between 0.1 and 1 Hz. only outputs 1 and 2 follow, output 1 is controllable up to frequencies of 3 Hz. If process input 3 cannot be used any more (actuator failure) only output 1 and 2 can be controlled up to 0.05 Hz, output 1 can be controlled up to 0.5 Hz. A controllability analysis can provide this useful information. For SISO processes (single input, single output) the frequency plots of the amplitude and phase behaviour provide enough information. At this moment it is not known how the controllability of MIMO processes (multiple inputs, multiple outputs) is analyzed. It is the goal of the author's master's degree project of which the report lays in front of you. The project was performed at Setpoint IPCOS (Best), a company that is aimed at the application of advanced process control. The Setpoint coordinator was ir. J. Ludlage and professor dr.ir. A.C.P.M. Backx was the university supervisor.

The following will be the tactic to arrive at a MIMO controllability analysis method. Assume that it is the client's wish to control all process outputs independently. This means that an ideal controller is wanted. It will become clear that it is impossible to realize it but that it is still useful to know what it looks like to find out what is maximally feasible. The words 'ideal' or 'perfect controller' will be used more than once here. One should realize that it exists as a concept and that it is only a means to achieve our goal.

The controller is bound to restrictions. One restriction is that the amplitude of the control signals is limited (a valve can't be more open than open). The knowledge about the process, including its inputs and outputs that is available in a model within the controller, contains errors. The stability of the controlled system and its performance have to be guaranteed for a given model uncertainty bound. These are two other restrictions. Where the perfect controller is restricted, concessions have to be made. Violations of the restrictions are caused by the wish to control a specific output or a group of outputs perfectly. From the frequency on that the restriction is passed, it will not be possible to control the specific output(s) independently any more. By doing so, the reason of the collision is clear and at the same time it is uncovered what is best achievable for the specific output.

To make the procedure that was sketched in rough outlines in the previous paragraph applicable, some things have to be investigated;

-what does the 'ideal' controller look like?

As restrictions were mentioned the stability and performance guarantees and the maximum amplitude of the control signals.

-how can the restrictions be related to the controller, the process model and the upper bound of the model error?

-how can the wish to control specific process outputs be related to the restrictions such that in case of violations, it can be derived what wish to control a specific process output is responsible?

The analysis method that is derived here can be applied for minimum phase processes only. Processes with zeros in the complex right half of the frequency plane cannot be analyzed. In the following chapters the tools that are needed are derived. For notational convenience most of the expressions will be for the static case (frequency=0). The indicators from which controllability can be concluded are calculated for each frequency point again. By doing so conclusions can be drawn for the behaviour over the whole discrete frequency spectrum from $0..\pi/T$ [rad/sec]. The price that is paid is that the method is applicable on minimum phase processes only.

Another important limitation of the method that is presented here is that it is studied only how uncertainty at the outputs restricts the controller. It must still be found out how the robustness demands for uncertainty at the inputs are translated to restrictions for the control of specific outputs.

In chapter 2 the reader is given an idea of what the typical MIMO control problem is and some relevant aspects of MIMO behaviour will be treated. In chapter 3 expressions are derived for the restrictions and the ideal controller that link them as directly as possible to the process. In chapter 4 it is derived how the wish to control specific outputs is related to the ideal controller which exists as a concept only. In chapter 5 it is wondered how the restrictions can be related via the controller to the wish to control specific outputs. When the restrictions are translated to information that can be used at the level of the outputs, a realizable controller can be derived from the ideal one by simply filtering. This will be done for the situation that the whole additive model error is reduced to uncertainty at the outputs (sensor uncertainty). The necessary filtering actions to adapt the ideal controller into a realizable one tell to what extent perfect control of each output is feasible. In chapter 6 the controllability analysis is performed for a 2x2 subsystem of an existing distillation column. In that case all behaviour can be calculated. Subsequently the behaviour of the whole 2x3 process will be analyzed.

2 Direction Behaviour of MIMO Processes and Some Relevant Aspects

In this chapter the reader is given an idea of what the big issue is in MIMO controllability that makes the transition from SISO controllability of the subloops to MIMO controllability of the whole system not a trivial one. It will become clear that, as for the SISO case, we can think of the MIMO transfer matrix as having a gain, but that its amplitude depends on the 'input direction' of the vector with which it is multiplied. The vector that results from the multiplication gives the 'output direction' that senses the specific gain. We will see that a small process gain (close to zero) will lead to problems because it demands large control signals. Not all outputs are equally important in practice. If the weight vector, in which the importance of all outputs is reflected, coincides with the output direction belonging to the largest process gain, then the wishes coincide with the process behaviour. If it coincides with the output direction with the very small gain then the wishes can come out not to be real. It is shown how model uncertainty can be modelled as uncertainty at the inputs and outputs respectively of which causes are indicated as well. In the last section it is explained why scaling of the transfer matrix should be performed and an example will be elaborated for different scaling criteria.

2.1 Direction Behaviour of MIMO Processes

At this moment it is known how a SISO process is analyzed. Plots of the amplitude ratio of the input and output signal of the process as a function of frequency and of the phase shift contain all the relevant information. Our goal in this report is to develop a method to analyze MIMO industrial processes. The following examples show the reader that MIMO analysis does not follow easily from SISO analysis. It may very well be possible to control each SISO subloop separately of a MIMO process perfectly but this does not imply that perfect control is possible for all outputs. To keep things simple, the examples will be static.

In figure 2.1 the structure of a 2x2 process is elaborated.

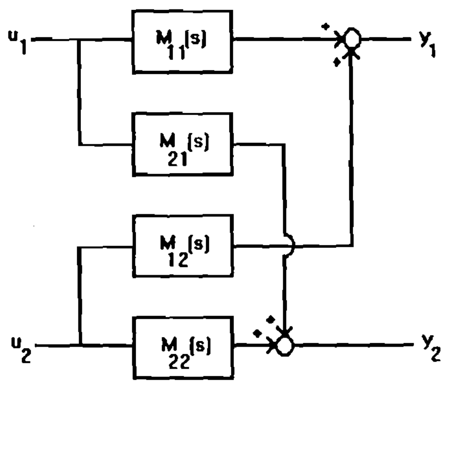


Figure 2.1 *SISO loops within a 2x2 system*

Equation (2-1) gives the transfer matrix and the mathematical relations between in- and outputs for the system of figure 2.1.

$$y = Mu$$

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix} \tag{2-1}$$

Example 1. *Suppose M is as follows:*

$$M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

All single loops given by a transfer of "1" or "-1" can be controlled perfectly per transfer (no phase shift and constant gain). Suppose we want to realize:

$$y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

then the vector of control signals (Manipulated Variables) is:

$$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Suppose we want:

$$y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

then the vector of control signals is:

$$u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example 2. *Suppose M is as follows:*

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

All single loops given by a transfer of "1" can be controlled perfectly. Suppose we want to realize:

$$y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The following control signals make $y_1=1$:

$$\forall_{\alpha \in \mathbb{R}} \quad u=\begin{bmatrix} \alpha \\ 1-\alpha \end{bmatrix}$$

But these control signals fix y_2 on 1 whilst -1 was wanted. In fact whatever the control signals are, $y_1=y_2$. A vector of control signals that realizes distinct values at the outputs does not exist. y_2 fully depends on y_1 and vice versa.

The process given in example 2 shows full row dependence. The angle between the rows, seen as vectors, is 0°. The rows of the process of example 1 are perpendicular. Their angle is 90° and their inner product is 0.

Now let us study a process which shows large row dependence:

Example 3. *Suppose M is as follows:*

$$M=\begin{bmatrix} 1 & 1 \\ 1 & 1+\varepsilon \end{bmatrix} \text{ with } \varepsilon=0.01$$

Suppose we want to realize:

$$y=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

then the vector of control signals is:

$$u=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Suppose we want:

$$y=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

then the vector of control signals is:

$$u=\begin{bmatrix} 201 \\ -200 \end{bmatrix}$$

We see that it is possible to control different outputs in different directions but that this can lead to large control signals. We will see in the subsequent chapters that the more dependent the vectors spanned by the rows of the process are, the larger the control signals have to be to realize distinct values at the outputs.

Let us now express the characteristics of the processes from examples 1 to 3 in terms of directions and gains. All directions will be denoted with normed vectors with length 1.

The process of example 1 transforms the (input) vector u to the direction $[1\ 0]^T$ in the (output) direction $[1/\sqrt{2}\ 1/\sqrt{2}]^T$ with a process gain $\sqrt{2}$. Input vector $[0\ 1]^T$ is transformed to the (output) direction $[1/\sqrt{2}\ -1/\sqrt{2}]^T$ with a process gain $\sqrt{2}$ as well.

The process of example 2 transforms the input vector $[1/\sqrt{2}\ 1/\sqrt{2}]^T$ to the output direction $[1/\sqrt{2}\ 1/\sqrt{2}]^T$ with a gain 2. The input vector that is perpendicular: $[1/\sqrt{2}\ -1/\sqrt{2}]^T$ is transformed to the output direction $[0\ 0]^T$. We could say that it is transformed to the (output) direction $[1/\sqrt{2}\ -1/\sqrt{2}]^T$ with a gain 0.

The process of example 3 transforms the input vector $\begin{bmatrix}\frac{200}{\sqrt{200^2+201^2}} & \frac{201}{\sqrt{200^2+201^2}}\end{bmatrix}$ to the (output) direction $[1/\sqrt{2}\ 1/\sqrt{2}]^T$ with a process gain 2.005. Input vector $\begin{bmatrix}\frac{-201}{\sqrt{200^2+201^2}} & \frac{200}{\sqrt{200^2+201^2}}\end{bmatrix}$ is transformed to the (output) direction $[-1/\sqrt{2}\ 1/\sqrt{2}]^T$ with a gain 0.005

We see that, as for the SISO case, we can think of the MIMO transfer matrixes as having gains, but that their magnitude depends on the input direction of the vector with which it is multiplied. The vector that results from the multiplication gives the output direction that senses the specific gain. To realize $[1\ -1]^T$ at the outputs in the process of example 3, the control signals in u need large amplitudes because the process itself has a small gain in that direction. For the process of example 2 it is impossible to control the output vector in the direction $\begin{bmatrix}1\\-1\end{bmatrix}$ because the process gain is 0.

The gains and directions as given before per example can be derived easily by performing a Singular Value Decomposition (SVD). It will be treated in chapter 4 extensively. An SVD is a means to characterize the gain behaviour of a MIMO transfer matrix. When it is performed for subsequent frequencies it can be seen as a generalization of the SISO Bode magnitude characteristics. The largest and smallest gains (singular values) of a matrix are determined together with their input and output directions which will be perpendicular and all the other mutual perpendicular in- and output directions with their gains that span the in- and output space completely.

Not all outputs are equally important in practice. As was said in the introduction, complete decoupling over the whole frequency band is impossible. It would be most convenient when the control of the more important outputs can be performed over a wider frequency range than the control of less important outputs. We have seen in example three that the control in the output direction $[1\ 1]^T$ costs far less control energy than in the direction $[1\ -1]^T$. When all disturbances affect the outputs in the direction $[1\ 1]^T$ than they can easily be suppressed but when they are in the direction $[1\ -1]^T$ then the disturbance reduction in one output must be sacrificed. If output 1 is more important than output 2 then output 2 will be sacrificed, even if all the setpoint inputs are in the direction $[1\ 1]^T$.

Specifications on the importance of the outputs of MIMO systems can be given in a weight vector. If for instance output 1 is four times more important than output 2 and it is understood that complete decoupling is impossible then it would still be interesting to know whether the process supports this ranking. This can be determined quickly by checking whether the weight vector $1/\sqrt{17}*[4\ 1]^T$ approximates the output direction that senses the largest process gain. If it approximates the output direction with the very small gain then the wishes can come out not to be real and would a more appropriate weighting vector have been: $1/\sqrt{17}*[4\ 1]^T$.

We have seen that direction behaviour is typical for MIMO processes. For output directions in which the process has a small gain, the control signals need a large amplitude to realize the wanted setpoint values. The larger the dependency between rows of the process, the closer the smallest singular value will be to zero. The weighting of outputs can also be translated in terms of output directions and gains.

2.2 Input and Output Uncertainty

Perfect models do not exist. A way to model uncertainty where physical causes exist for it as well is at the inputs and the outputs of the process. Uncertainty at the inputs exists in the actuators. One of its causes is hysteresis; the output signal ($\triangleq$ status valve: closed, open, 1/2 open) depends nonlinearly on the input signal. Another cause: the signals that are fed to the actuators stem from the digital controller. Since the actuators must be fed with analog signals, a digital to analog conversion has to take place. This causes an error of which the magnitude depends on the magnitude of the signal. The total uncertainty per input is 1% typically. The uncertainty at the outputs exists in the sensors. Measurement noise and an error in the required analog to digital conversion are causes. The output uncertainty is typically 5% per output.

Input and output uncertainty can be modelled in a mathematical convenient way. Figure 2.2 shows how the input and output uncertainty affect the outputs for the 2x2 case.

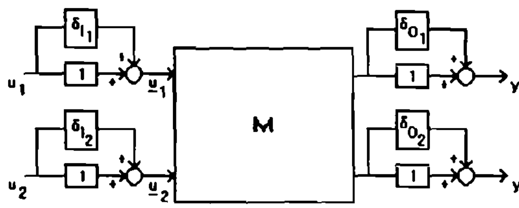


Figure 2.2 Input and Output Uncertainty for the 2x2 case

In (2.2-1) the input uncertainty is related to the total additive model error for the 2x2 case such that P=M+Δ.

$$M\underline{u} = M\begin{bmatrix} u_1+\delta_{I_1}u_1 \\ u_2+\delta_{I_2}u_2 \end{bmatrix} = Mu + M\Delta_I u = (M+\Delta)u = Pu \tag{2.2-1}$$

In (2.2-2) the output uncertainty is related to the total additive model error.

$$\begin{bmatrix} y_{p_1} + \delta_{O_1} y_{p_1} \\ y_{p_2} + \delta_{O_2} y_{p_2} \end{bmatrix} = Mu + \Delta_O Mu = (M + \Delta)u = Pu \tag{2.2-2}$$

In this section mathematically convenient ways for modelling uncertainty were presented that have a basis in reality. They will be used in our analysis of controllability.

2.3 Scaling

Before the controllability of a process is analyzed, a pseudo-freedom can be used to multiply the rows and columns with scaling factors. $M_{scaled}=D_2MD_1$ with D_1 and D_2 diagonal matrixes. If M is a mxn matrix then D_2 is a mxm matrix with which the rows of M are multiplied, D_1 is a nxn matrix which multiplies the columns. How should this freedom be used?

One could apply the scaling freedom for example to minimize the difference between the smallest and largest process gain. This is measured by the condition number that is the ratio of the largest and smallest gains or singular values.

As was treated in section 2.2, the input and output uncertainties affect the values in a relative way. For larger signals the absolute uncertainty is larger. Because of interaction, all outputs depend on more than one input. An input signal with a large value introduces a large error into the process. When this error affects an output with a small value then the 'signal to noise ratio' can be dramatically small. To overcome this problem a scaling can be applied that ensures that the amplitudes of all input and output signals are equal in order to make the absolute errors compatible. To calculate this scaling the domains of the values that the in- and output signals take on are needed.

Two scalings based on the previously described criteria are calculated here for a given process:

$$M = \begin{bmatrix} 2 & 0.5 \\ 0.2 & -0.05 \end{bmatrix} \tag{2.3-1}$$

For both cases suppose that the scalings are derived to perform the steady state disturbance reduction as good as possible.

(i) The condition number of the unscaled process: largest gain/smallest gain = 2.07/0.097 = 21.4

Choose $D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}$ and $D_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix}$ then $D_2MD_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and condition number(D_2MD_1)=$\sqrt{2}/\sqrt{2}$=1 Minimal! Interpretation: the scaling is such that there exists no difference in the effort to suppress disturbances affecting the output in whatever direction.

(ii) First determine what the maximum signals will be in steady state; the maximum output signals needed are given by the disturbance vector with maximum values that affects the output. Suppose all worst case disturbances at the outputs are given with: $d_y = \begin{bmatrix} \pm 1 \\ \pm 0.2 \end{bmatrix}$ then $D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ to make sure that all output signals have amplitudes of 1. D_1 is determined by the largest control signals that are needed to suppress d_y. To determine the signs of the elements of d_y a singular value decomposition is performed on the inverse of M:

$$M^{-1} = \begin{bmatrix} -0.24 & -0.97 \\ 0.97 & -0.24 \end{bmatrix} \begin{bmatrix} 10.35 & 0 \\ 0 & 0.48 \end{bmatrix} \begin{bmatrix} 0.088 & -0.996 \\ -0.996 & -0.088 \end{bmatrix} \qquad (2.3\text{-}2)$$

From this we derive that the disturbance that needs the largest control signals to suppress it is: $d_{y,worst} = \begin{bmatrix} 1 \\ -0.2 \end{bmatrix}$ Then $M^{-1} \begin{bmatrix} 1 \\ -0.2 \end{bmatrix} = \begin{bmatrix} -0.25 \\ 3 \end{bmatrix}$ So u(1) has an amplitude of 0.25 and u(2) of 3. The control signals coming from the controller can have amplitudes of 1 if they are multiplied with: $D_1 = \begin{bmatrix} 0.25 & 0 \\ 0 & 3 \end{bmatrix}$ .For this

scaling: $D_2 M D_1 = \begin{bmatrix} 0.5 & 1.5 \\ 0.25 & -0.75 \end{bmatrix}$
condition number(M_{scaled})=1.71/0.44=3.91

We see that the two criteria arrive at different scalings and that a minimal condition number need not be optimal under all circumstances.

2.4 Conclusion

We have seen that the MIMO process gain depends on the direction of the inputs. The relevant process gains including the directions to which they are related can be calculated by performing a singular value decomposition that will be treated in chapter 4. The larger the dependency between the rows of the process, the closer the smallest process gain or singular value will be to zero and the larger the control signals have to be to realize the wanted setpoint values at the process outputs. Input and output uncertainty are mathematically convenient ways for modelling uncertainty which have a basis in reality. They will be used in our analysis of controllability. The rows and columns of the process transfer matrix can be multiplied with scaling factors to make the process behaviour more favourable in terms of a specific criterion.

3 Ideal Controller and Restrictions

Our tactic in the derivation of the controllability analysis method is to derive the ideal controller, wonder how restrictions (limitations on the amplitude of the controller signals, stability and performance have to be guaranteed for a given model uncertainty bound) limit its realization and the wish to control what output(s) can be held responsible for this. By doing so it becomes clear what is maximally feasible for each output. The philosophy behind this is that the control of the whole system is given by the 'sum' of controllabilities of all outputs separately that are ordered in an absolute ranking.

It has to be investigated what the ideal controller that exists as a concept only (!), looks like and how it can be related to the process behaviour as directly as possible. For the restrictions, expressions need to be derived that link them as directly as possible to the (ideal) controller. This will be done in this chapter. In the following chapter it will be investigated how the effort to control specific outputs comes through in the ideal controller.

The procedure in this chapter will be to derive, for a control configuration that is as general as possible, expressions for the restrictions and to derive an expression from which the ideal controller can be derived. In section 3.1 this will be done for a common control configuration that includes a feedback loop. It will turn out that the expressions depend on the process and controller in a difficult, nonlinear way. In the subsequent section it will be shown that without loss of generality, the configuration can be changed into the Internal Model Control structure and that we arrive at expressions that are related to the process behaviour and controller more directly. To apply the IMC scheme the process has to be stable.

For notational convenience the transfers will be denoted with their static behaviour.

3.1 Expressions Based on the General Control Configuration

In figure 3.1 a common control structure together with the process is given. The feedback loop is necessary to reduce the deviation between setpoint input and process output. If control is perfect then the deviation e=0. The structure is based on the following assumptions which do not seriously affect the generality that was asked; the disturbances are modelled to affect the output only and all uncertainties are taken together in an additive model error: $P=M+\Delta$.

The controller signals (u) come from the controller C. From now on the behaviour of the controlled system will be referred to as "closed loop behaviour".

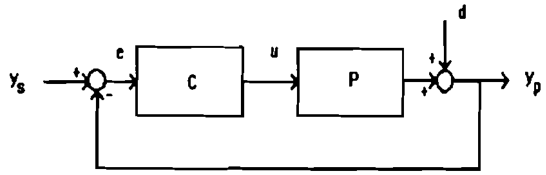


Figure 3.1	*The controlled process in closed loop*				
	P:	*Process*	*C:*	*Controller*	y_s: *setpoint input*
	y_p:	*output*	*e:*	*error*	*u: control signal*
	d:	*disturbance*			

The closed loop transfer function is built up from the transfers from all external inputs to the outputs. From this transfer the controller structure can be derived that realizes a specific transfer.

$$\begin{aligned} &y_s \rightarrow y_p\text{: } PC(I+PC)^{-1} = T_0 \\ &d \rightarrow y_p\text{: } (I+PC)^{-1} = S_0 = I - T_0 \\ &y_p = T_0 y_s + S_0 d \end{aligned} \tag{3.1-1}$$

For perfect control $y_p=y_s$. We see that if $T_0=I$ then $S_0=0$ (sensitivity for disturbances) such that: $y_p=y_s$. Apparently the controller C that realizes $PC(I+PC)^{-1}=I$ is the perfect or ideal controller that we are looking for. We observe already that it is related to the process in a nonlinear way and that it will be difficult to derive from this expression what the 'ideal' controller structure is.

Let us now derive expressions from which the violation (or not) of the restrictions by the controller can be concluded. The restrictions are:
1. limited amplitude of the control signals
2. robust stability
3. robust performance

The first two factors limit the space in which the controller can be found. The third expresses the consequences of the choice for a nonperfect controller for the deviation between input and output.

Now expressions will be derived that quantify the restrictions for the general closed loop configuration of figure 3.1.

To calculate the control signals we must know the transfer from all inputs to the control signal u:

$$(y_s - d) \rightarrow u\text{: } C(I+PC)^{-1} = P_c \tag{3.1-2}$$

We see that if we would know the ideal controller C, then still the controller signals depend on it in a difficult nonlinear way. It would be very difficult to translate the violation of the amplitude bound to the wish to control a specific output perfectly.

Let us now study how the robust stability demand is related to the process behaviour and the controller. When all setpoint inputs are zero we arrive at the situation of figure 3.2.

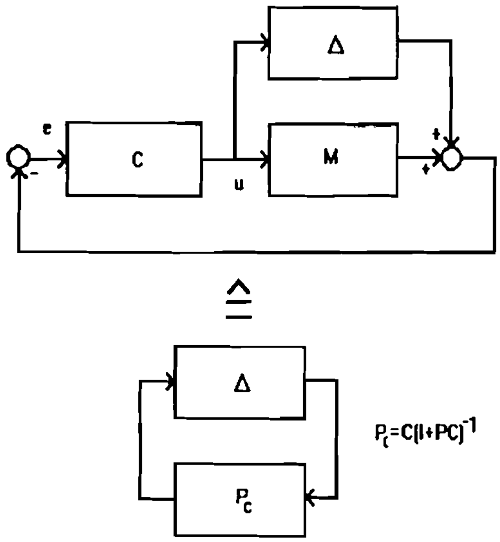


Figure 3.2   Model uncertainty influences stability

The open loop transfer is $P_c\Delta$ and is a square matrix. The closed loop transfer is given with $(I+P_c\Delta)^{-1}$. Assuming that $P_c\Delta$ is stable then the stability of the whole closed loop system of figure 3.1 is guaranteed when $(I+P_c\Delta)^{-1}$ is stable. This transfer is stable when its eigenvalues are positive. Since the eigenvalues of matrix X^{-1} are positive when the eigenvalues of matrix X are positive and vice versa, the demand that all eigenvalues of $I+P_c\Delta$ must be positive implies stability for the system of figure 3.1. Since:

$$I+P_c\Delta = Z(I+\Lambda)Z^{-1} \tag{3.1-3}$$

with Λ a diagonal matrix that contains all the eigenvalues of $P_c\Delta$ and with Z a unitary matrix, the stability demand is satisfied when all the entries of the diagonal matrix $I+\Lambda$ are positive which holds when:

$$\forall_i \; \lambda_i(P_c\Delta) > -1 \tag{3.1-4}$$

Singular values can also be used to guarantee stability. Since:

$$\sigma_{min}(P_c\Delta) \leq |\lambda_i(P_c\Delta)| \leq \sigma_{max}(P_c\Delta) \tag{3.1-5}$$

and from:

$$|\lambda_i| < 1 \rightarrow \lambda_i > -1 \tag{3.1-6}$$

it follows that as a sufficient condition for stability can be used:

$$\sigma_i(P_c\Delta) < 1 \tag{3.1-7}$$

But, because of the one way direction* in (3.1-6), conservatism is introduced.

To find out whether the stability demand is satisfied we have to calculate the eigenvalues or singular values of $C(I+PC)^{-1}\Delta$. Again we arrive at a nonlinear relation between a restriction and the process behaviour and controller.

The third -robust performance- demand expresses the consequences of the choice for a nonperfect controller for the deviation between input and output (nonperfect is stressed because if it would be perfect then there would be no deviation) in case the robust stability demand is satisfied:

$$(y_s-d)-(y_s-y_p):(I+(M+\Delta)C)^{-1}-(I+PC)^{-1} \tag{3.1-8}$$

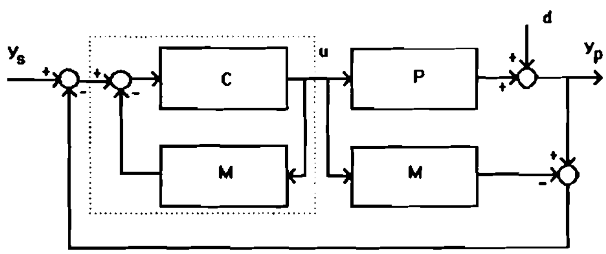
We have seen that the ideal controller which we need to know to derive our controllability analysis method, is not related easily to the process behaviour for the control structure of figure 3.1. Even if it would be known, the restrictions depend on it together with the process in a nonlinear way. Our conclusion here is that the expressions we derived thus far are not appropriate to base the controllability analysis on. In the next section it is tried to simplify the expressions.

3.2 Expressions based on the Internal Model Control Structure

Without loss of generality the standard configuration of figure 3.1 can be changed into the configuration of figure 3.3 by adding two blocks with the model M whose effects cancel each other [1]. The equivalence of the configurations is proven by:

$$\begin{aligned} Q &- C(I+MC)^{-1} \\ C &- Q(I-MQ)^{-1} \end{aligned} \tag{3.2-1}$$

A



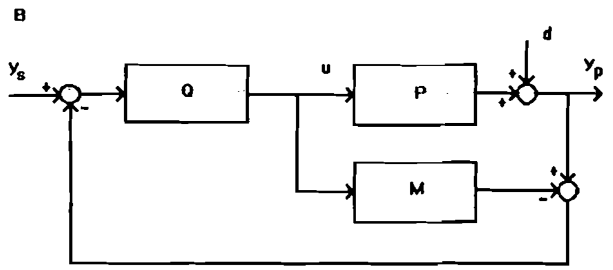


Figure 3.3 *Modification of the structure of figure 3.1 into an equivalent configuration*

The configuration of figure 3.3 is the Internal Model Control structure!

Now the relevant aspects of closed loop behaviour will be expressed in terms of the IMC scheme. It is assumed that M(s), P(s), Δ and Q(s) are stable and that the model is ideal: M=P. Under these conditions figure 3.3 can be simplified to figure 3.4.

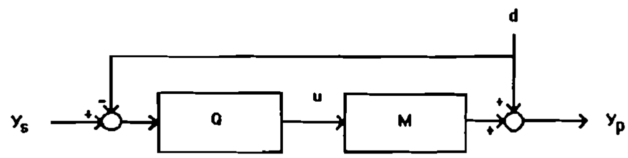


Figure 3.4 *Feedforward structure of the IMC scheme for M=P*

The feedback loop has disappeared under the assumption of an ideal model (P=M).

In equation (3.2-2) the closed loop transfer is given.

$$\begin{aligned} &y_s \rightarrow y_p\text{: } MQ = T \\ &d \rightarrow y_p\text{: } I - MQ = S \\ &y_p = Ty_s + Sd \end{aligned} \qquad (3.2\text{-}2)$$

We see that if T=I then S=0 which results in: $y_p=y_s$. But in that case the ideal controller is related to the process directly. It simply is its right inverse! Let us look how the restrictions are related to the controller and the process.

The control signals can be calculated from the transfer:

$$(y_s - d) \rightarrow u\text{: } Q \qquad (3.2\text{-}3)$$

This is a very direct relation between the control signals and the controller!

Robust stability is being guaranteed for the model uncertainty Δ as in figure 3.5 if the closed subloop of figure 3.5B that is the bottle neck for stability satisfies the following condition (ΔQ is always square):

$$\forall_i \ \lambda_i(\Delta Q(s)) > -1$$
$$\textit{or more conservative:} \qquad (3.2\text{-}4)$$
$$\sigma_{max}(\Delta Q(s)) \leq 1$$

One should not forget that it was assumed that Δ and Q on their own need also be stable.

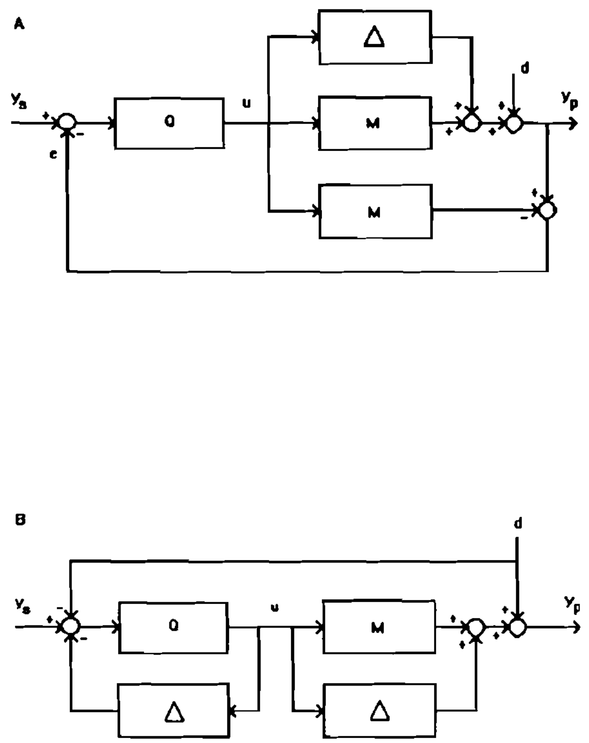


Figure 3.5 *Simplification of the IMC scheme including model uncertainty*

The third -robust performance- aspect which is not equal to 0 when the controller is not perfect (for derivation, see appendix 1) and in case the robust stability demand is satisfied:

$$(y_s-d)-(y_s-y_p):(I-MQ)(I+\Delta Q)^{-1}-S(I+\Delta Q)^{-1} \qquad (3.2\text{-}5)$$

The IMC control structure has resulted into favourable expressions. It was seen that the ideal controller that we need to know whether it is realizable or not, is related more directly to the process behaviour. It simply is the right inverse of the process. Another name is 'pseudo inverse'; $Q=M^+=M^T(MM^T)^{-1}$. When M is square we could state $Q=M^{-1}$ but because $Q=M^+$ is also valid for non square processes, we will use the last notation.

The control signals are linearly related to the controller. When the controller is stable then the stability of the whole controlled process depends on the eigenvalues of the product of additive model uncertainty and the controller. When robust stability is guaranteed the expression for robust performance is the least simple. As we will see in chapter 5, a specific approximation of the process inverse for the controller leads to some simplification.

3.3 Conclusion

The controllability analysis will be based on the IMC scheme because it learned us that the ideal controller which we need to know to be able to determine where the restrictions are violated, simply is the pseudo inverse of the process. When the inverse is stable then robust stability is guaranteed by a relatively easy expression. Whether or not robust performance is guaranteed can be determined with a less simple expression.

4 The Relation between the Ideal Controller and the Process Outputs

In this report we are deriving a controllability analysis method for MIMO processes. The method will make it possible for a given process to tell up to what frequency each output can be controlled using the given inputs. The tactic that is followed is to derive the controller that realizes perfect control for all outputs over the whole frequency band. We realize that its realization will not be possible in practice because physical restrictions limit the controller design space. Probably it will turn out that perfect control of the outputs is possible for a limited set of outputs up to a specific frequency bound only. This need not be a problem since the setpoint values for some outputs will depend on other outputs and disturbances will affect the output vector in fixed directions only. Some outputs are more important than others. In that case we are more interested in a specific response characteristic with for example minimum settling time.

In the previous chapter we found out what the controller looks like ideally, expressed in terms of the process behaviour as directly as possible. Also expressions were derived that relate the restrictions as directly as possible to the controller. Via the IMC scheme we found out that the controller that approximates the process inverse is the best controller. So the ideal controller would be the process inverse itself. We found out that when it can be guaranteed that the controller is stable, the restrictions on the amplitude of the control signals and the robust stability of the whole system can be satisfied when simple conditions are met. The condition for robust performance is less simple.

In this chapter we will focus our attention on the (ideal) controller that equals the process inverse and exists in theory only. In the first section we will wonder what process characteristics make the process inverse and thus the ideal controller unstable. Since the process had to be stable to apply the IMC scheme, we will for the second time limit the applicability of the analysis method that we are deriving by assuming the specific process characteristics not available in the processes that are studied. Then we are able to cover all restrictions with the relatively simple expressions that were derived in the previous chapter. In the following sections we will be looking for a decomposition of the ideal controller in which the effort to control specific outputs can be distinguished. We will try the singular value decomposition which was treated superficially in chapter 2, on its usefulness in this context. After the conclusion that we cannot discern the relation with specific outputs using the SVD but having a thorough idea of its merits after the treatment of simple examples, we will build up a decomposition that has the wanted characteristics. Based on a ranking of outputs which reflects their importance, we will arrive at an elegant decomposition that expresses the additional effort to control a specific output when all outputs that are higher in priority are controlled. This will be done by first studying the 2x2 case and next the general, nonsquare mxn case. It will turn out to be the QR decomposition. For notational convenience, all expressions will describe static behaviour only.

4.1 Process Characteristics that Limit Invertibility

In example 2 of chapter 3 we encountered a process that could not be controlled perfectly. Now we understand why: the full row dependence implies that the matrix is singular so that $Q=M^{+}$ does not exist.

The process of example 2 was static. Dynamic behaviour is studied here too. A zero in a MIMO transfer matrix M(s) causes full row or column dependence for a specific frequency. M drops below its normal rank [2]. If M(s) has a zero in the left half of the complex plain (LHP zero) then it will appear as a LHP pole in the inverse. Output signals from an inverse with LHP poles only are stable. If M(s) has a zero in the right half of the complex plain (RHP zero) then it will appear as a RHP pole in the inverse. Signals from an inverse with RHP poles are instable as a function of time. RHP zeros are denoted with 'nonminimum phase behaviour'.

In the subsequent sections we are interested in the relation between specific outputs and the process inverse (the ideal controller). Since nonminimum phase behaviour inhibits to invert the process leading directly to instable signals as a function of time, we assume that the processes for which the inverse is studied do not have nonminimum phase behaviour. In that case the restrictions treated in chapter 3 are the only criteria to see to what extent ideal behaviour (process inverse) can be realized for the controller which informs us about the controllability of the process as a whole.

4.2 Singular Value Decomposition

In section 2.1 we already encountered the singular value decomposition. It was explained that it is frequently used as the MIMO alternative for SISO Bode magnitude plots. In this section we will investigate, by applying it to the pseudo inverse process, whether we can develop indicators for the effort to control specific outputs perfectly. First it will be defined properly and for clarification applied to the example of chapter 2. Then it will be used to calculate the process inverse under the assumption of nonminimum phase behaviour and graphic representation forms will be treated to increase our insight in the relation between the process and the ideal controller. Finally, an example will be treated that proves that it is not the kind of decomposition that we are looking for.

4.2.1 Definition of Singular Value Decomposition

In this section it will be explained what a singular value decomposition is. In (4.2.1-1) the SVD of a nonsquare mxn matrix M is clarified [4].

$$
\begin{gathered}
M = U\Sigma V^T \text{ with} \\
U = [u_1 \; u_2 \; . \; . \; u_m] \wedge UU^T = U^T U = I \\
V = [v_1 \; v_2 \; . \; . \; v_n] \wedge VV^T = V^T V = I \\
u_i, v_i \text{ are column vectors with} \\
m, n \text{ elements respectively} \\
\Sigma = \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix}; \; m \geq n \\
\Sigma = [\Sigma_1 \; 0]; \; m \leq n \\
\Sigma_1 = diag(\sigma_1, \sigma_2, \ldots, \sigma_k); \; k = \min(m,n) \\
\overline{\sigma} = \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k = \underline{\sigma}
\end{gathered}
\tag{4.2.1-1}
$$

Both U and V are matrixes with full rank. After multiplication with an input vector (for M this is the vector containing control signals u) we arrive at a vector spanned by the columns of U. In other words: U spans the output space of M, V spans the input space. The singular values are the gains in the directions $u_1,..u_m$ for input vectors in the directions of $v_1,..v_n$ respectively. V divides the input vector over different gains (singular values) in different directions (columns of U).

To illustrate what was previously said, two extreme cases will be elaborated. For both cases it is assumed that m<n hence k=m: Case 1: the input vector coincides with the first vector of V. Case 2: the input vector coincides with the m-th vector of V.

$$Mv_1 = U\Sigma V^T v_1 = U\Sigma_1 \begin{bmatrix} 1 \\ 0 \\ . \\ 0 \end{bmatrix} = U \begin{bmatrix} \sigma_1 \\ 0 \\ . \\ 0 \end{bmatrix} = \sigma_1 u_1 = \bar{\sigma}\bar{u}$$

$$Mv_m = U\Sigma V^T v_m = U\Sigma_1 \begin{bmatrix} 0 \\ . \\ 0 \\ 1 \end{bmatrix} = U \begin{bmatrix} 0 \\ . \\ 0 \\ \sigma_m \end{bmatrix} = \sigma_m u_m = \underline{\sigma}\underline{u} \qquad (4.2.1\text{-}2)$$

The SVD of the process of example 2 in section 2.1:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad (4.2.1\text{-}3)$$

v_2^T gives the input direction that senses no gain. The interpretation is that a disturbance vector that affects the process output in the direction u_2 cannot be suppressed.

The condition number is related to the SVD. It passed by in section 2.3 as a scaling criterium:

$$cond(M) = \frac{\bar{\sigma}(M)}{\underline{\sigma}(M)} \qquad (4.2.1\text{-}4)$$

A large condition number indicates a large difference between the singular values and it means that depending on the output direction of the disturbance vectors there is a great difference in the ability to suppress them. This is not a pleasant characteristic if disturbances dó affect the output in all directions.

Now that we know what a SVD is, we will use it to calculate the pseudo inverse process.

4.2.2 Singular Value Decomposition and the Process Inverse

In this section we will express the process inverse in terms of the SVD under the assumption of nonminimum phase behaviour. Our insight in the relation between the process and the ideal controller will be increased by the graphic representation of the smallest and largest singular values. An example will prove that the SVD is, despite its useful sides, however not the kind of decomposition that we are looking for.

When M has minimum phase behaviour, M^+ can be expressed in terms of the SVD. In any case, U an V contain the relevant columns only; column 1 .. column m. Q expressed in terms of the SVD:

$$M = U\Sigma_1 V^T$$

when M nonminimum phase

$$Q = M^+ = M^T(MM^T)^{-1} = V\Sigma_1^{-1}U^T$$

$$\Sigma_1^{-1} = \begin{bmatrix} \sigma_1^{-1} & 0 & . & 0 \\ 0 & \sigma_2^{-1} & . & 0 \\ . & . & . & . \\ 0 & 0 & . & \sigma_m^{-1} \end{bmatrix} \qquad (4.2.2\text{-}1)$$

For nonsquare processes, null space can be added to this controller. We see that the input space of Q equals the output space of M and vice versa. In (4.2.2-2) the SVD of M is given for the 2x2 case. Q is expressed in terms of the SVD.

$$M = [u_1 \;\; u_2]\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} \;\; \rightarrow \;\; Q = [v_1 \;\; v_2]\begin{bmatrix} \sigma_1^{-1} & 0 \\ 0 & \sigma_2^{-1} \end{bmatrix}\begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix} \qquad (4.2.2\text{-}2)$$

For the 2x2 case we can represent these vectors and gains (singular values) graphically. Suppose that M is scaled such that the 2-norm of the vector of control signals ($=[\underline{u}_1 \; \underline{u}_2]^T$) is bounded to 1. Then all possible control vectors lay within the circle given in figure 4.1. If the control vector is in the direction of v_1 then it is amplified maximally (σ_1) in the output direction u_1.

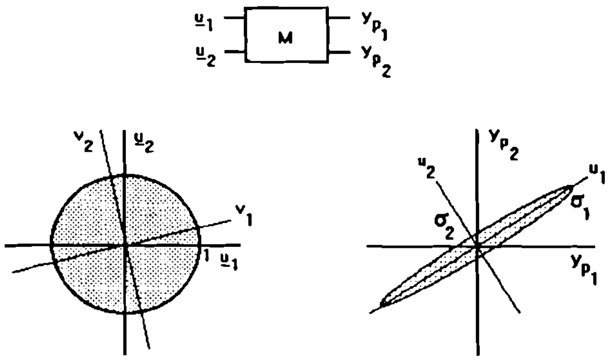


Figure 4.1 *Projection of the process input space onto the process output space*

Figure 4.2 shows the input and output space of the related process inverse and perfect controller. The input direction of M that has the smallest process gain (v_2) needs the largest amplification in the controller to realize the related setpoint input (y_s=u_2) at the output (y_p).

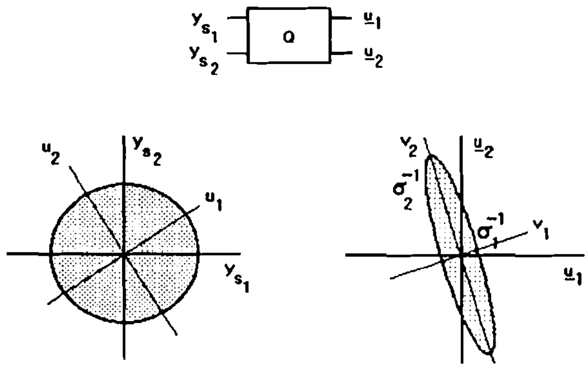


Figure 4.2 *Projection of the controller input space onto the controller output space*

The inversion of the smallest singular value of M gives us the largest singular value of the controller: $\sigma_{max_Q} = \frac{1}{\sigma_{min_M}}$ which implies that $cond(M) = \frac{\sigma_{max_M}}{\sigma_{min_M}} = \frac{\sigma_{min_M}^{-1}}{\sigma_{max_M}^{-1}} = \frac{\sigma_{max_Q}}{\sigma_{min_Q}} = cond(Q)$ This means that when the condition number of the ideal controller Q is known, it is known for the process as well. In section 4.3.1, the condition number of a general 2x2 controller will be calculated. It is not elaborated here because some theory has to be treated first.

From a plot of the largest singular value of Q as a function of frequency can be derived from what frequency on the restriction on the amplitude signals is violated. For a model uncertainty with a worst case structure it can also be used to judge where stability becomes a problem (see (3.2-4)), conservative condition). In figure 4.3A the two singular values of M are plotted as a function of frequency. In figure 4.3B the singular values of the related controller are plotted.

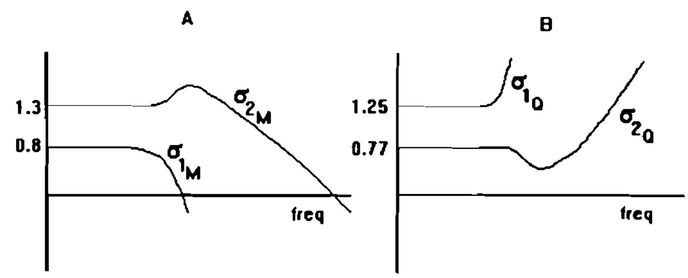


Figure 4.3 *Singular values of M and related Q as a function of frequency*

From figure 4.3 it becomes clear how a plot of the singular values can be useful to detect from what frequencies on the controller needs larger gains to control in a specific direction perfectly. For the controller of an mxn process a plot of its largest and smallest singular values gives already a lot of information. Elliptic surfaces as in figures 4.1 and 4.2 can be drawn as well.

From an SVD of the transfer matrix related to the robust performance restriction (see (3.2-5)), it can be derived for what input direction (y_s-d) the largest performance deterioration in what output direction (y_s-y_p) occurs when Q is nonperfect.

Despite all the useful applications for SVD we should not forget that we were actually looking for a decomposition of the process inverse or the ideal controller in which the effort to control each output perfectly comes out. (Remind: we were looking for a traceable relation between the restrictions and the wish to control each output perfectly. In this chapter we are trying to trace the relation between distinct outputs and the perfect controller (process inverse) and in the following chapter the relation between the restrictions and the outputs via the ideal controller.) Can a SVD help? The singular values of the controller express the effort to control in a specific *output direction* perfectly. Let us study the SVD of the following 3x3 process:

$$M = \begin{bmatrix} 0.58 & 0.58 & 0.58 \\ 0.71 & -0.71 & 0 \\ 0.91 & -0.25 & 0.35 \end{bmatrix} = U \Sigma V^T = \tag{4.2.2-3}$$

$$\begin{bmatrix} 0.41 & -0.81 & 0.41 \\ 0.57 & 0.58 & 0.58 \\ 0.71 & 0 & -0.71 \end{bmatrix} \begin{bmatrix} 1.42 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.0099 \end{bmatrix} \begin{bmatrix} 0.91 & -0.24 & 0.34 \\ -0.057 & -0.88 & -0.47 \\ 0.41 & 0.41 & -0.81 \end{bmatrix}$$

then using (4.2.2-1) it follows that:

$$Q = V\Sigma^{-1}U^T = V\begin{bmatrix} 0.71 & 0 & 0 \\ 0 & 1.00 & 0 \\ 0 & 0 & 100.78 \end{bmatrix} U^T \tag{4.2.2-4}$$

We see that the setpoint input direction for the controller and process output direction: $y_s^T=u_3=[0.41\ 0.58\ -0.71]^T$ demands tremendously large control signals. By which output is this caused?

We see that in the output directions of the process: $u_1=[0.41\ 0.57\ 0.71]^T$ and $u_2=[-0.81\ 0.58\ 0]^T$ can be controlled well for the related gains are limited. These directions are almost perpendicular ($\cos(\varphi)=0.0015$) and span the whole output space of y_{p1} and y_{p2}. When the control of y_{p1} and y_{p2} is more important than of y_{p3} then the additional wish to control y_{p3} independently results in a very badly conditioned controller. When we drop it the following 2x2 controller is left:

$$Q_{drop\,3} = \begin{bmatrix} 0.71 & 0.71 \\ 0.71 & -0.71 \end{bmatrix}\begin{bmatrix} 1.22 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.2.2-5}$$

This controller is realizable. The third row of the process of (4.2.2-3) is almost spanned by the first two.

The conclusion that y_{p3} causes the bad condition of the controller has cost some reasoning. This was only a 3x3 process. What if it would have been a badly conditioned 5x5 process? Our conclusion here is that an SVD is not an appropriate decomposition to uncover the effort to control specific outputs.

4.2.3 Conclusion

SVD is useful for the determination in what output direction control is most difficult. It cannot be seen directly how the wish to control one specific output of the process contributes to a small or large singular value of the process as a whole.

4.3 The Construction of the Ideal Controller

Being an essential part of our tactic we need to know the perfect controller. In our search for a traceable relation between the ideal controller (the pseudo-inverse process) and the wish to control specific outputs whilst the SVD not helping us any further, let us now build up the controller gradually; starting from a ranking which indicates the priority of the outputs, study the effort to control output 1 assuming that it is the only output of the process. Then add output 2 to the process and wonder: what is the effort to control output 2 perfectly provided that the perfect control of output 1 is not affected? Etcetera. First this scheme will be elaborated for a 2x2 process (2 inputs, 2 outputs). In the subsequent section the same will be done for the general, square or nonsquare, mxn case.

4.3.1 Construction of the 2x2 Ideal Controller

In chapter 3 it was derived that the closed loop behaviour of a controlled process can be represented as a feedforward structure for the nominal case that M=P. In figure 3.4 the structure was given. When control is perfect the disturbances are suppressed: S=0. Then $y_p=MQy_s$.

In (4.3.1-1) the 2x2 process is given. The row structure in the process model M is stressed because each row, multiplied with the vector of control signals results in the value of the output ($y_{pj}=M_ju$). Gain and direction behaviour are represented by M_g and E respectively.

$$M=\begin{bmatrix}M_1\\M_2\end{bmatrix}=\begin{bmatrix}m_1e_1\\m_2e_2\end{bmatrix}=\begin{bmatrix}m_1&0\\0&m_2\end{bmatrix}\begin{bmatrix}e_1\\e_2\end{bmatrix}=M_gE \tag{4.3.1-1}$$
$$\text{with } \|e_i\|=1 \text{ and } m_i=\|M_i\|_2$$

First control output 1 perfectly. For perfect control the controller ($=Q_1$) has to be the pseudo-inverse of M_1. So $Q_1=M_1^+$ or $M_1Q_1=1$:

$$M_1Q_1=m_1e_1Q_1=1 \rightarrow Q_1=m_1^{-1}e_1^T \tag{4.3.1-2}$$

m_1^{-1} gives the effort to realize y_{s_1} 1:1 in y_{p_1}. The expression derived in (4.3.1-2) for Q_1 is not in its most general form. Adding null space of M_1 does not change $M_1Q_1=1$.

$$Q_1=m_1^{-1}e_1^T+e_{1\perp}^T\alpha_1 \tag{4.3.1-3}$$

Now let us introduce output 2. For perfect control MQ=I. Or, into more detail:

$$MQ=\begin{bmatrix}M_1\\M_2\end{bmatrix}\begin{bmatrix}Q_1&Q_2\end{bmatrix}=\begin{bmatrix}M_1Q_1&M_1Q_2\\M_2Q_1&M_2Q_2\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix} \tag{4.3.1-4}$$

Since the inner product of perpendicular vectors is always 0, Q_2 must lay in the null space of M_1 such that $M_1Q_2=0$.

$$M_1Q_2=0 \rightarrow Q_2=e_{1\perp}^T\alpha_2 \tag{4.3.1-5}$$

To realize $M_2Q_1=0$, the freedom in Q_1 can be used (4.3.1-3). α_1 should be chosen such that $M_2Q_1=0$. With:

$$\begin{aligned}e_1e_2^T&=\cos(\angle(e_2,e_1))=\cos(\angle(e_1,e_2))=\cos(\varphi_{12})\\ e_2e_{1_\perp}^T&=\cos(\angle(e_2,e_{1_\perp}))=\sin(\angle(e_2,e_1))=\sin(\varphi_{12})\end{aligned} \tag{4.3.1-6}$$

α_1 follows from:

$$
\begin{gathered}
M_2 Q_1 = 0 \\
\Leftrightarrow \\
m_2 e_2 (m_1^{-1} e_1^T + e_{1_\perp}^T \alpha_1) = 0 \\
\Leftrightarrow \\
\alpha_1 = -m_1^{-1} \frac{e_2 e_1^T}{e_2 e_{1_\perp}^T} = -m_1^{-1} \frac{\cos(\varphi_{12})}{\sin(\varphi_{12})} = -m_1^{-1} \cot(\varphi_{12})
\end{gathered}
\tag{4.3.1-7}
$$

Since M_2Q_1 represents the influence of y_{s_1} on y_{p_2} , α_1 represents the effort to suppress this influence.

α_2 has to be such that $M_2Q_2=1$.

$$
\begin{gathered}
M_2 Q_2 = 1 \\
\Leftrightarrow \\
m_2 e_2 e_{1_\perp}^T \alpha_2 = 1 \\
\Leftrightarrow \\
\alpha_2 = m_2^{-1} \frac{1}{\sin(\varphi_{12})}
\end{gathered}
\tag{4.3.1-8}
$$

α_2 gives the effort to realize y_{s_2} 1:1 in y_{p_2} whilst y_{s_2} is not allowed to affect y_{p_1} . So the effort to realize Q_2 in the null space of M_1 is also indicated by α_2.

From the expressions for both α_1 and α_2 we derive that a small angle between the rows of M or, in other words, large dependency in M, results in a large effort to control the outputs independently. Of course large gains m_1 and m_2 can compensate for this and reduce the values of α_1 and α_2.

Q is made up by the two columns Q_1 and Q_2 in which α_1 and α_2 are substituted:

$$
\begin{gathered}
Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} m_1^{-1}(e_1^T - \cot\varphi_{12} e_{1_\perp}^T) & m_2^{-1} \sin^{-1}\varphi_{12} e_{1_\perp}^T \end{bmatrix} = \\
\begin{bmatrix} (e_1^T - \cot\varphi_{12} e_{1_\perp}^T) & \sin^{-1}\varphi_{12} e_{1_\perp}^T \end{bmatrix} \begin{bmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \end{bmatrix} = \\
\begin{bmatrix} e_1^T & e_{1_\perp}^T \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\cot\varphi_{12} & \sin^{-1}\varphi_{12} \end{bmatrix} \begin{bmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \end{bmatrix}
\end{gathered}
\tag{4.3.1-9}
$$

In (4.3.1-9) we derived an interesting decomposition of the process pseudo inverse or ideal controller. The first matrix is an orthonormal base (perpendicular vectors with length 1) for the output space of the controller which equals the input space of the process. The second matrix is a lower triangle matrix. Postmultiplication of the second matrix with the third, which contains the inverse gain behaviour, does not affect the lower triangle structure. All these matrixes together are nothing else than a QR decomposition [5] of the pseudo inverted process:

$$
\begin{aligned}
\left[e_1^T \; e_{1_\perp}^T\right] \begin{bmatrix} m_1^{-1} & 0 \\ -m_1^{-1}\cot\varphi_{12} & m_2^{-1}\sin^{-1}\varphi_{12} \end{bmatrix} &= \\
\left[q_{b_1}^T \; q_{b_2}^T\right] \begin{bmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{bmatrix} &= \\
Q_b R &
\end{aligned}
\qquad (4.3.1\text{-}10)
$$

R_{11} indicates the effort to realize y_{s_1} 1:1 at y_{p_1} . R_{21} is an indicator for the effort to suppress the influence of y_{s_1} on y_{p_2} . R_{22} represents the effort to realize y_{s_2} 1:1 in y_{p_2} and to suppress the influence of y_{s_2} on y_{p_1} .

Figure 4.5 is a visualization of what was previously said. Some explanation follows.

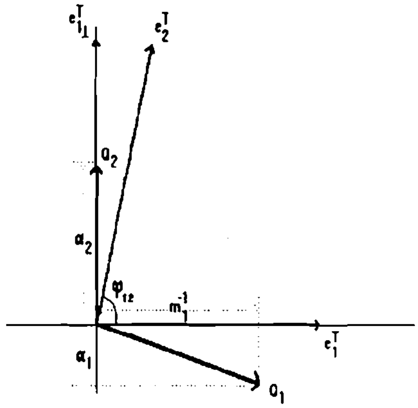


Figure 4.5 *Visualization of the controller of a 2x2 process*

e_1^T and $e_{1\perp}^T$ span the x- and y-axis respectively. e_1^T and e_2^T have an angle φ_{12} . Q_1 is spanned by a component in the direction of e_1^T and by a component in the direction of $e_{1\perp}^T$. Q_2 is realized in $e_{1\perp}^T$ only.

Now the foregoing expressions will be evaluated for an example (see (4.3.1-10)).

$$M=\begin{bmatrix}0.33 & 0\\ 2.88 & 0.87\end{bmatrix}=\begin{bmatrix}0.33[1\ \ 0]\\ 3[0.96\ \ 0.29]\end{bmatrix}=\begin{bmatrix}m_1e_1\\ m_2e_2\end{bmatrix}$$

$$m_1=0.33 \quad m_2=3 \quad \varphi_{12}=17^\circ \tag{4.3.1-11}$$

$$R_{11}=3 \quad R_{21}=-9.81 \quad R_{22}=1.14$$

$$Q=Q_bR=\begin{bmatrix}e_1^T & e_{1\perp}^T\end{bmatrix}R=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & 0\\ -9.81 & 1.14\end{bmatrix}$$

The amplitude of both R_{11} and R_{21} are relatively large. Causes are the small process gain m_1 and the small angle φ_{12} between e_1 and e_2. They result in a large factor α_1.

As was announced in section 4.2.2, a close approximation can be derived for the condition number of the 2x2 Q and M (equal, see section 4.2.2) using the perspective that was described before (see appendix 2):

$$\text{with } M=\begin{bmatrix}m_1 & 0\\ 0 & m_2\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix}$$

$$cond(M)=cond(Q)=\frac{\sigma_{1_Q}}{\sigma_{2_Q}}\approx\frac{1}{|\sin\varphi_{12}|}\sqrt{(\frac{m_2}{m_1})^2+(\frac{m_1}{m_2})^2+2\cos^2\varphi_{12}} \tag{4.3.1-12}$$

From this expression we can conclude that a small angle between the rows of M and a large difference between the gains results in a large condition number. Evaluated for the process of (4.3.1-11): cond(Q)=31.4. The large difference between m_1 and m_2 and the small angle cause a large condition number. For asymptotic situations the expression can be further simplified:

$$(i) \quad m_1 \gg m_2, \quad cond(M)=cond(Q)\approx\frac{1}{|\sin\varphi_{12}|}\frac{m_1}{m_2}$$

$$(ii) \quad m_2 \gg m_1, \quad cond(M)=cond(Q)\approx\frac{1}{|\sin\varphi_{12}|}\frac{m_2}{m_1} \tag{4.3.1-13}$$

$$(iii) \quad m_1 \approx m_2, \quad cond(M)=cond(Q)\approx\sqrt{2\left(\frac{1+\cos^2\varphi_{12}}{\sin^2\varphi_{12}}\right)}$$

Some examples:

(i)
$$M=\begin{bmatrix}10 & 0\\ 0 & 0.1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix},\quad cond(Q)=\frac{1}{|\sin\varphi_{12}|}\frac{m_1}{m_2}=\frac{1}{|\sin 90°|}\frac{10}{0.1}=100$$

(ii) $\varphi_{12}=90°$
$$M=\begin{bmatrix}1.1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -0.9\end{bmatrix},\quad cond(Q)=1.17=\sqrt{2} \tag{4.3.1-14}$$

(iii) $\varphi_{12}=0°$
$$M=\begin{bmatrix}1.1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 0.9\end{bmatrix},\quad cond(Q)=38.42\rightarrow\infty$$

We see that we have found a way for the 2x2 case to decompose the ideal Q in a way that the effort to control the distinct outputs is clear. The effort to control output 2 perfectly is the effort to control output 2 perfectly provided that the perfect control of output 1 is not affected. The decomposition simply is a QR decomposition in which Q is an orthonormal base and R a lower triangle matrix.

4.3.2 Construction of the mxn Ideal Controller

For the controllability analysis method we have to know how the effort to control distinct outputs fits in the ideal controller (pseudo inverse process) for the whole process. In this section this relation is derived based on the results of the 2x2 case for the general case of all square and nonsquare processes. In (4.3.2-1) the gain and direction behaviour of the process model M are split:

$$M=\begin{bmatrix}M_1\\ M_2\\ .\\ .\\ M_m\end{bmatrix}=\begin{bmatrix}m_1e_1\\ m_2e_2\\ .\\ .\\ m_pe_m\end{bmatrix}=\begin{bmatrix}m_1 & 0 & . & . & 0\\ 0 & m_2 & . & . & 0\\ . & . & . & . & .\\ . & . & . & . & .\\ 0 & 0 & . & . & m_m\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ .\\ .\\ e_m\end{bmatrix}=M_gE \tag{4.3.2-1}$$
$$m_i=\|M_i\|_2 \wedge \|e_i\|=1$$

In (4.3.2-2) the ideal controller is decomposed into an orthonormal base Q_b and a lower triangle matrix R.

$$Q=M^{*}=Q_bR=$$
$$\begin{bmatrix}q_{b_1}^T & q_{b_2}^T & . & q_{b_m}^T\end{bmatrix}\begin{bmatrix}R_{11} & 0 & . & 0\\ . & . & . & .\\ . & . & . & 0\\ R_{ml} & . & . & R_{mm}\end{bmatrix} \tag{4.3.2-2}$$

R_{11} indicates the effort to realize y_{s_1} 1:1 at y_{p_1} .

R_{22} indicates the effort to realize y_{s_2} 1:1 in y_{p_2} and to suppress the influence of y_{s_2} on y_{p_1} .

R_{mm} indicates the effort to realize y_{s_m} 1:1 in y_{p_m} and to suppress the influence

of y_{s_m} on y_{p_1} to $y_{p_{m-1}}$.

R_{21} indicates the effort to suppress the influence of y_{s_1} on y_{p_2} .

$R_{i+j,i}$, with j>0, indicates the effort to suppress the influence of y_{s_i} on $y_{p_{i+j}}$ whilst y_{s_i} not

disturbing y_{p_1} $y_{p_{i+j-1}}$.

Based on (4.2.2-4) we concluded that the SVD was not useful. Let us apply the QR decomposition on the same process here:

$$M=\begin{bmatrix}0.58 & 0.58 & 0.58\\ 0.71 & -0.71 & 0\\ 0.91 & -0.25 & 0.35\end{bmatrix},\quad Q=M^{+}=Q_bR=\begin{bmatrix}0.58 & 0.71 & 0.41\\ 0.58 & -0.71 & 0.41\\ 0.58 & 0 & -0.82\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 41.6 & 58.0 & -71.0\end{bmatrix} \qquad (4.3.2\text{-}3)$$

Row i of R represents the effort to control output i provided that the perfect control of more important outputs is not affected (lower indexed outputs). We see that there is hardly any effort in controlling output 1 and 2 perfectly. There is even no need to decouple ($R_{21}=0$). However the effort to control y_{p3} explodes (R_{31}, R_{32}, R_{33}).

In appendix 3 it is explained how the base Q_b is constructed and it is R is further decomposed. This decomposition facilitates us to express the different 'efforts' more explicitly.

We see that we have found a way for the mxn case to decompose the ideal Q in a way that the effort to control the distinct outputs is clear. The effort to control output i perfectly with i>1 is the effort to control output i perfectly provided that the perfect control of lower outputs is not affected. The decomposition simply is a QR decomposition in which Q is an orthonormal base and R is a lower triangle matrix. The R matrix can be further decomposed as can be read in appendix 3.

4.4 Conclusions

In this chapter we have seen that process characteristics that limit invertibility are RHP poles and delays, in other words nonminimum phase behaviour. The controllability analysis method that is developed here will not be applicable on processes with nonminimum phase behaviour. A singular value decomposition can be useful to detect in what output and input directions of the process and the ideal controller the smallest and largest gains appear. A QR-decomposition (Q an orthonormal base, R a lower triangle matrix) of the pseudo inverse process makes it possible to express the effort to control specific outputs. The R matrix can further be decomposed to increase our insight.

Our tactic in the derivation of the controllability analysis method is to derive the ideal controller, wonder how restrictions limit its realization and the wish to control what output(s) can be held responsible for this. By doing so it becomes clear what is best achievable per output. The philosophy behind this is that the control of the whole system is given by the 'sum' of controllabilities of all outputs separately.

In the previous chapter we found a decomposition of the ideal controller in which the effort to control specific outputs becomes clear. In chapter 3 we found out that the IMC scheme relates the restrictions as directly as possible to the process. The restrictions, that will be denoted with 'global' restrictions from now on, were: the maximum amplitude of the control signals and the demands for robustness to modelling errors of stability and performance. In this chapter we study how the restrictions come through per output which is precisely the information that we are looking for in our controllability analysis. A controller that performs as good as possible can be seen as a filtered version of the ideal one.

An important limitation of the treatment here is that all uncertainty is modelled at the outputs in a diagonal structure as was first seen in section 2.2. We will see that this uncertainty model fits well on the QR decomposition of the process inverse. For the input uncertainty a completely analogous treatment as in this chapter can be followed based on the dual case of QR decomposition of the process instead of decomposition of its inverse. The problem however lays in the translation of the resulting filter structure that is strongly related to the process inputs (and controller outputs) to the filter structure that is used in this chapter which is related more directly to the process outputs (or controller inputs). This will be a subject for future research.

The entries of the R matrix resulting from a QR decomposition of the pseudo-inverse process or ideal controller represent the effort to realize a setpoint value 1:1 at the output of the process or the effort to suppress its 'cross' influence on the other outputs. Here we will study for each global restriction separately how it limits the R entries and thereby the corresponding outputs. But first we have to define a framework to fit in the restrictions on the entries of R.

Since R has a lower triangle structure, the filter with which it is postmultiplied can be a lower triangle matrix as well. Then all entries of R are covered. The realizable controller can be expressed in terms of the ideal controller:

$$Q_{real} = Q_{ideal}F = Q_b RF = Q_b R_F$$

$$R_F = RF = \begin{bmatrix} R_{11} & 0 & . & 0 \\ R_{21} & R_{22} & . & . \\ . & . & . & 0 \\ R_{m1} & . & . & R_{mm} \end{bmatrix} \begin{bmatrix} f_{11} & 0 & . & 0 \\ f_{21} & f_{22} & . & . \\ . & . & . & 0 \\ f_{m1} & . & . & f_{mm} \end{bmatrix} \tag{5-1}$$

The lower triangle form for F is chosen because it covers all elements of R and the product RF results in a matrix with the same lower triangle structure. This maintains the direct relation with the process outputs (controller inputs) and, at the same time, it has not become more complicated to relate the entries of the product RF to the process inputs (controller outputs). We will see that the mechanism to 'fill in the entries' of F, is related to the ranking of the process outputs (see section 2.1).

F contains SISO filters. In the first two sections of this chapter it will be assumed that all filters are ideal. The procedure that we will follow here will be to derive for each global restriction what specific SISO filters it needs. We will find out that the restriction on the amplitude of the control signals and the demand for robust stability, all uncertainty being modelled as output uncertainty, are translated into filters that make the controller less ideal and that the robustness demand is less conservative than the limited amplitude demand. The robust performance demand is translated into keeping the ideal controller as much as possible intact, in other words no filtering.

In the conclusion of this chapter the controllability analysis method will be described and suggestions will be made for future research. In the following chapter the method will be applied on a process with three inputs and two outputs.

5.1 Maximum Amplitude of the Control Signals

In this section it is investigated what filtering the R matrix needs to satisfy the restriction on the control signals. We will see that the filters will be designed using a special mechanism. In (5.1-1) the control vector is calculated (see (3.2-3)) and an upper bound is derived for its length:

$$u = Q_{real} y_s = Q_b R_F y_s =$$

$$\left[q_{b_1}^T R_{F_{11}} + q_{b_2}^T R_{F_{21}} + .. + q_{b_m}^T R_{F_{m1}} \quad q_{b_2}^T R_{F_{22}} + q_{b_3}^T R_{F_{32}} + .. + q_{b_m}^T R_{F_{m2}} \quad . \quad q_{b_m}^T R_{F_{mm}} \right] y_s$$

$$\textit{with a scaling such that } |u(t)| \le 1 \textit{ and } |y_s(t)| \le 1 \textit{ and } y_s \textit{ worst case:}$$

$$u_{worst\ case} = q_{b_1}^T |R_{F_{11}}| + q_{b_2}^T (|R_{F_{21}}| + |R_{F_{22}}|) + q_{b_3}^T (|R_{F_{31}}| + |R_{F_{32}}| + |R_{F_{33}}|) + ..$$

$$+ q_{b_m}^T (|R_{F_{m1}}| + |R_{F_{m2}}| + .. |R_{F_{mm}}|) \tag{5.1-1}$$

$$\textit{so it holds that:}$$

$$\textit{worst case length of vector} : |u|^2 \le |R_{F_{11}}|^2 + (|R_{F_{21}}| + |R_{F_{22}}|)^2 + .. + (|R_{F_{m1}}| + |R_{F_{m2}}| + .. + |R_{F_{mm}}|)^2$$

$$\textit{if } u \textit{ contains } n \textit{ elements then } |u|^2_{max} \le n$$

$$|R_{F_{11}}|^2 + (|R_{F_{21}}| + |R_{F_{22}}|)^2 + .. + (|R_{F_{m1}}| + |R_{F_{m2}}| + .. + |R_{F_{mm}}|)^2 \le n$$

Each term of the left side of the last inequality represents the effort to control an output. It is based on the worst case setpoint input vector y_s that contains entries with values between -1 and +1. But how often is this setpoint input offered? It has resulted in a very conservative demand. The interpretation here will be the following; to restrict the amplitude of the control vector, all the amplitudes of all entries of R_F must be limited. We choose here for an upper bound 1 for all amplitudes of the elements of R_F. The question now is how we should choose the SISO filters of (5-1) such that the entries of R_F can keep their ideal values up to the highest frequency possible. Calculate R_F in case M has three outputs:

$$R_F = RF = \begin{bmatrix} R_{11} & 0 & 0 \\ R_{21} & R_{22} & 0 \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} f_{11} & 0 & 0 \\ f_{21} & f_{22} & 0 \\ f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} R_{11}f_{11} & 0 & 0 \\ R_{21}f_{11}+R_{22}f_{21} & R_{22}f_{22} & 0 \\ R_{31}f_{11}+R_{32}f_{21}+R_{33}f_{31} & R_{32}f_{22}+R_{33}f_{32} & R_{33}f_{33} \end{bmatrix} \tag{5.1-2}$$

Because we are interested in the fact how to filter each entry of R as much as is necessary, we rewrite:

$$R_F = \begin{bmatrix} R_{11}f_{11} & 0 & 0 \\ R_{21}(f_{11}+\frac{R_{22}}{R_{21}}f_{21}) & R_{22}f_{22} & 0 \\ R_{31}(f_{11}+\frac{R_{32}}{R_{31}}f_{21}+\frac{R_{33}}{R_{31}}f_{31}) & R_{32}(f_{22}+\frac{R_{33}}{R_{32}}f_{32}) & R_{33}f_{33} \end{bmatrix} = \begin{bmatrix} R_{11}f_{11} & 0 & 0 \\ R_{21}\tilde{f}_{21} & R_{22}f_{22} & 0 \\ R_{31}\tilde{f}_{31} & R_{32}\tilde{f}_{32} & R_{33}f_{33} \end{bmatrix} \tag{5.1-3}$$

Now it is clear for each entry of R what filter or sum of filters affects it. We see that all diagonal R_{ii}'s are affected by one filter f_{ii}. The elements on the second diagonal, $R_{i+1,i}$, are affected by the sum of the diagonal filters f_{ii} and a weighted filter $f_{i+1,i}$. Later we will have a closer look at these weights. The entries on the third diagonal (here only one: R_{31}) are affected by a sum of the previous filters that were encountered in the same column (f_{ii} and $f_{i+1,i}$ now another weight and a weighted new filter $f_{i+2,i}$. Etc.) In all these expressions, f_{ii} will never be weighted. Again we state here that each R_{ji} must be filtered as much as is necessary. We would rather not filter but since we must limit the amplitudes of all R_{ji} to 1, we have to. As a function of frequency the condition of the process and therefore the amplitudes of the entries of R only get worse. Therefore it will be clear that all filters on the diagonal and the weighted sums of filters between braces must have low pass characteristics. We can already conclude that the diagonal filters f_{ii} áre LPF as far as the restriction on the amplitude of the control signals is concerned. In figure 5.1 the filter and the filtered and unfiltered version of the amplitude of R_{11} as an example for all R_{ii} are given. For simplification the filter is assumed to be ideal.

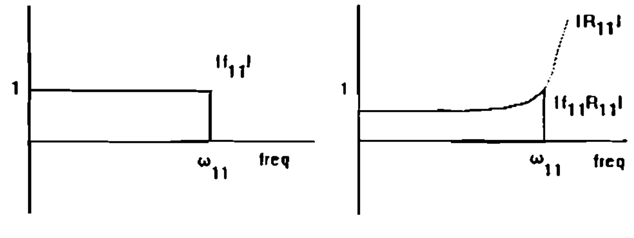


Figure 5.1 *Filtering of R_{11} and more generally all R_{ii}*

If we fit f_{11} on R_{11}, will freedom be lost to filter R_{21} precisely as much as is necessary and not more? What if R_{21} keeps an amplitude smaller than 1 over the whole frequency band whilst R_{11} needs filtering from 2 [rad/sec]? The answer lays in f_{21}. We can choose it such that R_{21} senses less filtering than R_{11}. The total filtering f^*_{21} sensed by R_{21}:

$$f^*_{21} = f_{11} + \frac{R_{22}}{R_{21}} f_{21} \tag{5.1-4}$$

Suppose R_{21} does not need to be filtered at all then the following choice for f_{21} realizes this:

$$f_{21} = \frac{R_{21}}{R_{22}}(1-f_{11}) \text{ then } f^*_{21} = f_{11} + \frac{R_{22}}{R_{21}} f_{21} = 1 \tag{5.1-5}$$

The choice for a filter for R_{21} can be generalized:

with f^*_{21} representing the wanted filter characteristic for R_{21}:

$$\textit{filter sensed by } R_{21}\text{:}\ f_{11} + \frac{R_{22}}{R_{21}} f_{21} \quad \textit{choose } f_{21} = \frac{R_{21}}{R_{22}}(f^*_{21} - f_{11});\ \textit{then} \tag{5.1-6}$$

$$f_{11} + \frac{R_{22}}{R_{21}} f_{21} = f^*_{21}$$

To get an idea how f_{21} looks for this choice it is depicted in figure 5.2. It is assumed that the cutoff frequency ω_{21} is smaller than ω_{11}.

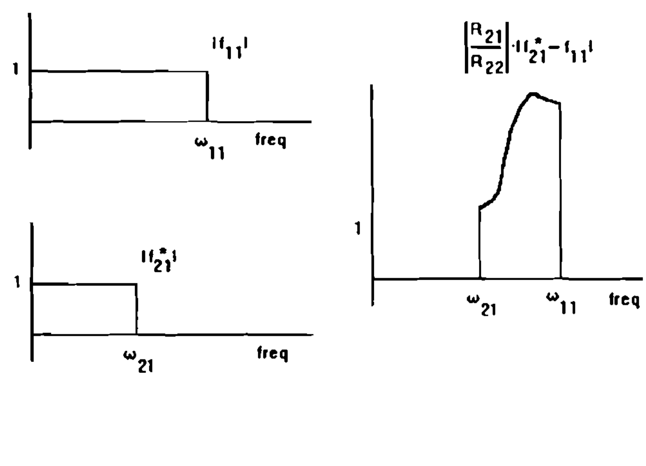


Figure 5.2 Band filter f_{21}

f_{21} is the subtraction of two low pass filters amplified with a factor which will be treated later. This results in a band filter.

We saw that the choice for an optimal filter (as far as the amplitude of the control signals is concerned) for R_{11} did not limit us in finding the right filter for R_{21}. With f_{22} fitting R_{22} best, f_{32} is chosen analogously with f^*_{32} suppressing R_{32} where it exceeds 1.

$$\begin{aligned} &choose\ f_{32} = \frac{R_{32}}{R_{33}}(f^{*}_{32} - f_{22});\\ &then\ f_{22} + \frac{R_{33}}{R_{32}} f_{32} = f^{*}_{32} \end{aligned} \tag{5.1-7}$$

Now the filter on the third diagonal: f_{31}. The filtering that is sensed by R_{31} (see (5.1-3)):

$$f_{11} + \frac{R_{32}}{R_{31}} f_{21} + \frac{R_{33}}{R_{31}} f_{31} \tag{5.1-8}$$

with f_{11} and f_{21} satisfying R_{11} and R_{21} best, f_{31} must be such that in order to limit R_{31} best, it establishes a low pass action for (5.1-8):

$$\begin{aligned} &f_{31} = \frac{R_{31}}{R_{33}}(f^{*}_{31} - f_{11} - \frac{R_{32}}{R_{31}} f_{21}) =\\ &\frac{R_{31}}{R_{33}}(f^{*}_{31} - f_{11}) - \frac{R_{32}}{R_{31}}\frac{R_{21}}{R_{22}}(f^{*}_{21} - f_{11}) \end{aligned} \tag{5.1-9}$$

with f^{*}_{31} the appropriate low pass filter for R_{31}. f_{31} is a weighted sum of two band filters.

With all the filters chosen optimally for the 3x3 case, as far as the global restriction on the amplitude of the control vector is concerned, we can express R_F simpler in low pass filters only:

$$R_F = \begin{bmatrix} R_{11} f_{11} & 0 & 0 \\ R_{21} f^{*}_{21} & R_{22} f_{22} & 0 \\ R_{31} f^{*}_{31} & R_{32} f^{*}_{32} & R_{33} f_{33} \end{bmatrix} \tag{5.1-10}$$

After the derivation of the SISO filters for the 3x3 case we can formulate a general mechanism for filter design as far as the restriction on the amplitude of the control signals is concerned. The filter matrix F in (5-1) is filled per diagonal:

- design the SISO low pass filters $f_{i,i}$ that suit the main diagonal entries of R best.
- design the SISO low pass filters $f^{*}_{i+1,i}$ that suit the entries on the second diagonal of R best.
- substitute the designed filters into the expression $f_{i+1,i} = \frac{R_{i+1,i}}{R_{i+1,i+1}}(f^{*}_{i+1,i} - f_{i,i})$ to obtain the filters $f_{i+1,i}$ on the second diagonal of F
- design the SISO low pass filters $f^{*}_{i+2,i}$ that suit the entries on the third diagonal of R best.
- substitute the designed filters $f_{i,i}$, $f_{i+1,i}$ and $f^{*}_{i+2,i}$ into the expression $f_{i+2,i} = \frac{R_{i+2,i}}{R_{i+2,i+2}}(f^{*}_{i+2,i} - f_{i,i}) - \frac{R_{i+2,i+1}}{R_{i+2,i+2}} f_{i+1,i}$ to obtain the filters $f_{i+2,i}$ on the third diagonal of F.

Generally: the filters $f_{i+j,i}$ on the j+1-th diagonal of F (with j≥2) follow from $f^*_{i+j,i}$ (must be designed properly) and f_{ii} to $f_{i+j-1,i}$ by the following expression:

$$f_{i+j,i} = \frac{R_{i+j,i}}{R_{i+j,i+j}}(f^*_{i+j,i} - f_{i,i}) - \frac{1}{R_{i+j,i+j}}\sum_{k=1}^{j-1} R_{i+j,i+k}\, f_{i+k,i} \qquad (5.1\text{-}11)$$

The proof follows by induction. What is the characteristic of this filter $f_{i+j,i}$ in the j+1-th diagonal? It can be proven that it can be rewritten as a weighted sum of band filters analogous to f_{31} (see (5.1-9)). These band filters look like: $f^*_{i+k,i}-f_{ii}$ (with k=1 to j-1). It will depend on the characteristic of each $R_{i+k,i}$ whether this subfilter starts or ends at ϖ_{ii}. The amplitude of the weighted sum of all these filters that represents $f_{i+j,i}$ will probably result in a large bulb to the left of ϖ_{ii} and a smaller one or none at all to the right since R_{ii} will in general require less conservative filtering than the elements beneath it in the same column of R. The cause is that the effort to realize a setpoint input 1:1 at the corresponding process output requires a smaller gain than the effort to suppress the influence of the same input in less important outputs in general.

Before we will look at the consequences of this filter structure for nominal performance (we will treat robust performance in section 5.3), let us first study the weights that we encountered in the lower diagonal filters f_{ji} that determine the height of the bulb.

In appendix 3 the lower triangle matrix R which resulted from the QR decomposition of the pseudo-inverse process was decomposed further (see section 4.3.2). This knowledge we can use here to express the filter weights more directly in terms of the process. We will see that the expressions contain canonical angles. The canonical angles that are used here are the smallest angles that exist between a vector and a subspace (see appendix 3, figure 3.1). In appendix 3 it is derived that $R=Q_l Q_\alpha Q_g$. With Q_l (appendix 3, equation (3.4)) and Q_g (3.6) both diagonal matrices and Q_α (4.3) being a lower diagonal matrix. Remind:

$$\text{with } |e_i| = |q_{b_i}| = 1$$

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = R^{-1}\begin{bmatrix} q_{b_1} \\ q_{b_2} \\ q_{b_3} \end{bmatrix} \qquad (5.1\text{-}12)$$

$$\text{such that:}$$

$$Q_{ideal} = \begin{bmatrix} q_{b_1}^T & q_{b_2}^T & q_{b_3}^T \end{bmatrix} R$$

In (5.1-13) the alternative expressions for the SISO filter weights in the 3x3 case, based on the decomposition of R in appendix 3, are given:

$$for\ f_{21} = \frac{R_{21}}{R_{22}}(f_{21}^{*} - f_{11}):$$
$$\frac{R_{21}}{R_{22}} = -\frac{m_2}{m_1} e_2 q_{b_1}^T = -\frac{m_2}{m_1} e_2 e_1^T = -\frac{m_2}{m_1}\cos\varphi_{12}$$

$$for\ f_{32} = \frac{R_{32}}{R_{33}}(f_{32}^{*} - f_{22}):$$
$$\frac{R_{32}}{R_{33}} = -\frac{m_3}{m_2}\frac{1}{e_2 q_{b_2}^T} e_3 q_{b_2}^T \qquad (5.1\text{-}13)$$

$$for\ f_{31} = \frac{R_{31}}{R_{33}}(f_{31}^{*} - f_{11}) - \frac{R_{32}}{R_{31}}\frac{R_{21}}{R_{22}}(f_{21}^{*} - f_{11}):$$
$$\frac{R_{31}}{R_{33}} = -\frac{m_3}{m_1}\frac{1}{e_2 q_{b_2}^T}(e_3 q_{b_1}^T e_2 q_{b_2}^T - e_3 q_{b_2}^T e_2 q_{b_1}^T)$$
$$\frac{R_{32}}{R_{33}}\frac{R_{21}}{R_{22}} = \frac{m_3}{m_1}\frac{1}{e_2 q_{b_2}^T} e_2 q_{b_1}^T$$

$e_2 q_{b_2}^T$ can be expressed in the canonical angle: $e_2 q_{b_2}^T = \sin\varphi_{2x}$

All sines of canonical angles $e_i q_{b_i}^T = \sin\varphi_{ix}$ can take on values between -1 and +1. The same holds for the cosines of ordinary angles between the two vectors: $e_i q_{b_j}^T = \cos(\angle(e_i, q_{b_j}))$ Bad condition of M is given here by small canonical and ordinary angles. We can derive the following upper bounds for the amplitudes of the filter weights (see (5.1-13)):

$$for\ f_{21}:$$
$$\left|\frac{R_{21}}{R_{22}}\right| \le \frac{m_2}{m_1}$$

$$for\ f_{32}$$
$$\left|\frac{R_{32}}{R_{33}}\right| \le \frac{m_3}{m_2}\frac{1}{|e_2 q_{b_2}^T|} \qquad (5.1\text{-}14)$$

$$for\ f_{31}:$$
$$\left|\frac{R_{31}}{R_{33}}\right| \le \frac{m_3}{m_1}\frac{1}{|e_2 q_{b_2}^T|}\left|e_3 q_{b_1}^T 0 - e_3 q_{b_2}^T 1\right| \le \frac{m_3}{m_1}\frac{|e_3 q_{b_2}^T|}{|e_2 q_{b_2}^T|} \le \frac{m_3}{m_1}\frac{1}{|e_2 q_{b_2}^T|} \quad \text{since when } e_2 q_{b_2}^T = \sin\varphi_{2x} = \sin\varphi_{12} = 0 \text{ then } e_2 q_{b_1}^T = \cos\varphi_{12} = 1$$
$$\left|\frac{R_{32}}{R_{33}}\frac{R_{21}}{R_{22}}\right| \le \frac{m_3}{m_1}\frac{1}{|e_2 q_{b_2}^T|}$$

We see that the amplitude of filter f_{21} strongly depends on m_2/m_1. f_{32} strongly depends on m_3/m_2 and factor $\sin^{-1}\varphi_{2x}$. f_{31} depends on m_3/m_1 and factor $\sin^{-1}\varphi_{2x}$. So large differences in process gain and a small angle between the rows of M result in large filter gains. It is expected that this result for the 3x3 case can be generalized to the following for the mxn process and the related mxm matrix R and filter F:

the amplitudes of the filter weights in f_{ji} are all dominated by m_j/m_i and the sines of *the canonical angles:* $e_2 q_{b_2}^T$ *to* $e_{l-1} q_{b_{l-1}}^T$

The proof of this using induction is a subject for future research.

Now let us study as the last subject in this section what the consequences are of the derived filter structure for nominal performance using the example of a 2x2 process. In (3.2-2) the transfers for the nominal case were given. Here we study the matrix S=I-MQ as a measure for nominal performance. With $Q=Q_{real}=Q_{ideal}F=Q_bRF$ it follows for the 2x2 case that:

$$S=I-MQ=I-F=\begin{bmatrix}1-f_{11} & 0\\ -f_{21} & 1-f_{22}\end{bmatrix} \qquad (5.1\text{-}15)$$

Example:
For this example the process is chosen such that R_{21} *needs filtering most,* R_{22} *needs filtering least. As a consequence* $\varpi_{21}<\varpi_{11}<\varpi_{22}$. *From (4.3.1-10) we can derive the process behaviour that fits on this profile; the rows of M must be rather dependent (*$\cos\varphi_{12}$ *relatively large,* $\sin\varphi_{12}$ *relatively small),* m_1 *is very small and* m_2 *is very large. Based on ideal filters the sensitivity is given here by the amplitudes of its three lower triangle entries. They are depicted in the following figure:*

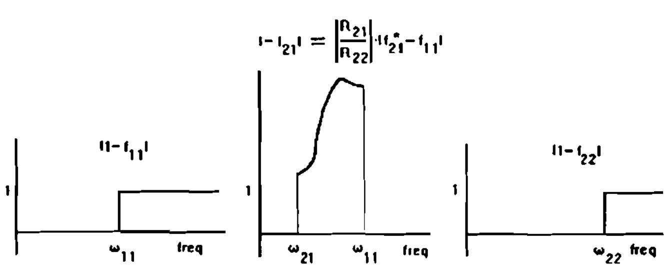


Figure 5.2a *Amplitudes of the entries of S*

The following will hold for the steady state ($y_s=\underline{0}$*). We conclude that for frequencies higher than* ϖ_{11}, y_{p1} *starts being sensible for disturbances that affect* y_{p1}. *For frequencies between* ϖ_{21} *and* ϖ_{11}, y_{p2} *is extremely sensible for disturbances that affect* y_{p1}. *For frequencies between* ϖ_{11} *and* ϖ_{22}, y_{p2} *is not sensible at all and higher than* ϖ_{22} *its sensitivity is 1.*

The large amplitude of $|-f_{21}|=|f_{21}|$ *is caused by the fact that* m_2 *is much larger than* m_1. *The consequence is that for frequencies between* ϖ_{21} *and* ϖ_{11} *the influence of disturbances in* y_{p1} *on* y_{p2} *is very large. This can be overcome by choosing a lower cutoff frequency for* f_{11}, *more precisely choose* $f_{11}:=f^*_{21}$ *or* $\varpi_{11}:=\varpi_{21}$ *then* $f_{21}=0$ *(5.1-6). The price that is paid is that* y_{p1} *stops being controllable at a lower cutoff frequency which is perhaps not acceptable because*

y_{p1} is more important than y_{p2}. It would have been more favourable for the system when output 2 was more important than output 1.

In this section we have seen that the global restriction on the control signals can be translated into the demand that the amplitude of all entries of the R matrix which stems from the QR decomposition of the controller must be limited. A special filter design mechanism was derived in order to choose the SISO filters such that the entries of R are limited precisely as much as is necessary. For a simple 2x2 process we studied the consequences for nominal performance and how by sacrificing performance for one output, the performance in another output can be improved. The filter weights can be a subject of future research.

5.2 Robustness of Stability for Model Errors Caused by Diagonal Output Uncertainty

In this section we will study how the second global restriction on the controller introduced by the demand for stability for modelling errors can be translated into a specific filter structure such that the relation with the outputs is clear. In this section the model uncertainty that was modelled in chapter 3 as an additive model error is modelled as diagonal output uncertainty. In chapter 2 the causes of this uncertainty were treated and the reason why we are interested in it separately was explained.

$\Delta=\Delta_0 M$. We will find out that this description of Δ fits very well on our QR decomposition to the outputs of the pseudo-inverse process. In (5.2-1) the stability demand which was derived in section 3.2 is given:

$$\lambda_i(\Delta Q) > -1 \tag{5.2-1}$$

All eigenvalues of ΔQ have to be determined. First rewrite ΔQ:

$$\Delta Q_{real} = \Delta_o M Q_{real} = \Delta_o M Q_{ideal} F = \Delta_o F \tag{5.2-2}$$

But $\Delta_0 F$ is a lower triangle matrix of which the eigenvalues are found easily at the diagonal:

$$\textit{m outputs, n inputs, m<n:}$$

$$\Delta_o F = \begin{bmatrix} \delta_1 & 0 & . & 0 \\ 0 & \delta_2 & . & . \\ . & . & . & 0 \\ 0 & . & 0 & \delta_m \end{bmatrix} \begin{bmatrix} f_{11} & 0 & . & 0 \\ f_{21} & f_{22} & . & . \\ . & . & . & 0 \\ f_{m1} & . & . & f_{mm} \end{bmatrix} = \begin{bmatrix} \delta_1 f_{11} & 0 & . & 0 \\ \delta_2 f_{21} & \delta_2 f_{22} & . & . \\ . & . & . & 0 \\ \delta_m f_{m1} & . & . & \delta_m f_{mm} \end{bmatrix} \tag{5.2-3}$$
$$\Leftrightarrow$$
$$\lambda_i(\Delta_o F) = \delta_i f_{ii}$$

So the diagonal SISO filters determine the stability:

$$\begin{gathered} \textit{for } i=1 \textit{ to } m: \\ \delta_i f_{ii} > -1 \end{gathered} \tag{5.2-4}$$

When f_{ii} has an amplification ≤ 1 and if all $\delta_i>-1$ then stability is guaranteed over the whole frequency band.

The restriction on the amplitude of the control signals was translated into a low pass filter identity for all main diagonal filters f_{ii}. We have found out that robust stability for output uncertainty comes through in a less conservative demand.

5.3 Robustness of Performance for Model Errors Caused by Diagonal Output Uncertainty

In this section we will study how the third global restriction on the controller, introduced by the demand that the performance of the controlled system must be guaranteed to a certain extent in case model errors affect the process model, can be translated into a specific filter structure. The model error is modelled completely as diagonal uncertainty at the outputs.

In (3.2-5) the performance was expressed in the deviation vector between setpoint inputs and process outputs. Once again:

$$y_s-y_p=S(I+\Delta Q)^{-1}(y_s-d) \qquad \text{with } S=(I-MQ) \tag{5.3-1}$$

We studied nominal performance in section 5.1 for a 2x2 case already. We concluded that performance can be deteriorated seriously by large differences in filter gain ($m_2>>m_1$). By sacrificing performance in the control of one output, the performance for the other output could be improved. Now we will elaborate robust performance. We begin with the 2x2 case:

$$\begin{gathered}
S(I+\Delta Q)^{-1}=(I-MQ)(I+\Delta Q)^{-1}= \\
(I-MQ_{ideal}F)(I+\Delta_o MQ_{ideal}F)^{-1}=(I-F)(I+\Delta_o F)^{-1}= \\
\begin{bmatrix}1-f_{11} & 0\\ -f_{21} & 1-f_{22}\end{bmatrix}\begin{bmatrix}1+\delta_1 f_{11} & 0\\ \delta_2 f_{21} & 1+\delta_2 f_{22}\end{bmatrix}^{-1}=\begin{bmatrix}(1-f_{11})(1+\delta_1 f_{11})^{-1} & 0\\ -f_{21}(1+\delta_2)(1+\delta_1 f_{11})^{-1}(1+\delta_2 f_{22})^{-1} & (1-f_{22})(1+\delta_2 f_{22})^{-1}\end{bmatrix} \\
=\begin{bmatrix}a & 0\\ -c & b\end{bmatrix} \\
\Leftrightarrow \\
\begin{bmatrix}y_{s_1}-y_{p_1}\\ y_{s_2}-y_{p_2}\end{bmatrix}=\begin{bmatrix}a & 0\\ -c & b\end{bmatrix}\begin{bmatrix}y_{s_1}-d_1\\ y_{s_2}-d_2\end{bmatrix}
\end{gathered} \tag{5.3-2}$$

Later, we will study these entries a, b and c by plotting their amplitudes. In steady state $y_{s1}=y_{s2}=0$. Then 'a' represents the sensitivity of process output 1 for disturbances that affect output 1. 'b' Represents the sensitivity of process output 2 for disturbances at process output 2. 'c' Represents the sensitivity of process output 2 for disturbances at process output 1. Ideally they are zero.

A measure that combines the influence of a, b and c is the largest singular value of $S(I+\Delta Q)^{-1}$. By performing a singular value decomposition we can determine it. The restriction introduced by the robust performance demand could be that σ_1 is not allowed to be larger than a given bound. How small it must be depends on the amplitude of the disturbance vector d and what the maximal deviation at the output is allowed to be .

We will calculate the largest singular value for the 2x2 case:
The singular values of matrix X are calculated by taking the square roots of $\lambda_i(X^TX)$:

$$\left|(S(I+\Delta_o F)^{-1})^T(S(I+\Delta_o F)^{-1})-\lambda I\right|=\left|\begin{bmatrix}a^2+c^2 & -bc\\ -bc & b^2\end{bmatrix}-\lambda I\right|=0$$
$$\Leftrightarrow$$
$$\lambda_{1,2}=a^2+b^2+c^2\pm\sqrt{a^4+b^4+c^4+2b^2(a^2+c^2)-2a^2c^2}$$
$$\Leftrightarrow$$
$$\sigma_1=\sqrt{a^2+b^2+c^2+\sqrt{a^4+b^4+c^4+2b^2(a^2+c^2)-2a^2c^2}} \tag{5.3-3}$$

$$\textit{with}:$$
$$a=(1-f_{11})(1+\delta_1 f_{11})^{-1}$$
$$b=(1-f_{22})(1+\delta_2 f_{22})^{-1}$$
$$c=f_{21}(1+\delta_2)(1+\delta_1 f_{11})^{-1}(1+\delta_2 f_{22})^{-1}$$

Robust performance is best if a, b, and c are zero. We see confirmed that $\sigma_1=0$ in that case as well. What filter structure establishes this? We see:

$$\begin{aligned}&\bullet\ a=b=0 \Leftrightarrow f_{ii}=1 \textit{ and } |\delta_i|<1\\ &\bullet\ c=0 \Leftrightarrow f_{21}=0\end{aligned} \tag{5.3-4}$$

But this means that F=I. No filtering! Of course this is a very trivial result. F=I is possible only when the restriction on the amplitude of the control signals does not require any filtering which is unlikely to occur and when the robust stability demand is satisfied. We have seen that the limitation of the control signals required a low pass filter identity for f_{ii} and a band filter identity for f_{21}. The second restriction, introduced by the stability demand and under the assumption that uncertainty can be modelled as a diagonal at the outputs, was translated into less conservative SISO filters.

Now let us study the consequences of the LPF identities of f_{ii} for the factors 'a' and 'b'. We will see that the characteristic of the LPFs around the cutoff frequency is critical here. That is why we introduce a more realistic LPF characteristic. After the cutoff frequency the characteristic goes asymptotically to zero. The higher the order of the filter the steeper the slope. When uncertainty at the outputs is estimated at 10% of the amplitude and both input and output signals are scaled at an amplitude of 1 then the largest amplitude of the error that could occur is $|\delta_i|=0.1$. Its sign is worst case and it remains as large as possible if $\delta_i=-0.1e^{-j\arg(f_{ii})j}$ In figure 5.3B, f_{11} is given. The amplitude of 'a' is given in figure 5.3A. Everything that holds for 'a', holds for 'b' too.

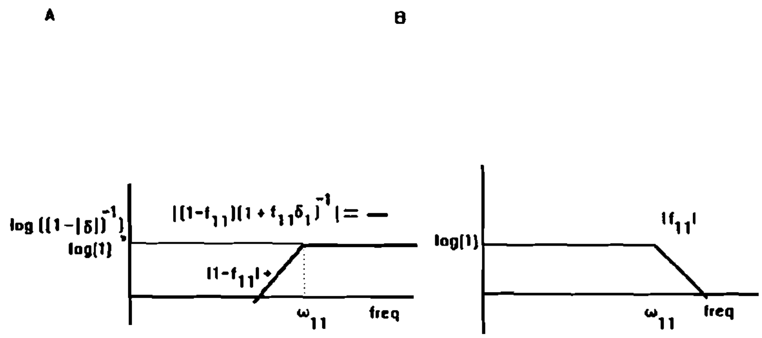


Figure 5.3 *Typical characteristic of the amplitude of the diagonal elements a and b of $S(I+\Delta_o F)^{-1}$*

From figure 5.3A we derive that the output y_{p1} starts becoming sensitive for disturbances around ϖ_{11}. The exact 'starting point' depends on the filter order. The higher the filter order, the steeper its slope. However higher order filters result in an overshoot and perhaps even a damped oscillation for frequencies higher than the cutoff frequency and could seriously damage the performance. We conclude here that choosing the filter order and designing it is critical.

If we compare figure 5.3 to the corresponding diagonal characteristics of S in figure 5.2a which represented the nominal performance by the entries of S, we conclude that the diagonal output uncertainty does not introduce new effects for the diagonal elements of S that could damage performance.

Let us focus our attention on 'c' now. 'c' Represents the sensitivity of process output 2 for disturbances at process output 1. It is given by:

$$c = f_{21}(1+\delta_2)(1+\delta_1 f_{11})^{-1}(1+\delta_2 f_{22})^{-1} \tag{5.3-5}$$

We see that all filters are combined in it. We can make c=0 for $f_{21}=0$. f_{21} was given by:

$$f_{21} = \frac{R_{21}}{R_{22}}(f^*_{21}-f_{11}) - \frac{m_2}{m_1}\cos\varphi_{12}(f_{11}-f^*_{21}) \tag{5.3-6}$$

with φ_{12} the angle between the unit length vectors of the process model M and m_1, m_2 the length of the row vectors (4.3.1-1). Depending on the condition of M and m_1 and m_2 specifically, the amplitude of f_{21} can become tremendously large if $f^*_{21} \neq f_{11}$. We have already seen this for nominal performance (figure 5.2a). Worst case $\varpi_{21} < \varpi_{11}$. If f_{11} is chosen such that it equals f_{21} then $f_{21}=0$ and the bulb disappears in 'c'. In words: performance at output 1 is sacrificed to improve the performance for output 2. In figure 5.5A the characteristic of the amplitude of c for $f_{21}=1$ is given. Again the model uncertainties are chosen worst case. The characteristic of f_{21} such that the amplitude restriction for R_{21} is satisfied, is given in figure 5.5B. The actual characteristic of $|c|$ can be derived by passing the characteristic of figure 5.5A through the filter of 5.5B.

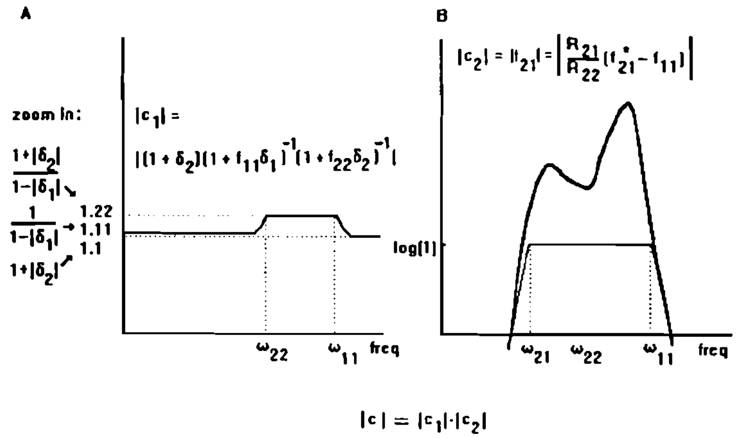


Figure 5.5 *Determination of the worst case characteristic for the amplitude of 'c'*

We derive from figure 5.5A that the model uncertainty at the output will never cause a serious deterioration of performance via 'c'. As for the nominal case (figure 5.2a), the filter f_{21} can be the bottle neck. We conclude that the diagonal output uncertainty does not introduce new effects that could damage performance as far as 'c' is concerned. We arrived at the same conclusion for the diagonal entries of S. Our global conclusion for robust performance is: if the performance for the nominal case is all right then robust performance will be all right as well, if all uncertainty can be modelled as diagonal output uncertainty and the amplitude of the uncertainty is limited.

In section 2.3 scaling of the process was treated. As one of the possible criteria was mentioned the minimal condition number. In (4.3.1-12) an expression was given for the condition number of the 2x2 ideal controller it equals the condition number of the process. We see that the number is minimal for $m_1=m_2$. When $m_2<<m_1$ results in a large condition number for certain. But for $m_2<<m_1$, R_{21}/R_{22} is minimal. So a scaling such that the condition number is minimal is not an optimal scaling for performance.

In this section we found out that the demand for robustness of performance is translated into a filter that equals the identity; no filtering at all. Of course we could have expected this since performance is best for the unfiltered ideal controller. When the SISO filters are chosen as the restriction on the amplitude of the control signals and the robust stability demand forces us to do, we see that robust performance is negligible worse than nominal performance for a diagonal output uncertainty of 10%. It is easily seen that sacrificing performance in one output can result in improved performance for an other output. Since the filters determine the performance, their design is critical. The minimal condition number criterium is not a favourable criterium for performance.

5.4 Conclusion

In this chapter we have seen for the case that the modelling error was assumed to be fully caused by output uncertainty, how the SISO filters of the lower triangle filter matrix with which the lower triangle matrix of the ideal controller is postmultiplied, must be chosen in order to achieve a controller that satisfies all restrictions. The less the filtering, the better the performance. However, to limit the amplitude of the control signals and to satisfy the robust stability demand, filtering is necessary. The negative consequences for performance of a specific SISO filter, which were studied for the 2x2 case only, can be limited when the process is scaled favourably.

In this chapter we will apply the insights derived so far on an industrial process; it will be a distillation column with three inputs and 2 outputs.

After the process has been scaled such that input and output signals have an amplitude 1, the QR decomposition will be applied on the pseudo-inverse process which represents the ideal controller. The entries of R that indicate whether the first restriction on the amplitude of the control signals is satisfied, will be plotted. Since the entries of R are optimal, the null space that belongs to processes with more inputs than outputs cannot be used to decrease specific entries. For the ideal controller it is determined where the first restriction is violated. SISO filters are derived that limit as much as is necessary.

After the filters have been designed that satisfy the first restriction, nominal performance is investigated. Then it is wondered whether the choice of these filters conflicts with the restrictions introduced by demanding robust stability. We found out already that the demand for stability for a model error that is fully based on diagonal output uncertainty and with an amplitude of the SISO error that is smaller than 1, is never translated into more conservative filters than the filters that satisfy restriction 1. The robust performance restriction (again based on diagonal output uncertainty only) is in conflict with the filter structure that was derived to satisfy the first and second restriction. Robust performance is checked by plotting the entries of the sensitivity matrix. The only freedom that can be left (the restriction on the amplitudes of the control signals considered as hard restrictions) is to choose the filters of more important outputs more conservatively to gain performance in less important outputs. No freedom will be left when the more important output requires more conservative filtering than the less important output.

In the next section the C6/C7 distillation column is analyzed. First we will analyze a 2x2 subsystem, next the whole 2x3 process.

6.1 C6/C7 Splitter

In figure 6.1 the C6/C7 splitter is represented graphically. The outputs are the qualities of top and bottom product. The inputs are the reflux, reboil and top pressure.

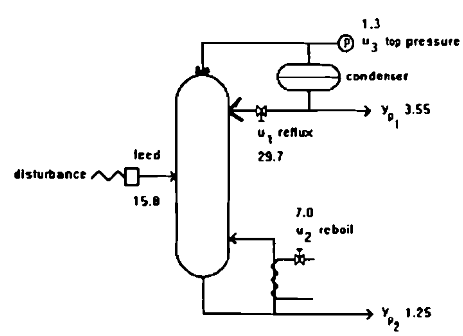


Figure 6.1 *C6/C7 splitter*

Two discrete state space models are given. The first is the process model given with A_m, B_m, C_m and D_m. It models the transfer from the three inputs to the two outputs. The state vector contains four states:

$$\begin{gathered} x(kT+T)=A_m x(kT)+B_m u(kT) \\ y(kT)=C_m x(kT)+D_m u(kT) \\ dimension(A_m)=(4,4),\ dimension(B_m)=(4,3) \\ dimension(C_m)=(2,4),\ dimension(D_m)=(2,3) \end{gathered} \tag{6.1-1}$$

The second model is the disturbance model given with A_f, B_f, C_f and D_f. It models the transfer from the feed variations to the two outputs. The following holds:

$$\begin{gathered} dimension(A_f)=(4,4),\ dimension(B_f)=(4,1) \\ dimension(C_f)=(2,4),\ dimension(D_f)=(2,1) \end{gathered} \tag{6.1-2}$$

In (6.1-3) the relation is given between the discrete state space models and the frequency dependent transfer matrix:

$$M(j\omega)=C(e^{j\omega T}I-A)^{-1}B+D \tag{6.1-3}$$

Despite the complex entries for frequencies≠0, the pseudo inverse and its QR-decomposition can still be calculated. When we are interested in the magnitude of a complex entry of R we take the length of the associated vector in the complex plane of numbers.

The scaling will be here such that the control signals and the output signals take on values between -1 and +1 in the steady state. See section 2.3

Important limitations are here that the process is assumed to be minimum phase and that the whole additive model error is caused by diagonal uncertainty at the outputs.

In the following section first the 2x2 subsystem spanned by the reflux and reboil inputs and both the outputs will be analyzed. In the subsequent section this will be done for the whole process. The reason to do this is to gain insight into the usefulness for the system as a whole of the third input, the one with which the top pressure can be manipulated. It will be certain that it improves the condition of the process. It will be interesting to find out how much more the 2x3 system is controllable than the 2x2 system is.

6.1.1 Analysis of the C6/C7 Splitter without Excitation of the Pressure Input

In this section we will analyze the process with excitation via the reflux and reboil only.

First an appropriate scaling is determined. See section 2.3. In figure 6.1 the nominal values for all inputs and outputs are mentioned. The feed is nominally 15.8. Suppose a disturbance is superposed with an amplitude of ±40% of the nominal value: 6.5. Then the disturbance model tells us that d_y=[-0.084 -0.8948]^T. Round this off: d_y=[-0.1 -1]^T. Then:

$$D_2=\begin{bmatrix}10 & 0\\ 0 & 1\end{bmatrix} \tag{6.1.1-1}$$

It follows that:

$$D_1 = \begin{bmatrix} 7.36 & 0 \\ 0 & 1.68 \end{bmatrix} \tag{6.1.1-2}$$

Compared to the nominal values of the control signals in the steady state given in figure 6.1 these values are not too large.

The process that we will analyze here for subsequent frequencies is given with:

$$M(j\omega) = D_2(C_m(e^{j\omega T}I - A_m)^{-1}B_m^* + D_m^*)D_1 \tag{6.1.1-3}$$

with B_m^* and D_m^* containing the first two columns of B_m and D_m

Restriction 1

In figure 6.2 the amplitudes of R_{11}, R_{21} and R_{22} are given as a function of frequency.

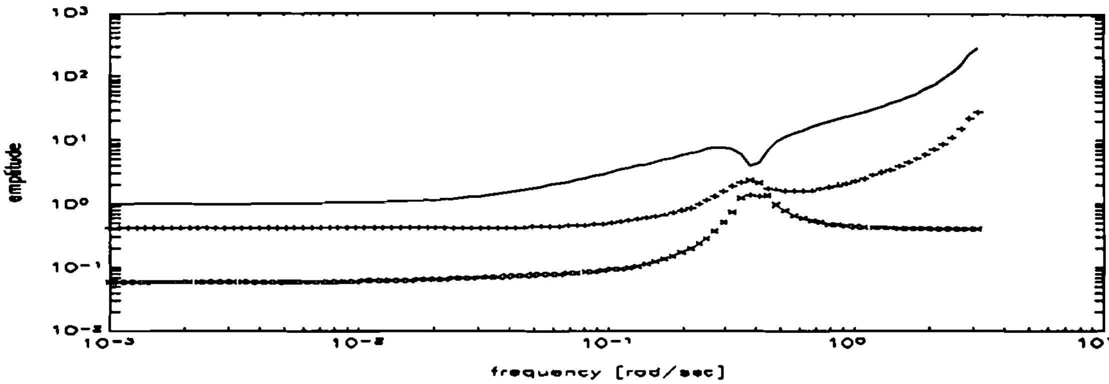


Figure 6.2 $|R_{11}|=x$, $|R_{21}|=+$ *and* $|R_{22}|=-$

Over the whole, discrete frequency spectrum from 0 to π/T (with T=1) we want to limit all amplitudes of R_{ij} in order to limit the amplitude of the control signals. As was proposed in section 5.1 all the amplitudes of the elements of R must be limited to a value 1. We use the following filters:

$$f^1 = \frac{1-p_1}{z-p_1} = \frac{1-p_1}{e^{j\omega T}-p_1}$$

$$f^2 = \frac{1-p_1}{z-p_1}\frac{1-p_2}{z-p_2} \tag{6.1.1-4}$$

$$f^3 = \frac{1-p_1}{z-p_1}\frac{1-p_2}{z-p_2}\frac{1-p_3}{z-p_3}$$

From figure 6.2 we derive that $\varpi_{11} \approx 0.32$ [rad/sec], $\varpi_{21} \approx 0.22$ and $\varpi_{22} \approx 0.01$. We apply a simplification on the filters by assuming that $p_1=p_2=p_3$. With the cutoff frequency given by $(1-p)/(1+p)$ which is for all filters the same. However, the higher the order, the steeper the slope at the cost of overshoot. Without choosing order yet, we can say that $p_{11}=0.52$, $p_{21}=0.64$ and $p_{22}=0.98$. After some iterations the filter constants for which the entries of R are limited precisely enough were derived. In the amplitudes of R_{22} and R_{21}, both related to the control of output 2, a peak appears downwards and in the amplitude of R_{11} a peak upwards around ϖ=0.4 [rad/sec]. Using equation (4.3.1-10) to derive an explanation we find out that in 0.4 [rad/sec] m_1 must be very small and the condition of the process must be less unfavourable than in the frequencies in the neighbourhood (given by the (canonical) angle between the rows of the process) m_2 could keep the same value as it had for lower frequencies. The explanation was checked and is proven to be true.

In figure 6.3 the filter and the filtered and unfiltered version of R_{22} which needs filtering most, is presented. The filter is second order and has values p_{22}=0.95 for which the cutoff frequency $\varpi_{22} \approx 0.26$ [rad/sec].

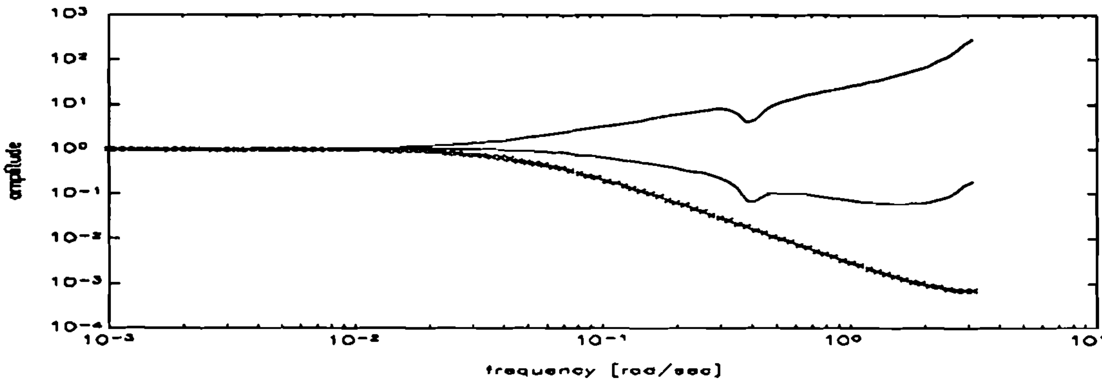


Figure 6.3 $|f_{22}R_{22}|$=--, $|R_{22}|$=-, $|f_{22}|$=x

In figure 6.4 the filtered and unfiltered version of R_{21} is represented. The filter f^*_{21} is third order with p_{21}=0.66 for which the cutoff frequency $\varpi_{21} \approx 0.20$ [rad/sec].

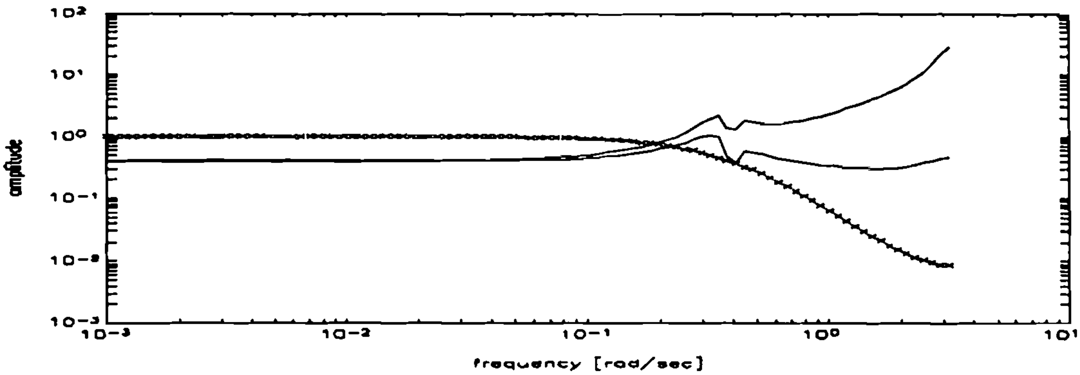


Figure 6.4 $|f^*_{21}R_{21}|$=--, $|R_{21}|$=-, $|f^*_{21}|$=x

In figure 6.5 the filtered and unfiltered version of R_{11} is represented. The filter f_{11} is third order with p_{11}=0.64 for which the cutoff frequency $\varpi_{11} \approx 0.22$ [rad/sec]. However this value surprises since the estimated value that was derived from figure 6.2 was 0.32 [rad/sec]. Probably the peak is so steep that it requires a higher order filter that is tuned more accurately.

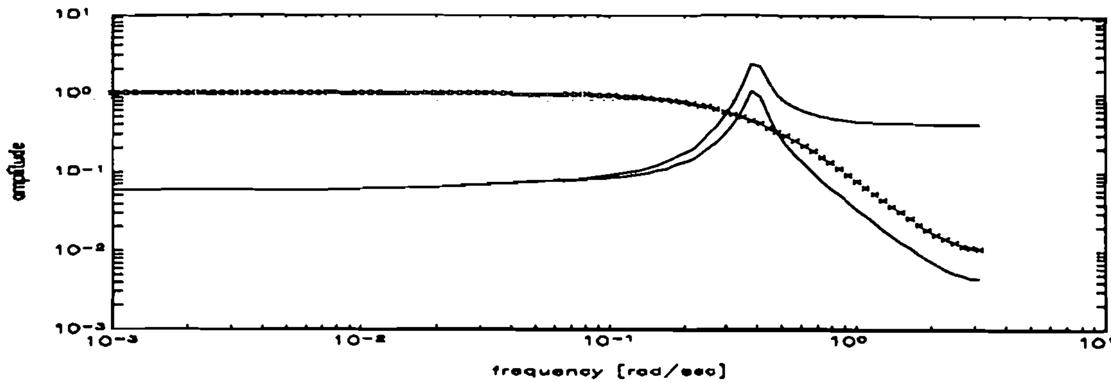


Figure 6.5 *$|f_{11}R_{11}|$=--, $|R_{11}|$=-, $|f_{11}|$=x*

We conclude already that since $\varpi_{21}\approx\varpi_{11}$ and f_{*21} and f_{11} having the same order, the influence of f_{21} on the nominal performance will be limited to a small frequency band. Let us study nominal performance by the amplitudes of the entries of the sensitivity matrix S=I-F:

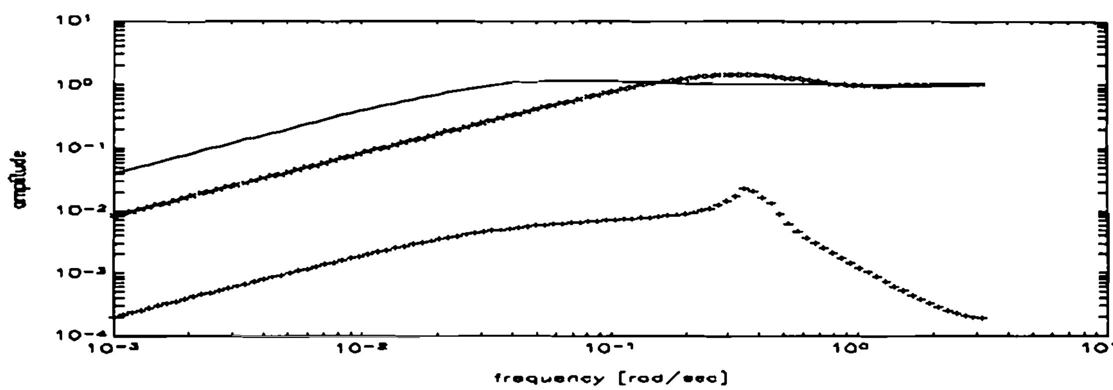


Figure 6.6 *$|1-f_{11}|$=x, $|f_{21}|$=+, $|1-f_{22}|$=-*

Compare figure 6.6 with figure 5.2a (ideal filtering assumed). As we expected, $|f_{21}|$ does not cause any problems. The part of the amplitude of f_{21} that is based on the process gains per output (m_2/m_1) will be smaller than 0.4 over the whole frequency band. The real bottle neck is $|1-f_{22}|$. Because of the low cutoff frequency ϖ_{22}, output 2 will be very sensible for disturbances that affect output 2. Over the whole frequency band output 2 will not be sensitive for disturbances at output 1 because $|f_{21}|$ remains small. Output 1 is for low frequencies not sensible for disturbances in output 1 and by definition for no frequency at all for disturbances in output 2 (caused by the lower triangle structure). For frequencies higher than 0.15, output 1 is extra sensitive for disturbances in output 1 because of the overshoot caused by the high order filter. In figure 6.3 we see that filter f_{22} could be less conservative resulting in improved performance at output 2.

Restriction 2

In figure 6.7 the eigenvalues of ΔQ are plotted for a worst case output uncertainty $\Delta_o = \begin{bmatrix} -0.1e^{-\varphi(f_{11})} & 0 \\ 0 & -0.1e^{-\varphi(f_{22})} \end{bmatrix}$. Since for frequency=π/T [rad/sec] with T=1, both eigenvalues are larger than -1 and stability is guaranteed.

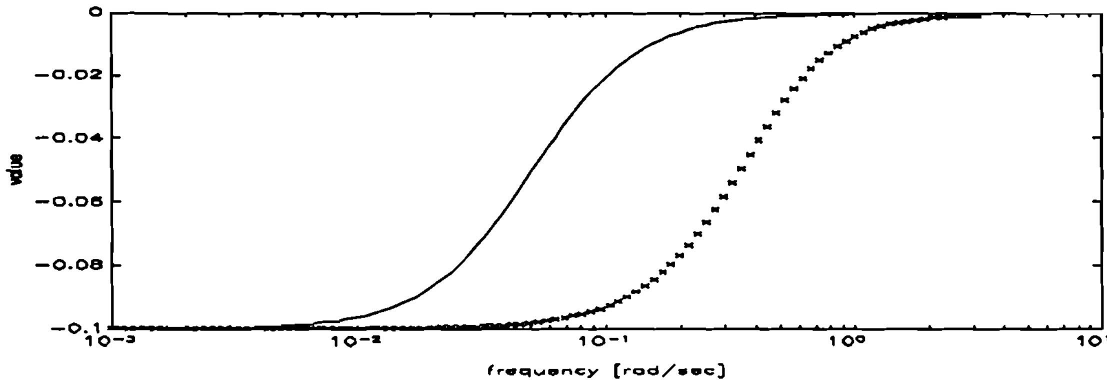


Figure 6.7 *The eigenvalues of ΔQ*

In the characteristics of λ_1 and λ_2, the LPFs f_{11} and f_{22} can be recognized (see (5.2-3)). As was concluded in the previous chapter, stability is not a problem when the SISO filters are chosen such that the restriction on the control signals is satisfied.

Restriction 3

In figure 6.8 the amplitudes of the entries 'a', 'b' and 'c' of the robust sensitivity matrix $S(I+\Delta Q)^{-1}$ are plotted for the worst case output uncertainty $\Delta_o = \begin{bmatrix} -0.1e^{-\varphi(f_{11})} & 0 \\ 0 & -0.1e^{-\varphi(f_{22})} \end{bmatrix}$.

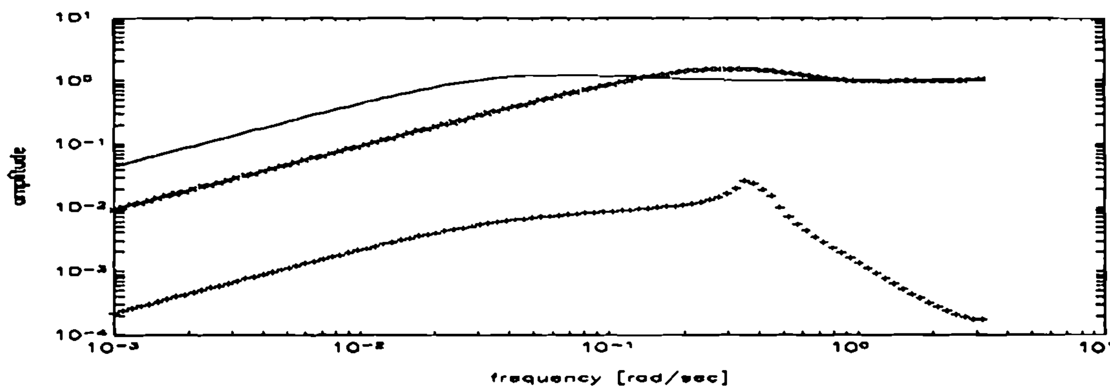


Figure 6.8 *Contents of sensitivity matrix $S(I+\Delta_o F)^{-1}$: amplitudes of 'a'=x, 'b'=- and 'c'=+*

If we compare figure 6.8 to figure 6.6 (nominal performance) then we see hardly any difference. This was predicted in section 5.3. The diagonal output uncertainty does not cause serious problems for performance. If nominal performance is acceptable then robust performance will be acceptable too. To study the difference closer, the largest singular values of the nominal sensitivity and the robust sensitivity are depicted on a semi-log-scale:

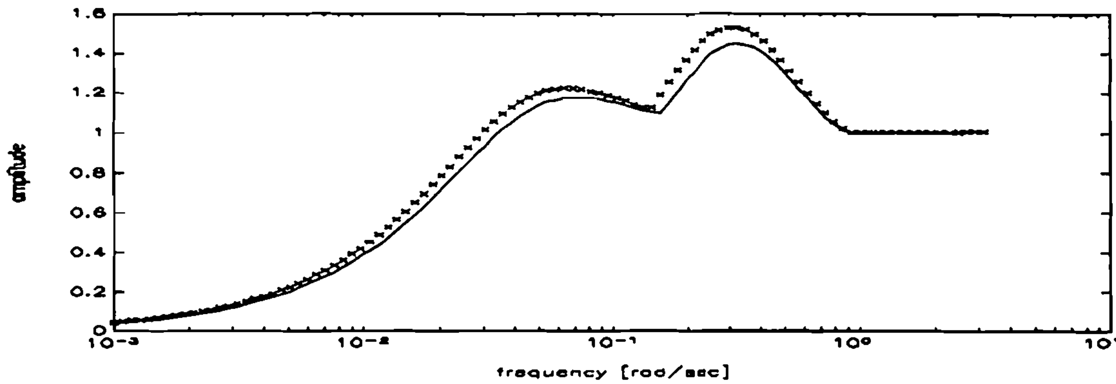


Figure 6.9 *Largest singular values of nominal sensitivity (=-) and robust sensitivity (=x)*

For ϖ=0.4 the deviation is maximal: 0.15. We conclude that robust performance is determined mainly by nominal performance.

6.1.2 Conclusion

The use of filters that was necessary to satisfy the restriction on the amplitude of the control signals has resulted in a severe deterioration of nominal performance. Robust performance has almost the same characteristics as nominal performance. Robust stability was satisfied. The bottle neck for this process is the disturbance reduction at output 2 caused by disturbances that affect output 2. Nothing can be done about this. No performance in output 1 can be sacrificed to improve.

6.1.3 Analysis of the whole C6/C7 Splitter

In this section the same analysis as in section 6.1.1 will be performed for the C6/C7 process when the pressure can be varied as well (see figure 6.1). Only small variations in the pressure input are allowed which must be reflected in the scaling. The nonsquare system introduces null space which can be 'used' to obtain a favourable scaling. Because the input weight for u_3 has to be small, the input weights for u_1 and u_2 will closely resemble the input scaling for the 2x2 process.

Nothing changes to the output scaling:

$$D_2=\begin{bmatrix}10 & 0\\ 0 & 1\end{bmatrix} \qquad (6.1.3\text{-}1)$$

Using null space we derive at the following input scaling:

$$D_1=\begin{bmatrix} 7.82 & 0 & 0 \\ 0 & 1.77 & 0 \\ 0 & 0 & 0.0377 \end{bmatrix} \tag{6.1.3-2}$$

The process that we will analyze here for subsequent frequencies is given with:

$$M(j\omega)=D_2(C_m(e^{j\omega T}I-A_m)^{-1}B_m+D_m)D_1 \tag{6.1.3-3}$$

Restriction 1

In figure 6.10 the amplitudes of R_{11}, R_{21} and R_{22} of the whole process are given as a function of frequency.

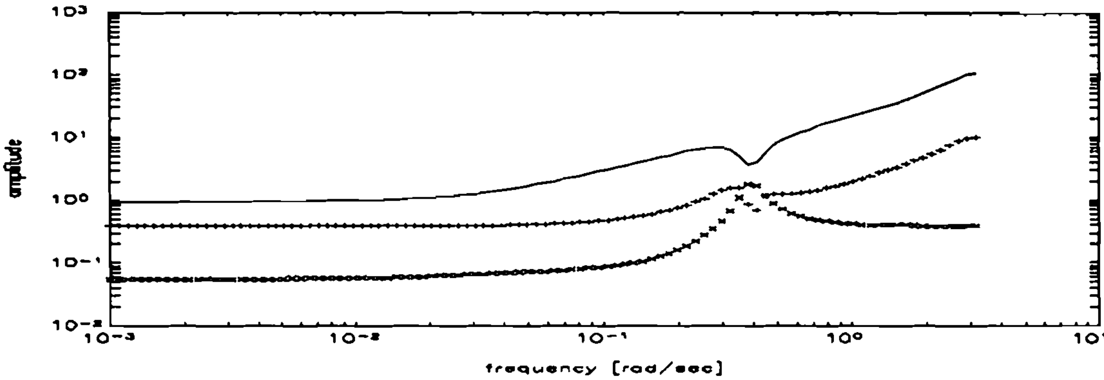


Figure 6.10 *$|R_{11}|$=x, $|R_{21}|$=+ and $|R_{22}|$=- for the whole process*

Compared to figure 6.2, figure 6.10 shows a small improvement.

For the whole process we derive the same cutoff frequencies and filter parameters as for the 2x2 case: $\varpi_{11}\approx 0.32$ [rad/sec], $\varpi_{21}\approx 0.22$ and $\varpi_{22}\approx 0.01$. Without choosing order yet, we can say that $p_{11}=0.52$, $p_{21}=0.64$ and $p_{22}=0.98$. After some iterations the filter constants for which the entries of R are limited precisely as much as is necessary were derived. The same peaks appear around $\varpi=0.4$ in the amplitudes of R_{22}, R_{21} and R_{11}. They have the same explanation as for the 2x2 case.

In figure 6.11 the filter and the filtered and unfiltered version of R_{22} which needs filtering most, is presented. The filter is first order and has a value $p_{22}=0.98$ for which the cutoff frequency $\varpi_{22}\approx 0.01$ [rad/sec].

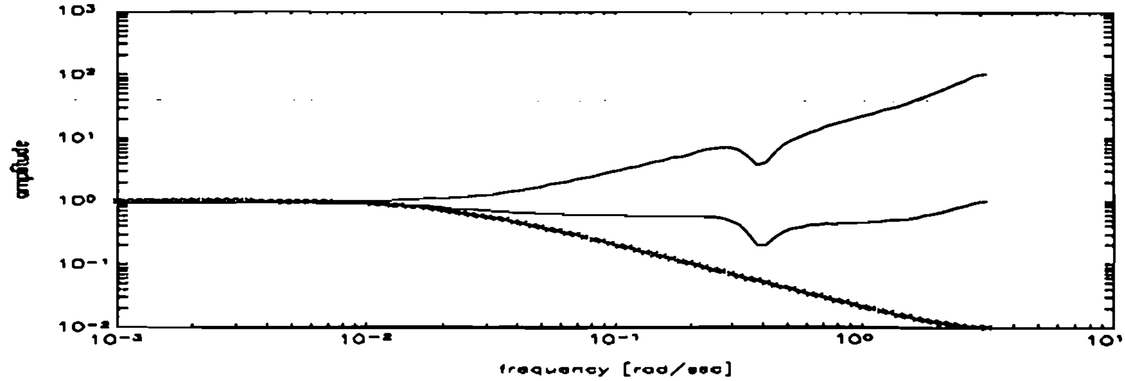


Figure 6.11 $|f_{22}R_{22}|$=--, $|R_{22}|$=-, $|f_{22}|$=x

In figure 6.12 the filtered and unfiltered version of R_{21} are represented. The filter f^*_{21} is second order with p_{21}=0.65 for which the cutoff frequency $\varpi_{21}\approx0.21$ [rad/sec].

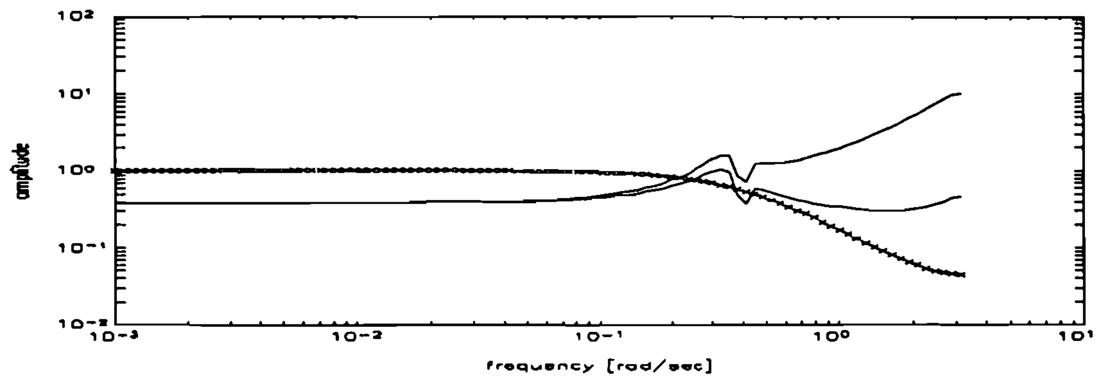


Figure 6.12 $|f^*_{21}R_{21}|$=--, $|R_{21}|$=-, $|f^*_{21}|$=x

In figure 6.13 the filtered and unfiltered version of R_{11} are represented. The filter f_{11} is third order with p_{11}=0.56 for which the cutoff frequency $\varpi_{11}\approx0.28$ [rad/sec].

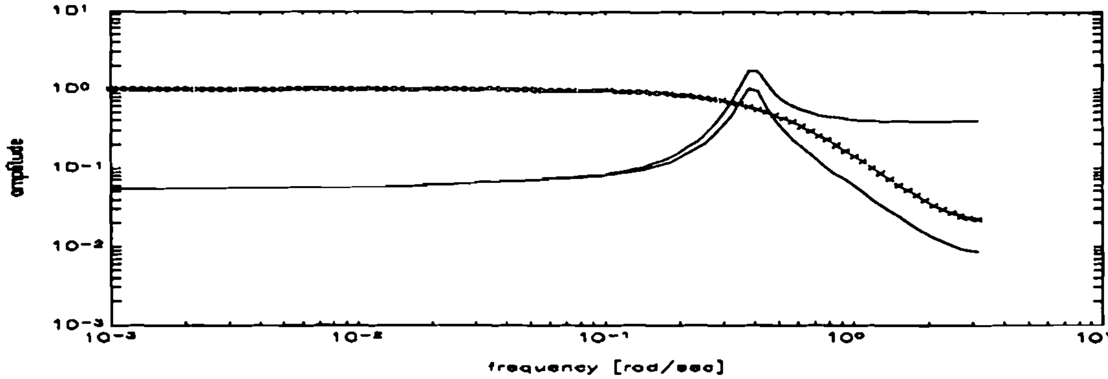


Figure 6.13 $|f_{11}R_{11}|$=--, $|R_{11}|$=-, $|f_{11}|$=x

Compared to the 2x2 filtering we arrive for the 2x3 case at a less conservative filtering because of the lower filter orders for f_{22} and f^*_{21}. Now the difference in cutoff frequencies between ϖ_{21} and ϖ_{11} is larger together with a difference in filter order. The influence of f_{21} on the nominal performance will be sensed over a broader frequency band but still, the part of the amplitude of f_{21} that is based on the process gains per output (m_2/m_1) will be smaller than 0.4. Remind: f_{21} is the filter that is needed to filter R_{21} (the effort to suppress the influence of y_{s1} on y_{p2}) precisely as much as is necessary. This filter comes through as a negative influence on performance of which the amplitude can be very unfavourable depending on the process gains per output. Let us study nominal performance by the amplitudes of the entries of the sensitivity matrix S=I-F:

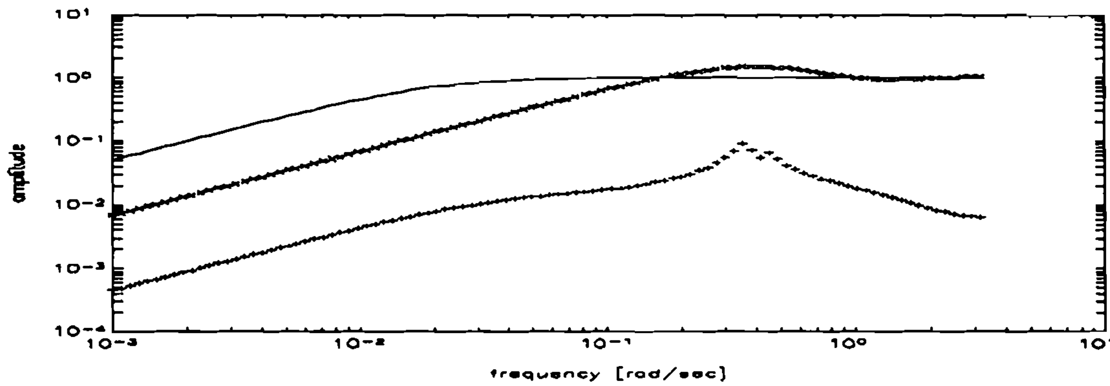


Figure 6.14 *Nominal Performance:* $|1\text{-}f_{11}|$=x, $|f_{21}|$=+, $|1\text{-}f_{22}|$=-

Compare figure 6.14 to figure 6.6. As we expected, $|f_{21}|$ is sensed over a wider range. Its contribution however is still negligible. $|1\text{-}f_{11}|$ is a little bit better than for the 2x2 case. This was expected because of the higher cutoff frequency for the 2x3 case. Again, the bottle neck is $|1\text{-}f_{22}|$. Compared to figure 6.6, we can derive that a high price is paid to get rid of the overshoot of a higher order filter since f_{22} for the 2x3 case is first order. Because of the low cutoff frequency ϖ_{22}, output 2 will be very sensible for disturbances that affect output 2. The same comments that were made on figure 6.6 hold for figure 6.14. We should wonder here whether a second or third order filter for f_{22} with higher cutoff frequency would not have been better. The result here is that the nominal performance is even worse than the 2x2 case which must be due to a failure in filter design.

Restriction 2
In figure 6.15 the eigenvalues of ΔQ are plotted for a worst case output uncertainty $\Delta_o = \begin{bmatrix} -0.1e^{-\varphi(f_{11})} & 0 \\ 0 & -0.1e^{-\varphi(f_{22})} \end{bmatrix}$. Since for frequency=$\pi/T$ [rad/sec] with T=1 both eigenvalues are larger than -1 and therefore stability is guaranteed.

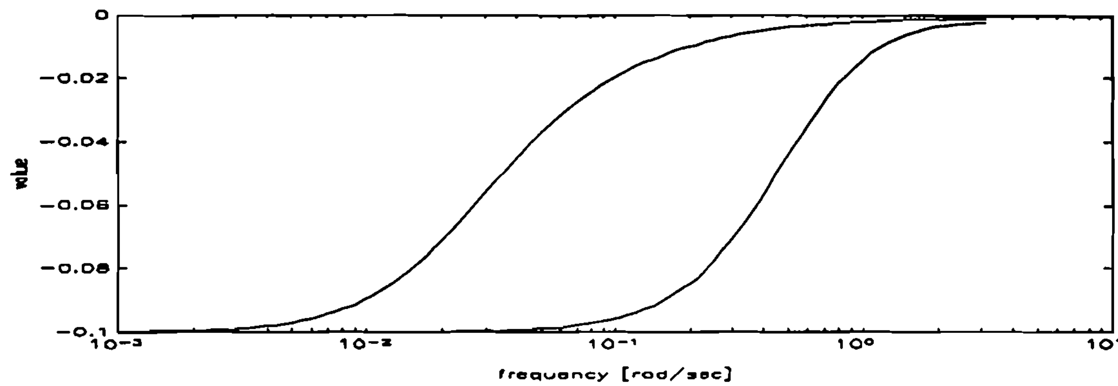


Figure 6.15 The eigenvalues of ΔQ

Again the LPF f_{11} and f_{22} can be recognized (see (5.2-3)). Stability is not a problem when the SISO filters are chosen such that the restriction on the control signals is satisfied.

Restriction 3
In figure 6.16 the amplitudes of the entries 'a', 'b' and 'c' of the robust sensitivity matrix $S(I+\Delta Q)^{-1}$ are plotted for the worst case output uncertainty $\Delta_o = \begin{bmatrix} -0.1e^{-\varphi(f_{11})} & 0 \\ 0 & -0.1e^{-\varphi(f_{22})} \end{bmatrix}$.

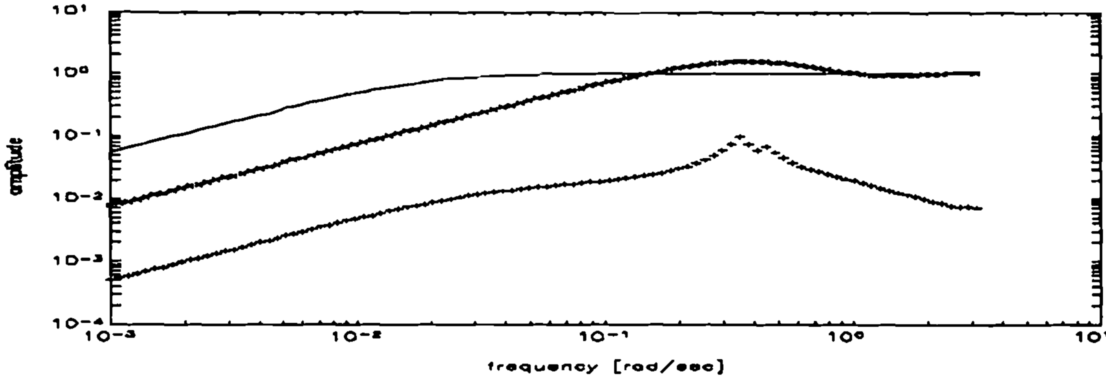


Figure 6.16 *Contents of sensitivity matrix $S(I+\Delta_o F)^{-1}$: amplitudes of 'a'=x, 'b'=- and 'c'=+*

Again there is hardly any difference between nominal and robust performance. To study the difference closer, the largest singular values of the nominal sensitivity and the robust sensitivity are depicted on a semi-log-scale:

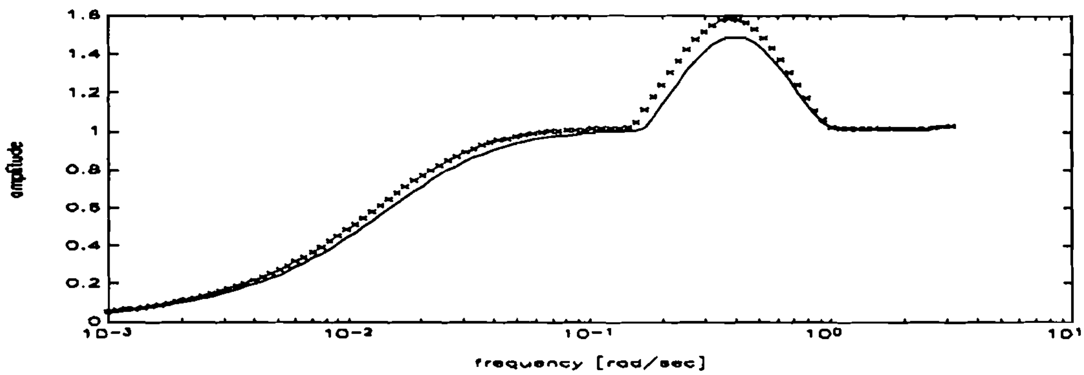


Figure 6.17 *Largest singular values of nominal sensitivity (=-) and robust sensitivity (=x)*

Now the maximal deviation is 0.2. Again we conclude that robust performance is determined mainly by nominal performance. The 'bulb' is caused by the third order filter f_{11}.

As a final check for the 2x3 process let us look at the amplitudes of the SISO transfers between controller input 1 and 2 (y_{s1}, y_{s2}) respectively and the third controller output (u_3, pressure input of the process) to determine whether the scaling has been such that $|u_3| \leq 1$:

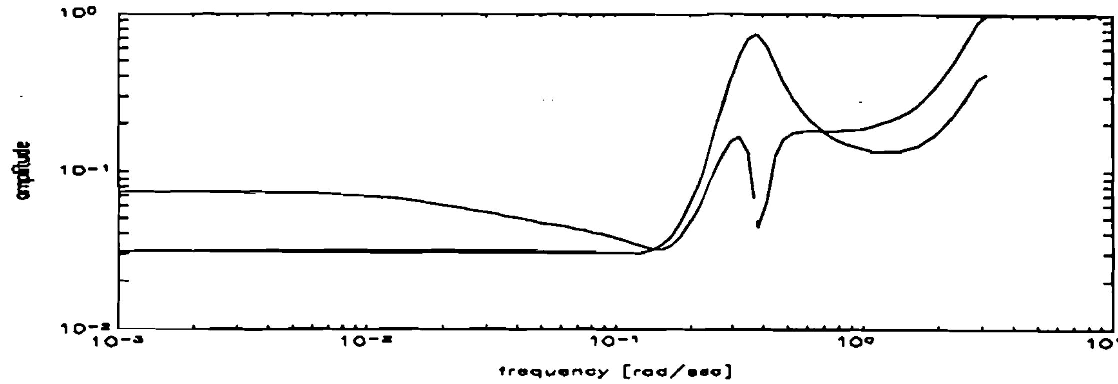


Figure 6.18 *Transfers to the pressure input*
y_{s1} *to* u_3 = -; y_{s2} *to* u_3 = --

Both amplitudes remain smaller than 1 so the third process input will be handled with care. The characteristics look familiar. It is easily derived from the SISO transfers in the last row of $Q=Q_bR_F$ (5-1) that the first transfer is a weighted sum of $R_{11}f_{11}$ and $R_{21}f^*_{21}$, and that the second transfer is given with $R_{22}f_{22}$. All these transfers are depicted in figures 6.11 to 6.13.

6.1.4 Conclusion

Despite the fact that it was limited in its use, it is doubtful whether the third input has resulted in a process that is better controllable. The doubt is caused by the filter design. The filters were very roughly tuned here. The addition of the third input to the process should have resulted in a better controllable process. Since this was not the outcome of the analysis the filters should be redesigned.

6.2 Conclusion

In chapter 5 the filter mechanism was given that is applied on the ideal controller in order to make it realizable in such a way that it is clear how the restrictions come through per output. In this chapter the filters were designed for two processes in order to satisfy the restrictions and after this had been done it was wondered what was left of performance per output. If the performance is bad for a specific output and the restrictions are hard then the conclusion can be drawn that the process is not controllable in that output. But, conclusions on the controllability of a process should be based on process behaviour only and never on poorly tuned filters as was done in this chapter. If we would have a perfect filter toolbox at our disposal then we can remain as close as possible to the process behaviour in our analysis. Perfect filters need not be designed because they exist already as a concept.

It is concluded here that an efficient way of analyzing the controllability of a process is, without taking characteristics into consideration that are not related to process behaviour, to base it on perfect filters.

7 Conclusions

In this report a controllability analysis method was derived which cannot be applied on processes with nonminimum phase behaviour (RHP zeros) and on process models for which the uncertainty cannot be modelled as a diagonal at the outputs. The generalization of the method such that it can be applied on these processes as well, could be subject of future research.

The analysis method is completely performed in the frequency domain where it is based on the violations of restrictions. The assumption is made that if the behaviour of a magnitude satisfies boundaries in the frequency domain, the related magnitude in the time domain will also be limited. Despite the fact that counterexamples can be given, it is the rule of thumb on which this method is based.

First the analysis method is described, then the subjects for future research to improve and generalize the method will be treated.

*scale the process such that all in- and output signals take on values between -1 and +1
*perform QR decompositions for subsequent frequency points from 0 to π/T on the pseudo inverse process or the ideal controller such that $Q_{ideal}=Q_bR$ with R a lower triangle matrix and Q_b an orthonormal base
*plot the amplitudes of all entries of R
*look at the plots and determine for each entry of R (R_{ji}) at what frequency it exceeds the value 1 which will be the cutoff frequency to satisfy both the restrictions on the amplitude of the control signals and the robust stability demand.
*determine, with the filter mechanism that was treated in section 5.1, based on the LPFs that were derived in the previous step all SISO filters $f_{i+j,i}$ that fit into the lower triangle matrix F. The low pass filtering of the entries of R is established by the multiplication R*F.
*to study how badly the filtering affected nominal performance, plot all the amplitudes of the entries of S=I-F. For frequencies where the amplitude of S(i+j,i) is not equal to 0, output i+j will be sensitive for disturbances that affect output i. How sensitive the output is, is determined by the amplitude of S(i+j,i) at that specific frequency which is determined by the filter weights.
*to study the consequences for robust performance the actions will be the same as in the previous step for the matrix $S(I+\Delta F)^{-1}$

Subjects for future research:
-derivation of general expressions for the entries of the robust performance matrix $S(I+\Delta F)^{-1}$
-derivation of an upper bound for the sum of weights of filter $f_{i+j,i}$

List of Symbols

P	transfer matrix of the process
M	transfer matrix of the process model
M_i	row number i of M
m_i	length of row vector M_i
e_i	unit vector representing direction of M_i
M^+	right/pseudo-inverse of M such that $MM^+=I$
u	vector of control signals
y_s	vector of setpoint inputs to the controlled system
y_p	vector of process outputs
d	vector of disturbances at y_p
Δ	additive model uncertainty of M
Δ_I	diagonal input uncertainty
Δ_O	diagonal output uncertainty
σ_i	singular value: 'gain' sensed in a specific input direction to a specific output direction
Q_bR	QR-decomposition of a matrix Q_b: orthonormal base, R: lower triangle matrix

References

[1] Morari, M. (1983). *Design of Resilient Processing Plants-III. A General Framework for the Assessment of Dynamic Resilience.* Chemical Engineering Science, Vol. 38, No. 11, pp. 1881-1891.

[2] Holt, B.R. and M. Morari (1985). *Design of Resilient Processing Plants-VI. The Effect of Right-Half-Plane Zeros on Dynamic Resilience.* Chemical Engineering Science, Vol. 40, No. 1, pp. 59-74.

[3] Per Ake Wedin (1987?). *On Angles between Subspaces of a Finite Dimensional Inner Product Space.* Source unknown.

[4] Morari, M. and E. Zafiriou (1989). *Robust Process Control.* Prentice-Hall, Inc.

[5] Golub, G.H. and C.F. van Loan (1989). *Matrix Computations.* The Johns Hopkins University Press.

Appendix 1

If we replace M+Δ by P in figure 3.5A then:

$$
\begin{gathered}
e = d \pm (P-M)Q(y_s - e) \Rightarrow \\
e = d + \Delta Q y_s - \Delta Q e \Rightarrow \\
e = (I + \Delta Q)^{-1}(d + \Delta Q y_s)
\end{gathered}
\tag{1}
$$

The last expression we can substitute into:

$$
\begin{gathered}
y_p = d + PQ(y_s - e) \Rightarrow \\
y_p = d + (M+\Delta)Qy_s - (M+\Delta)Q(I+\Delta Q)^{-1}(d + \Delta Q y_s) \Rightarrow \\
y_p = MQy_s + \Delta Q y_s - (M+\Delta)Q(I+\Delta Q)^{-1}(d + \Delta Q y_s) + d
\end{gathered}
\tag{2}
$$

With T=MQ and S=I-MQ:

$$
y_p = Ty_s + Sd + S\Delta Q(I+\Delta Q)^{-1}(y_s - d) \tag{3}
$$

The first two terms in the right half of (3) give the nominal behaviour (M=P) of y_p. With:

$$
\Delta Q(I+\Delta Q)^{-1} = I - (I+\Delta Q)^{-1} \tag{4}
$$

(3) can be rewritten:

$$
\begin{gathered}
y_p = Ty_s + Sd + S(I - (I+\Delta Q)^{-1})(y_s - d) \Rightarrow \\
y_p = (T+S)y_s + Sd - Sd - S(I+\Delta Q)^{-1}(y_s - d) \Rightarrow \\
y_s - y_p = S(I+\Delta Q)^{-1}(y_s - d)
\end{gathered}
\tag{5}
$$

Appendix 2

In (4.3.1-9) we derived:

$$
Q = \left[m_1^{-1}(e_1^T - \cot\varphi_{12} e_{1_\perp}^T) \quad m_2^{-1}\sin^{-1}\varphi_{12} e_{1_\perp}^T \right] \tag{1}
$$

The singular values equal the square roots of the eigenvalues of:

$$
Q^T Q = \frac{1}{\sin^2\varphi_{12}} \begin{bmatrix} \frac{1}{m_1^2} & -\frac{1}{m_1 m_2}\cos\varphi_{12} \\ -\frac{1}{m_1 m_2}\cos\varphi_{12} & \frac{1}{m_2^2} \end{bmatrix} \tag{2}
$$

which can be calculated by the following expression:

$$
\begin{gathered}
\det(Q^T Q - \lambda I) = 0 \\
\textit{with } \gamma = \sin^2\varphi_{12}: \\
\det(Q^T Q - \lambda I) = \left(\frac{1}{m_1^2} - \gamma\lambda\right)\left(\frac{1}{m_2^2} - \gamma\lambda\right) - \frac{1}{(m_1 m_2)^2}\cos^2\varphi_{12} = \\
(\gamma\lambda)^2 - \left(\frac{1}{m_1^2} + \frac{1}{m_2^2}\right)\gamma\lambda + \frac{1}{(m_1 m_2)^2}\sin^2\varphi_{12} = 0 \\
\textit{solutions}: \\
\gamma\lambda_{1,2} = \frac{1}{2m_1^2}\left(1 + \left(\frac{m_1}{m_2}\right)^2 \pm \sqrt{\left(1 + \left(\frac{m_1}{m_2}\right)^2\right)^2 - 4\left(\frac{m_1}{m_2}\right)^2 \sin^2\varphi_{12}}\right.
\end{gathered}
\tag{3}
$$

$$with\ \alpha = \frac{m_1}{m_2};$$

$$\sigma_1 = \sqrt{\lambda_1} = \frac{1}{\sqrt{2}}\frac{1}{m_1 \sin\varphi_{12}}\sqrt{1+\alpha^2}\sqrt{1+\sqrt{1-4(\frac{\alpha}{1+\alpha^2})^2 \sin^2\varphi_{12}}} \tag{4}$$

$$\sigma_2 = \sqrt{\lambda_2} = \frac{1}{\sqrt{2}}\frac{1}{m_1 \sin\varphi_{12}}\sqrt{1+\alpha^2}\sqrt{1-\sqrt{1-4(\frac{\alpha}{1+\alpha^2})^2 \sin^2\varphi_{12}}}$$

such that:

$$cond(Q) = \frac{\sqrt{1+\sqrt{1-4(\frac{\alpha}{1+\alpha^2})^2 \sin^2\varphi_{12}}}}{\sqrt{1-\sqrt{1-4(\frac{\alpha}{1+\alpha^2})^2 \sin^2\varphi_{12})}}} \tag{5}$$

Let us try to simplify this expression:

$$cond^2(Q) = \frac{1+\sqrt{1-x}}{1-\sqrt{1-x}} = \frac{1}{x}(1+2\sqrt{1-x}+1-x) = -1+\frac{2}{x}(1+\sqrt{1-x})$$

$$with\ \sqrt{1-x} = \sum_{k=0}^{\infty}(-1)^k \begin{pmatrix} \frac{1}{2} \\ k \end{pmatrix} x^k : \tag{6}$$

$$cond^2(Q) = \frac{4}{x} - 2\ ;\ \ with\ x = 4(\frac{\alpha}{1+\alpha^2})^2 \sin^2\varphi_{12} :$$

$$cond^2(Q) = \frac{1}{\alpha^2 \sin^2\varphi_{12}}(1+2\alpha^2\cos^2\varphi_{12}+\alpha^4)$$

this leads to:

$$cond(Q) = \frac{1}{|\sin\varphi_{12}|}\sqrt{\frac{1}{\alpha^2}+\alpha^2+2\cos^2\varphi_{12}}\quad or:$$

$$cond(Q) = \frac{1}{|\sin\varphi_{12}|}\sqrt{(\frac{m_2}{m_1})^2+(\frac{m_1}{m_2})^2+2\cos^2\varphi_{12}} \tag{7}$$

Appendix 3

The ideal mxn controller which is the pseudo inverse of the mxn process will be derived in four steps. The product MQ will gradually assimilate the form of the identity matrix. The practical usefulness of this derivation is the further decomposition of the lower triangle matrix R in $Q_{ideal}=Q_bR$ (see section 4.3.2).

Step 1

The gain is extracted per output. Matrix E contains the direction behaviour.

$$M = \begin{bmatrix} M_1 \\ M_2 \\ . \\ . \\ M_m \end{bmatrix} = \begin{bmatrix} m_1 e_1 \\ m_2 e_2 \\ . \\ . \\ m_m e_m \end{bmatrix} = \begin{bmatrix} m_1 & 0 & . & . & 0 \\ 0 & m_2 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & m_m \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ . \\ . \\ e_m \end{bmatrix} = M_g E \tag{1.1}$$

$$m_i = \|M_i\|_2 \wedge \|e_i\| = 1$$

Step 2
The absolute ranking of the outputs must be guaranteed. This is done by an orthonormal base Q_b for which it holds that the upper triangle of the product MQ_b without the diagonal is filled with zeros:

$$MQ_b = \begin{bmatrix} * & 0 & . & . & 0 \\ * & * & 0 & . & 0 \\ . & . & . & . & . \\ * & . & . & * & 0 \\ * & . & . & . & * \end{bmatrix} \tag{2.1}$$

Q_b is an orthonormal base. This means that $Q_bQ_b^T$=I. Multiplying the left and right side of (2.1) with Q_b^T gives:

$$MQ_bQ_b^T = \begin{bmatrix} * & 0 & . & . & 0 \\ * & * & 0 & . & 0 \\ . & . & . & . & . \\ * & . & . & * & 0 \\ * & . & . & . & * \end{bmatrix} Q_b^T \Leftrightarrow M^T = Q_b \begin{bmatrix} * & * & . & . & * \\ 0 & * & . & . & . \\ . & . & . & . & . \\ . & . & . & * & . \\ 0 & . & . & 0 & * \end{bmatrix} \tag{2.2}$$

(2.2) Suggests that an easy way to construct Q_b is by applying a QR decomposition on M [5]. To deepen our insight, Q_b is studied here.

All lower triangle entries of EQ_b including the diagonal elements are smaller than 1 and larger than -1. They are the inner products of vectors that have the same size (=1) and form angles of 0 to 180 degrees. Complete freedom exists to choose $q_{b_1}^T$ such that:

$$EQ_b = \begin{bmatrix} e_1 \\ e_2 \\ . \\ . \\ e_m \end{bmatrix} \begin{bmatrix} q_{b_1}^T & q_{b_2}^T & . & . & q_{b_m}^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & . & . & 0 \\ * & * & 0 & . & 0 \\ . & . & . & . & . \\ * & . & . & * & 0 \\ * & . & . & . & * \end{bmatrix} \tag{2.3}$$

Or $q_{b_1}^T = e_1^T$. Because $e_1 q_{b_2}^T = 0$, $q_{b_2}^T$ is perpendicular to e_1^T . Because $e_1 q_{b_3}^T = e_2 q_{b_3}^T = 0$, $q_{b_3}^T$ is perpendicular to e_1^T and e_2^T . Etc. Hence the following must hold:

$$\begin{aligned} &q_{b_2}^T \in \langle e_1^T \rangle^\perp \\ &q_{b_3}^T \in \langle e_1^T, e_2^T \rangle^\perp \\ &\quad .. \\ &q_{b_j}^T \in \langle e_1^T, e_2^T, \ldots, e_{j-1}^T \rangle^\perp; \quad for\ j \geq 4 \end{aligned} \tag{2.4}$$

Apart from the conditions mentioned in (2.4) we have to meet the demand that Q_b is an orthonormal base:

$$\begin{aligned} &q_{b_i} q_{b_i}^T = 1 \\ &q_{b_i} q_{b_j}^T = 0; \quad for\ i \neq j \end{aligned} \tag{2.5}$$

To make EQ_b look as much as possible like I, the $q_{b_j}^T$'s should be such that the diagonal elements of EQ_b are as close to 1 as possible. $e_j q_{b_j}^T$ is maximal when the angle between e_j and $q_{b_j}^T$ is as small as possible. When this angle is as small as possible then the angle between e_j and the subspace spanned by the first j-1 rows of E must also be as small as possible since the sum of both angles is 90^0. The second angle is the canonical angle [3]. This is realized when $e_j q_{b_j}^T$ is chosen as the perpendicular projection of e_j on the subspace $\langle e_1^T,..,e_{j-1}^T\rangle^\perp$ which was the subspace in which $e_j q_{b_j}^T$ could be realized. The following figure clarifies:

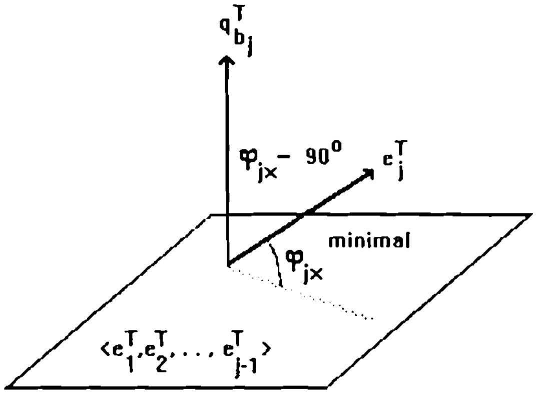


Figure 3.1 *Visualization of the canonical angle between vector e_j and a subspace, the subspace being represented by a plane*

When $\|$ denotes perpendicular projection and it is understood that $\langle e_1^T,..,e_j^T\rangle^\perp = \langle q_{b_1}^T,..,q_{b_j}^T\rangle^\perp$ then the best choices for the $q_{b_j}^T$'s are:

$$
\begin{gathered}
q_{b_1}^T = e_1^T \\
q_{b_2}^T = \alpha_2 e_2^T \| \langle e_1^T \rangle^\perp = \alpha_2 e_2 \| \langle q_{b_1}^T \rangle^\perp \\
q_{b_3}^T = \alpha_3 e_3^T \| \langle e_1^T, e_2^T \rangle^\perp = \alpha_3 e_3^T \| \langle q_{b_1}^T, q_{b_2}^T \rangle^\perp \\
\cdot\cdot \\
q_{b_j}^T = \alpha_j e_j \| \langle e_1^T, e_2^T, \ldots, e_{j-1}^T \rangle^\perp = \alpha_j e_j^T \| \langle q_{b_1}^T, q_{b_2}^T, \ldots, q_{b_{j-1}}^T \rangle^\perp; \quad \textit{for } j \geq 4
\end{gathered}
\tag{2.6}
$$

Now that the background of Q_b is better understood, a QR-decomposition will be used to calculate it.

Step 3
Now the controller Q will be derived based on Q_b such that:

$$
MQ=\begin{bmatrix} 1 & 0 & . & . & 0 \\ * & 1 & 0 & . & 0 \\ . & . & . & . & . \\ * & . & * & 1 & 0 \\ * & . & . & * & 1 \end{bmatrix} \tag{3.1}
$$

First the product EQ_b has to be adapted to a matrix which contains ones only at the diagonal. This is done by multiplication of each diagonal element with its inverse:

$$
EQ_bQ_l=\begin{bmatrix} e_1 \\ e_2 \\ . \\ . \\ e_m \end{bmatrix}\begin{bmatrix} q_{b_1}^T, q_{b_2}^T, \ldots q_{b_m}^T \end{bmatrix}\begin{bmatrix} (e_1q_{b_1}^T)^{-1} & 0 & . . & 0 \\ 0 & (e_2q_{b_2}^T)^{-1} & . . & 0 \\ . & . & . . & . \\ . & . & . . & . \\ 0 & 0 & . . & (e_mq_{b_m}^T)^{-1} \end{bmatrix}=\begin{bmatrix} 1 & 0 & . & . & 0 \\ * & 1 & 0 & . & 0 \\ . & . & . & . & . \\ * & . & * & 1 & 0 \\ * & . & . & * & 1 \end{bmatrix} \tag{3.2}
$$

In words: the diagonal elements of Q_l are correction factors for the perpendicular components in the vectors in e_i^T and $q_{b_i}^T$. The elements of Q_l are inner products which can be expressed in canonical angles:

$$
\begin{gathered}
e_jq_{b_j}^T=|e_j||q_{b_j}|\cos(\angle(e_j,q_{b_j}))=\cos(90^0-\angle(e_j,\langle e_1,e_2,\ldots,e_{j-1}\rangle))= \\
\sin(\angle(e_j,\langle e_1,e_2,\ldots,e_{j-1}\rangle))=\sin\varphi_{jx} \\
\varphi_{jx} \textit{ is a canonical angle}
\end{gathered} \tag{3.3}
$$

Q_l can be written differently:

$$
Q_l=\begin{bmatrix} (e_1q_{b_1}^T)^{-1} & 0 & . . & 0 \\ 0 & (e_2q_{b_2}^T)^{-1} & . . & 0 \\ . & . & . . & . \\ . & . & . . & . \\ 0 & 0 & . . & (e_mq_{b_m}^T)^{-1} \end{bmatrix}=\begin{bmatrix} \sin^{-1}\varphi_{1x} & 0 & . . & 0 \\ 0 & \sin^{-1}\varphi_{2x} & . . & 0 \\ . & . & . . & . \\ . & . & . . & . \\ 0 & 0 & . . & \sin^{-1}\varphi_{mx} \end{bmatrix} \tag{3.4}
$$

Q_l expresses the difficulty in building up the orthonormal base Q_b from the rows of E. Since the product EQ_b is a lower diagonal matrix, Q_l represents the costs of the control according to the ranking; it is guaranteed that higher indexed outputs will not affect lower indexed outputs e.g. y_{s_m} will not affect y_{p_1} . From Q_l (3.4) we derive that the cost of achieving this becomes a problem for small canonical angles. Then $1/\sin(\varphi_{ix})$ becomes large. The smaller the canonical angle between a row of E and its lower indexed rows, the larger the dependence in the subspace that is spanned by the combination of rows. This situation can occur even when the 'normal' angles between pairs of rows are not small.
To adapt MQ_bQ_l to (4.1), it is post-multiplied with the inverse matrix of M_g which will be called Q_g.

$$MQ = M_g E Q_b Q_f Q_g = \begin{bmatrix} m_1 & 0 & . & . & 0 \\ 0 & m_2 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & m_m \end{bmatrix} \begin{bmatrix} 1 & 0 & . & . & 0 \\ * & 1 & 0 & . & 0 \\ . & . & . & . & . \\ * & . & * & 1 & 0 \\ * & . & . & * & 1 \end{bmatrix} \begin{bmatrix} m_1^{-1} & 0 & . & . & 0 \\ 0 & m_2^{-1} & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & m_m^{-1} \end{bmatrix} \tag{3.6}$$

$$= \begin{bmatrix} 1 & 0 & . & . & 0 \\ * & 1 & 0 & . & 0 \\ . & . & . & . & . \\ * & . & * & 1 & 0 \\ * & . & . & * & 1 \end{bmatrix}$$

Step 4A Additively

As a final step in the evolution of the product MQ to I, the lower triangle interaction of the product is removed. In other words: change Q is such a way that the lower triangle excluding the diagonal is filled with zeros. The interaction is removed per element. This can be done in two ways. In step 4B the second way will be treated.

Let us first study the removal of lower triangle interaction-element MQ(i,j) of the product MQ in (3.6). The interaction amounts: $m_2 e_2 q_{b_1}^T (e_1 q_{b_1}^T)^{-1} m_1^{-1}$ Its removal is best established by adding a vector in the direction of $q_{b_2}^T$ to the specific column of the controller. Of all vectors from the base Q_b the inner product of $q_{b_2}^T$ and e_2 is largest and therefore requires the lowest amplification possible. In general, the "cheapest" way to remove interaction element MQ(i,j) is to add $\alpha_{ij} q_{b_i}^T$ to the j-th column of Q.

α_{ij}'s exist such that the following Q achieves perfect control of M:

$$Q = Q_b Q_f Q_g + Q_b A \text{ with}$$

$$A = \begin{bmatrix} 0 & . & . & . & 0 \\ \alpha_{21} & . & . & . & . \\ \alpha_{31} & \alpha_{32} & . & . & . \\ . & . & . & . & . \\ \alpha_{m1} & . & . & \alpha_{m,m-1} & 0 \end{bmatrix} \tag{4.1}$$

The advantage of this description is that α_{ij} makes MQ(i,j)=0. Now (4.1) is written differently:

$$Q = Q_b Q_l Q_g + Q_b A = Q_b (Q_l Q_g + A) =$$

$$Q_b Q_l (I + Q_l^{-1} A Q_g^{-1}) Q_g = Q_b Q_l Q_\alpha Q_g$$

with

$$Q_\alpha = \begin{bmatrix} 1 & 0 & . & . & 0 \\ e_2 q_{b_2}^T \alpha_{21} m_1 & . & . & . & . \\ e_3 q_{b_3}^T \alpha_{31} m_1 & e_3 q_{b_3}^T \alpha_{32} m_2 & . & . & . \\ . & . & . & . & 0 \\ e_m q_{b_m}^T \alpha_{m1} m_1 & e_m q_{b_m}^T \alpha_{m2} m_2 & . & e_m q_{b_m}^T \alpha_{m,m-1} m_{m-1} & 1 \end{bmatrix} \tag{4.2}$$

By induction the elements α_{ij} can be calculated and substituted in Q_α:

$$Q_\alpha = Q_{LI} Q_I = \begin{bmatrix} e_1 q_{b_1}^T & 0 & . & . & . & 0 \\ -e_2 q_{b_1}^T & e_2 q_{b_2}^T & . & . & . & . \\ -e_3 P(2,2) q_{b_1}^T & -e_3 q_{b_2}^T & . & . & . & . \\ . & -e_4 P(3,3) q_{b_2}^T & . & e_{m-2} q_{b_{m-2}}^T & . & . \\ . & . & . & -e_{m-1} q_{b_{m-2}}^T & e_{m-1} q_{b_{m-1}}^T & 0 \\ -e_m P(2,m-1) q_{b_1}^T & -e_m P(3,m-1) q_{b_2}^T & . & -e_m P(m-1,m-1) q_{b_{m-2}}^T & -e_m q_{b_{m-1}}^T & e_m q_{b_m}^T \end{bmatrix} Q_I \tag{4.3}$$

with

$$P(i,j) = \prod_{k=i}^{j} P(k) = \prod_{k=i}^{j} (I - q_{b_k}^T (e_k q_{b_k}^T)^{-1} e_k)$$

In Q_{LI} "oblique projections" are introduced. $P(k)x^T$ is the projection of vector x^T along $q_{b_k}^T$ on the subspace $e_{k_\perp}^T$. Q_{LI} can best be observed per column. Each entry must be seen as the inner product of a row of E and a sequence of oblique projections of a column of Q_b. $Q_{LI}(i+1,j)$ follows from $Q_{LI}(i,j)$ by taking the inner product of row e_{i+1} with the oblique projection P(i) of the projected part of $Q_{LI}(i,j)$. Element $Q_\alpha(i+j,i)$ (with i and j>1) indicates the effort to suppress the influence of y_{s_i} on y_{p_i} up to and including $y_{p_{i+j}}$. The sequence of oblique projections that is necessary to achieve this can be interpreted as the effort to achieve a new goal (the control of a new output) without loosing gain in other directions. An oblique projection does not loose gain in the direction along which it is projected.

In section 4.3.2 we applied the QR decomposition on the pseudo inverse process in the mxn case being the ideal controller which resulted into $Q_{ideal}=Q_bR$ with R a lower triangle matrix. We have seen here that we can further decompose R: $R=Q_lQ_\alpha Q_g$ with Q_l (3.4) and Q_g (3.6) both diagonal matrices and Q_α (4.3) being a lower diagonal matrix.

Step 4A Multiplicatively

The second way of removing lower triangle interaction. Steps 1-3 resulted in:

$$Q_b Q_l Q_g = \left[q_{11}^T \; q_{22}^T \; . \; . \; q_{mm}^T \right] \tag{4.4}$$

Remove the interaction between q_{11} and e_2 by adding a component in the direction $q_{b_2}^T$:

$$\begin{gathered} m_2 e_2 (q_{11}^T + q_{b_2}^T \alpha_1) = 0 \Leftrightarrow \alpha_1 = -(e_2 q_{b_2}^T)^{-1} e_2 q_{11}^T \\ q_{21}^T = (I - q_{b_2}^T (e_2 q_{b_2}^T)^{-1} e_2) q_{11}^T \end{gathered} \tag{4.5}$$

q_{21} is an improved version of q_{11}. By induction it follows that:

$$q_{ji}^T = \prod_{k=i+1}^{j} (I - q_{b_k}^T (e_k q_{b_k}^T)^{-1} e_k) q_{ii}^T \tag{4.6}$$

In (4.6) an "oblique projection" is introduced. $(I - q_{b_k}^T (e_k q_{b_k}^T)^{-1} e_k)$ is the projection along $q_{b_k}^T$ on the subspace $e_{k\perp}^T$.

When all interaction is removed the ideal Q is derived:

$$\begin{gathered} with \; P(i,j) = \prod_{k=i}^{j} (I - q_{b_k}^T (e_k q_{b_k}^T)^{-1} e_k) \\ Q_{ideal} = \left[q_{m1}^T \; q_{m2}^T \; . \; . \; q_{mm}^T \right] = \\ \left[P(2,m) q_{11}^T \; P(3,m) q_{22}^T \; . \; . \; q_{mm}^T \right] = \\ \left[P(2,m) q_{b_1}^T \; P(3,m) q_{b_2}^T \; . \; P(m) q_{b_{m-1}}^T \; q_{b_m}^T \right] Q_l Q_g \end{gathered} \tag{4.7}$$