MASTER

Investigation of the structure of spatially interconnected systems on a half plane

Verheijen, C.J.

Award date:
2007

TECHNISCHE UNIVERSITEIT EINDHOVEN
Department of Mathematics and Computing Science

MASTER'S THESIS

Investigation of the Structure of Spatially Interconnected Systems on a Half Plane

C.J. Verheijen

Supervisor: Prof.Dr. A.A. Stoorvogel (TU/e)

Eindhoven, June 2007

Contents

Nomenclature

Matrices/operators:

A_{tt}	system matrix (temporal state $\rightarrow$ temporal state)
A_{ts}	system matrix (temporal state $\rightarrow$ spatial state)
A_{st}	system matrix (spatial state $\rightarrow$ temporal state)
A_{ss}	system matrix (spatial state $\rightarrow$ spatial state)
B_{td}	system matrix (temporal state $\rightarrow$ disturbance)
B_{sd}	system matrix (spatial state $\rightarrow$ disturbance)
C_{yt}	system matrix (output $\rightarrow$ temporal state)
C_{ys}	system matrix (output $\rightarrow$ spatial state)
D_{yd}	system matrix (output $\rightarrow$ disturbance)

Operators:

S_l	left shift operator
S_r	right shift operator
S_d	downward shift operator
S_u	upward shift operator
Δ_S	spatial shift operator
$\tilde{\Delta}_S$	spatial shift operator (extended definition)
σ	spatial shift operator (Fourier transformation)

Λ	input operator
$\tilde{\Lambda}$	input operator (extended definition)

Δ	Laplace operator
∂	partial derivative

Variables/vectors:

c	input vector
c_f	element of the input vector
c_u	horizontal velocity component of the input vector $[m/s]$
c_v	vertical velocity component of the input vector $[m/s]$
c_p	pressure component of the input vector $[N/m^2]$
d	disturbances (external inputs)
p	pressure $[N/m^2]$
u	horizontal velocity $[m/s]$
$\tilde{u}$	horizontal velocity (extended) $[m/s]$
v	vertical velocity $[m/s]$
$\tilde{v}$	vertical velocity (extended) $[m/s]$
x_t	system state (temporal)
x_s	system state (spatial)
y	controlled output

Indices:

l	left
r	right
u	up
d	down
n	number of time steps
t	temporal
s	spatial

Coordinates:

t	time $[s]$
s_1	horizontal coordinate (spatial) $[m]$
s_2	vertical coordinate (spatial) $[m]$
x	horizontal coordinate (temporal) $[m]$
y	vertical coordinate (temporal) $[m]$

Constants:

α	constant $[s^{-1}]$
β	constant $[s^{-1}]$
γ	constant $[s^{-1}]$
δ	constant $[s^{-1}]$
θ	constant $[s^{-1}]$
λ	constant $[s^{-1}]$
τ	constant $[s^{-1}]$
φ	constant $[s^{-1}]$

ν	kinematic viscosity $[m^2/s]$
μ	viscosity $[kg/ms]$
ρ	air density $[kg/m^3]$

m_l	multiplicity of left shift operators $[-]$
m_r	multiplicity of right shift operators $[-]$
m_d	multiplicity of downward shift operators $[-]$
m_u	multiplicity of upward shift operators $[-]$
M	maximum pressure $[N/m^2]$
N	number of rows $[-]$

U	horizontal velocity (linearized) $[m/s]$
U_0	horizontal velocity $[m/s]$
V	vertical velocity (linearized) $[m/s]$
V_0	vertical velocity $[m/s]$

Acknowledgements

This is my final report for the course Applied Mathematics at Eindhoven University of Technology. First of all, I would like to thank my supervisor, prof.dr. A.A. Stoorvogel, for his support during this period and his patience with me. I thank prof.dr.ir. J. de Graaf, dr. J.M.L. Maubach, dr.ir. J.H.M. ten Thije Boonkkamp and dr. I.S. Pop for their advice. Then I thank dr. H.G. ter Morsche and dr.ir. L.C.G.J.M. Habets for taking place in my committee. Last but not least, I would like to thank my parents for their support.

Stijn Verheijen
June 2007

Overview

This paper is in many ways a continuation of the final report of B.P. Tilma at Delft University of Technology. In the report of Tilma [10], a two-dimensional flow problem is formulated in the framework of a spatially invariant interconnected system. Such a system consists of similar subsystems which interact with their nearest neighbors. A spatially interconnected system takes advantage of the spatial structure of a system. The spatially interconnected system is introduced in the article of R. D'Andrea and G.E. Dullerud [3]. We will first shortly outline some properties of a spatially interconnected system formulated in this article, after that we take a look at the way such a system is used in the report of Tilma and to which restrictions boundary conditions can lead with respect to the spatial structure of the system. Finally, we will formulate our goals and give an outline of this report.

Spatially interconnected systems

There exist many systems which consist of subsystems which have a similar behavior. These subsystems, or basic building blocks, interact with their nearest neighbors. When we apply the classical finite elements methods to such a system in order to obtain a mathematical model, we do not take advantage of the spatial invariance of such a system. In the article of D'Andrea and Dullerud [3] is investigated how one can benefit from the properties of these spatially interconnected systems. All building blocks consist of a finite dimensional, linear time-invariant system which is governed by the same state-space equations. To take advantage of the spatial structure of this spatially interconnected system, the spatial shift operator is introduced. With this operator it is possible to shift information in space. If we are for instance working on a plane, we can introduce horizontal and vertical shift operators to shift information. With use of these operators, the state-space equations of every basic building block is similar, and we can decrease the system's dimension significantly.

Restrictions to spatial structure

In the interconnected system in [3], no boundary conditions are taken into account, except for a periodic boundary condition. The question is if it is possible to maintain the spatial structure of the model if we have to define some boundary conditions. In the report of Tilma in [10], a two-dimensional flow problem is considered. A Poiseuille flow over the wing of an airplane is investigated, this wing is a solid wall which is equipped with actuators and sensors. The actuators are in fact blowing and suction devices which are integrated in the surface of the wing. The sensors measure data on which we determine if the devices have to blow or to suck air and at which velocity. The blowing and suction devices are very small micro electro mechanical systems (MEMS), they have a characteristic length between 1 μm and 1 mm. The devices are used to control the velocity distribution of the flow just outside a boundary layer. For some background information about control theory, we refer to [10].

The wing of the airplane is presented as a horizontal wall in the model. The flow problem is described by the linearized incompressible Navier-Stokes equations. A discretization of these equations is made on a rectangular grid. Afterwards, the model is implemented in the spatially interconnected system, which was introduced in [3]. Since the flow problem of Tilma is defined on the upper half plane, some boundary conditions have to be defined. Because of these conditions, a part of the spatial structure of the model gets lost. More specific, the problem does not fully fit anymore in the framework of the spatially interconnected system in [3]. The spatial structure is kept for the horizontal direction, but for the vertical direction the classical finite elements methods is applied because of the boundary. The reason is that the state-space equations of building blocks at the boundary differ from those at the interior of the half plane, because the influence of the devices have to be taken into account.

Objectives

In our report a two-dimensional flow problem on a half plane is considered. This problem is in many ways similar to the one in [10], however, we are not interested in a specific flow, like the Poiseuille flow in [10], but we will focus on the structure of the problem. We formulate the two main goals of this report. The first one is to construct a model in which we take full advantage of the spatial structure of the problem, even if we have to formulate some boundary conditions. We investigate if it is possible to use the vertical shift operators on the half plane, and if this leads to complications.

The second goal has to do with the way the pressure variable in the Navier-Stokes equations is handled. The Navier-Stokes equations do fit in the framework of the spatially interconnected system, but in [10] the pressure is calculated like it is an exogenous input. Our purpose is to take the interaction between the velocity variables and the pressure into account in our model. We use the continuity equation of the Navier-Stokes equations as a tool.

Outline of this report

In chapter 1 the spatially interconnected system on a plane is introduced. We describe the components such a system consists of and give an example in illustration. Furthermore the shift operator is introduced, this operator plays a very important role in our model since it is responsible for the spatial structure. At the end of the first chapter the general solution to the problem is given. We mention that we do not take any boundary conditions into account in this chapter. In chapter 2 we investigate the properties of the interconnected system when it is applied to a half plane instead of a plane. This is necessary since our flow problem is defined on a half plane. It is clear that we have to pay special attention to the boundary in this case, which is the horizontal wall. A discretization of the Navier-Stokes equations is made in chapter 3. We examine how we can take full advantage of the spatial structure of the interconnected system and investigate if we have to make adaptations to our model to assure that the problem is well-defined. Hereby, we use the definitions formulated in chapter 2. We will see that the boundary plays a crucial role in this question. We draw some conclusions in chapter 4 and investigate in which way some properties of the air plane might have influence on our model. Finally, we make some recommendations for future research.

Chapter 1

Interconnected system on a plane

In this report Navier-Stokes equations are described in a system of spatially invariant interconnected systems. In this chapter we investigate such a system when it is applied to a plane. Note that we will *not* define any boundary conditions in this chapter, since we are only interested in the spatial structure of the model at this stage. We do not want to obtain complications which definitely occur when we define boundary conditions. In section 1.1 we introduce spatially invariant interconnected systems and pay special attention to the variables such a system contains and to the different equations it is composed of. In the following sections we examine the interconnected system part by part. In section 1.1.1 we first take a closer look at the temporal state, in which we can find out how to derive the state of the system. The spatial state (section 1.1.2) gives us information about the individual subsystems an interconnected system is composed of, how the state of a subsystem is being influenced by variables of its surrounding subsystems and how these variables are shared by the system. Also the spatial shift operator is introduced in this section. The temporal and spatial state together form the dynamics of the system. The last part of the interconnected system is the output, a user defined part which can be found in section 1.1.3. In section 1.2 a small example of an interconnected system is given in order to get a better understanding of the connection between the temporal and spatial state. In section 1.3 we extend the definition of the spatial shift operator and look how this affects the example in the section before. Finally we give the general solution of the state of the system in section 1.4.

1.1 Introduction

Interconnected systems are often based on identical copies of basic building blocks. Such a block consists of a finite-dimensional linear time-invariant system governed by the following state-space equations [3]:

$$\begin{aligned}
\begin{bmatrix} \dot{x}_t(t,\mathbf{s}) \\ \Delta_S x_s(t,\mathbf{s}) \\ y(t,\mathbf{s}) \end{bmatrix} &= \begin{bmatrix} A_{tt} & A_{ts} & B_{td} \\ A_{st} & A_{ss} & B_{sd} \\ C_{yt} & C_{ys} & D_{yd} \end{bmatrix} \begin{bmatrix} x_t(t,\mathbf{s}) \\ x_s(t,\mathbf{s}) \\ d(t,\mathbf{s}) \end{bmatrix} \\
x_t(0,\mathbf{s}) &= x_{t,0}(\mathbf{s})
\end{aligned} \tag{1.1}$$

In the interconnected system in (1.1), the temporal vector x_t presents the state of a building block, or subsystem. Vector x_s consists in general of variables of surrounding subsystems, which have influence on the state of the building block, we will call x_s the spatial vector. In the interest of clarity, we will call the subsystem with state x_t the central or main subsystem. Vector d presents the exogenous signals and y is a user defined output vector of the system. Δ_S is an operator which has the scape of a diagonal matrix; it contains spatial shift operators, these are used to make a connection between the building blocks. A more detailed explanation of this operator can be found in section 1.1.2. Finally, t stands for the time and $\mathbf{s}$ presents the discrete spatial coordinates of a system. The interconnected system can be split up into three equations:

$$\dot{x}_t(t,\mathbf{s}) = (A_{tt}x_t)(t,\mathbf{s}) + (A_{ts}x_s)(t,\mathbf{s}) + (B_{td}d)(t,\mathbf{s}) \tag{1.2}$$

$$(\Delta_S x_s)(t,\mathbf{s}) = (A_{st}x_t)(t,\mathbf{s}) + (A_{ss}x_s)(t,\mathbf{s}) + (B_{sd}d)(t,\mathbf{s}) \tag{1.3}$$

$$y(t,\mathbf{s}) = (C_{yt}x_t)(t,\mathbf{s}) + (C_{ys}x_s)(t,\mathbf{s}) + (D_{yd}d)(t,\mathbf{s}) \tag{1.4}$$

Equation (1.2) presents the temporal state, in which the state of a building block is derived. In the spatial state (1.3) is outlined which variables of surrounding subsystems have influence on the state of a building block and how these are connected with the main subsystem. Finally, equation (1.4) stands for the output of the system, this part can be filled in by the user. In the following sections, these three parts of the system will be explained.

1.1.1 The temporal state

The first part of this interconnected system is the temporal state, which consists of the following equation:

$$\dot{x}_t(t,\mathbf{s}) \quad = \quad (A_{tt}x_t)(t,\mathbf{s}) + (A_{ts}x_s)(t,\mathbf{s}) + (B_{td}d)(t,\mathbf{s}) \qquad (1.5)$$

Consider a building block, the central subsystem, with state x. This state consists of variables which evolve in time according to the differential equation in (1.5) with external influences of surrounding subsystems coming through spatial vector x_s and influences of exogenous signals like disturbances coming through d. The temporal vector x_t can contain quantities like the velocity. Note that a variable of our central subsystem can only be part of x_t if we know something about its derivative in time, simply because (1.5) is a differential equation. In chapter 3 we will see that for this reason the pressure variable moves to the spatial vector x_s. This vector x_s also consists of all variables of the surrounding subsystems which have influence on any variable of the state of the main subsystem; x_s can contain variables like velocity or pressure.

It might be a bit confusing whether a variable belongs to the temporal vector x_t or to the spatial vector x_s. For most variables this depends on the subsystem in which we are interested. In other words, which subsystem do we consider as our central or main subsystem. Suppose we have a one-dimensional interconnected system and the temporal state of any subsystem consists of the horizontal velocity u. Suppose the horizontal velocities at the left and right cell of a subsystem have influence on the evolution of this horizontal velocity, and suppose the same holds for the pressure p of the subsystem itself and of those at the left and right cell. Note that we do not have a differential equation for the pressure. Then we have the following elements for the temporal and spatial vector if we consider the subsystem with index s as our main subsystem:

$$\begin{aligned} u(t,s) \quad &\rightarrow \quad \text{part of temporal vector } x_t \\ u(t,s-1),\, u(t,s+1) \quad &\rightarrow \quad \text{part of spatial vector } x_s \\ p(t,s-1),\, p(t,s),\, p(t,s+1) \quad &\rightarrow \quad \text{part of spatial vector } x_s \end{aligned} \qquad (1.6)$$

Consider the same interconnected system as above, but now assume that the subsystem with spatial coordinate $s+1$ is our central subsystem, in other words the subsystem of which we want to investigate the evolution of its state. In this case we get:

$$\begin{aligned} u(t,s+1) &\rightarrow \text{part of temporal vector } x_t \\ u(t,s),\, u(t,s+2) &\rightarrow \text{part of spatial vector } x_s \\ p(t,s),\, p(t,s+1),\, p(t,s+2) &\rightarrow \text{part of spatial vector } x_s \end{aligned} \tag{1.7}$$

Note that $p(t,s)$ and $p(t,s+1)$ are in both cases part of the spatial vector, because we do not have any information about the derivative of the pressure and therefore this variable cannot be part of the temporal state. In the interest of clarity we will from now on assume that our main subsystem has spatial coordinate s, if this coordinate is left out. More about the spatial vector x_s can be found in the next section.

1.1.2 The spatial state

The second part of the system describes the spatial state and contains the following equation:

$$(\Delta_S x_s)(t,\mathbf{s}) = (A_{st}x_t)(t,\mathbf{s}) + (A_{ss}x_s)(t,\mathbf{s}) + (B_{sd}d)(t,\mathbf{s}) \tag{1.8}$$

Spatial vector x_s in equation (1.8) can be regarded as a small library, containing all variables which have any influence on the evolution in time of state x_t of the central subsystem and therefore have to be shared by the system. These variables are connected with the central subsystem by the spatial shift operators in operator Δ_S. This operator has the scape of a diagonal matrix which consists of spatial shift operators and can be seen as a tool to shift information in space, like $\dot{x}_t$ shifts information in time. Suppose we are working on a horizontal line, so our interconnected system is one-dimensional. To be able to connect the variables of all subsystems with each other, we need two spatial shift operators: an operator S_l to shift information from the left to the right and an operator S_r to shift information from the right to the left. We define these operators as follows [3], [8]:

$$\begin{aligned} (S_l x)(t,s) &= x(t,s-1) \\ (S_r x)(t,s) &= x(t,s+1) \end{aligned} \tag{1.9}$$

Unless indicated otherwise, we will assume that the state x of every subsystem consists of only one variable, namely x itself. This is done in order to

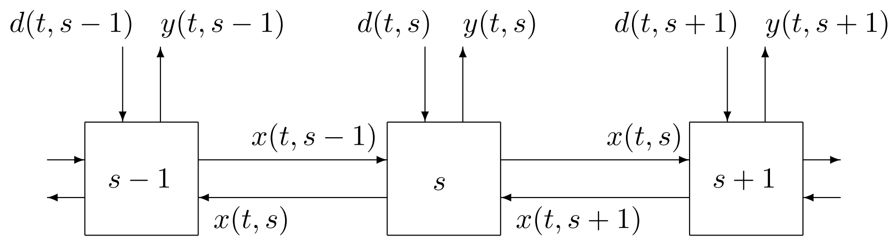

Figure 1.1: *One-dimensional system with single shift operators*

keep the explanations clear.

Shift operator S_l is in fact the inverse of operator S_r. Operators S_l and S_r are the horizontal shift operators. Suppose the central subsystem of our one-dimensional system has spatial coordinate s. As mentioned before, a variable $x(t, s)$ might be a part of the state of the central subsystem.

Consider Figure 1.1, in which a part of a one-dimensional system is depicted. We can make the following connections in this picture with use of the left and right shift operator:

$$\begin{aligned} (S_r x)(t, s-1) &= x(t, s) \\ (S_r x)(t, s) &= x(t, s+1) \\ (S_l x)(t, s+1) &= x(t, s) \\ (S_l x)(t, s) &= x(t, s-1) \end{aligned} \tag{1.10}$$

The number of shift operators in Δ_S depends on the number of variables which have influence on the evolution of the state of the central subsystem and on the location of these variables with respect to this main subsystem. Suppose a subsystem two spatial steps at the right of the main building block contains a variable which has influence on the state of this central subsystem, so this variable has to be shared by the system. Then we need two shift operations to connect this variable with the state:

$$(S_l x)(t, s+2) = x(t, s+1) \tag{1.11}$$
$$(S_l x)(t, s+1) = x(t, s) \tag{1.12}$$

The connection of a variable of subsystems which are not directly connected with the main subsystem, like equation (1.11), appears in matrix A_{ss}, since both $x(t, s+1)$ and $x(t, s+2)$ are a component of the spatial vector x_s. Operations which connect a variable of a surrounding subsystem with a variable of the state of the main building block, like in equation (1.12), appear in matrix A_{st}. Connections to variables of the state of the main subsystem which are part of the spatial vector x_s instead of the temporal vector x_t, will always appear in A_{ss}. In section 1.3 we will see that it is also possible to connect the variables of any surrounding subsystem with just one spatial operation with the main block, by using multiple - or higher order - shift operators, like S_r^2. However, in this section we restrict ourselves to single shift operations. Define the number of used left shift operators as m_l and the number of right shifts as m_r. In a one-dimensional system, the operator Δ_S then looks like this:

$$\Delta_S = \begin{pmatrix} S_l I_{m_l} & 0 \\ 0 & S_r I_{m_r} \end{pmatrix} \tag{1.13}$$

Besides, the way the equations in the spatial state are ordered has no influence on the system, so a strict order of the operators is equation (1.13) is not essential. The definition of the shift operator can be expanded to the two-dimensional case, which will be necessary since we are working on a plane. The spatial coordinate $\mathbf{s}$ now consists of a horizontal and a vertical coordinate, which we will call s_1 and s_2, respectively. Next to the left and right shift operator, we also define a downward shift operator S_d and an upward operator S_u:

$$\begin{aligned}
(S_l x)(t, s_1, s_2) &= x(t, s_1 - 1, s_2) \\
(S_r x)(t, s_1, s_2) &= x(t, s_1 + 1, s_2) \\
(S_d x)(t, s_1, s_2) &= x(t, s_1, s_2 - 1) \\
(S_u x)(t, s_1, s_2) &= x(t, s_1, s_2 + 1)
\end{aligned} \tag{1.14}$$

We can immediately see that for the vertical shift operators holds that S_d and S_u are each others inverse on a plane. The number of shift operators used in the equations for the downward and upward direction are defined as m_d and m_u, respectively. The operator in equation (1.13) is expanded likewise, we get:

$$\Delta_S = \begin{pmatrix} S_l I_{m_l} & 0 & 0 & 0 \\ 0 & S_r I_{m_r} & 0 & 0 \\ 0 & 0 & S_d I_{m_d} & 0 \\ 0 & 0 & 0 & S_u I_{m_u} \end{pmatrix} \tag{1.15}$$

Again, a strict order of operators is not required in (1.15). In some fields, the definition of the shift operators differs from those used in this report, for example in photography. If we apply our operator S_l to all pixels of a picture, the value of any cell depends on the neighbor at the left, but the image will be shifted one pixel to the right. For this reason, in photography it would be logical to switch the definition for the horizontal and vertical operators S_r and S_l used in this report, and to do the same for the vertical shift operators. However, in this report we will stick to the definition used in [3] and [10].

With the temporal and spatial state, the dynamics of the interconnected system are described.

1.1.3 Output of the system

In the third row the output of the system is described:

$$y(t, \mathbf{s}) = (C_{yt}x_t)(t, \mathbf{s}) + (C_{ys}x_s)(t, \mathbf{s}) + (D_{yd}d)(t, \mathbf{s}) \tag{1.16}$$

The output vector y is used to deduce information from the system. It is a user defined vector which depends on the composition of the matrices C_{yt}, C_{ys} and D_{yd}. In the following chapter, in which we consider interconnected systems on a half plane, we introduce an input vector c which affects the system. The output y and the input c are closely related to each other, because with the input we can influence our output and vice versa.

1.2 Example of an interconnected system

In this section we consider an example of an interconnected system, in order to get a better understanding of the connection between the temporal and spatial state. Suppose we have the following two-dimensional system equation, in which we use spatial shift operators:

$$\dot{x} \;=\; \frac{1}{16}(S_r + S_l - 2)(S_u + S_d - 2)x \tag{1.17}$$

In the interest of clarity, the variables t and $\mathbf{s}$ have been omitted as well as an output y and a disturbance d. Furthermore, we assume that the state x of any subsystem consists of just one variable. In this example, the state of the eight surrounding subsystems have influence on the state of the main subsystem when it evolves in time. We can rewrite the equation as follows:

$$\dot{x} \;=\; \frac{1}{4}x - \frac{1}{8}(S_r + S_l + S_u + S_d)x + \frac{1}{16}(S_rS_u + S_rS_d + S_lS_u + S_lS_d)x \tag{1.18}$$

In the interest of easy referring, the variables which will be part of x_s are renamed according to their spatial position with respect to our main subsystem. In the remaining of this report, we will call this the *spatial notation*. For example, the variable of the subsystem at the upper right of the central subsystem is renamed as follows:

$$x_{ru} = x_{ru}(t, s_1, s_2) := (S_rS_ux)(t, s_1, s_2) = x(t, s_1 + 1, s_2 + 1) \tag{1.19}$$

The subscript $_{ru}$ is placed because we can reach this subsystem from our main subsystem by first taking a spatial step to the right with S_r and then a step in upward direction with S_u. In Figure 1.2 all variables which have influence on the state of the main subsystem are depicted.

Now the spatial state of the interconnected system can be described. Note that there are several possibilities (in this case $2^4 = 16$) to define matrix A_{ss}, because the surrounding subsystems which can not directly be linked to the central subsystem - those at the corners - all can be connected with this main subsystem in two different ways. For example, variable x_{ld} can be connected with x through x_d and through x_l. Suppose we connect x_{ld} and x_{lu} through x_l with the central subsystem, and x_{rd} and x_{ru} through x_r, then we get the following equations for the spatial state:

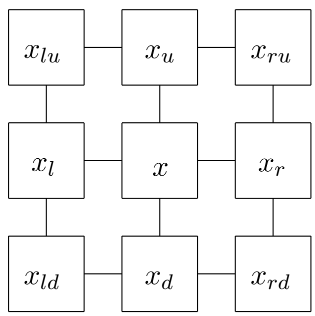

Figure 1.2: *The spatial notation of all variables is made according to their position with respect to the temporal state of the main subsystem*

$$\begin{array}{lllllll} S_r x_l & = & x & \quad & S_u x_{rd} & = & x_r \\ S_l x_r & = & x & & S_d x_u & = & x \\ S_u x_d & = & x & & S_d x_{lu} & = & x_l \\ S_u x_{ld} & = & x_l & & S_d x_{ru} & = & x_r \end{array} \tag{1.20}$$

For the equation of the temporal state, we use the spatial notation in Figure 1.2:

$$\dot{x} = \frac{1}{4}x - \frac{1}{8}(x_l + x_r + x_d + x_u) + \frac{1}{16}(x_{ld} + x_{rd} + x_{lu} + x_{ru}) \tag{1.21}$$

This results in the temporal and spatial state of the interconnected system:

$$
\begin{pmatrix} \dot{x}_t \\ \Delta_S x_s \end{pmatrix} = \begin{pmatrix} A_{tt} & A_{ts} \\ A_{st} & A_{ss} \end{pmatrix} \begin{pmatrix} x_t \\ x_s \end{pmatrix} \Rightarrow
$$

$$
\begin{pmatrix} \dot{x} \\ S_r x_l \\ S_l x_r \\ S_u x_d \\ S_u x_{ld} \\ S_u x_{rd} \\ S_d x_u \\ S_d x_{lu} \\ S_d x_{ru} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{16} & \frac{1}{16} & -\frac{1}{8} & \frac{1}{16} & \frac{1}{16} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ x_l \\ x_r \\ x_d \\ x_{ld} \\ x_{rd} \\ x_u \\ x_{lu} \\ x_{ru} \end{pmatrix} \tag{1.22}
$$

The system above contains the following system matrices A_{tt}, A_{ts}, A_{st} and A_{ss}, and operator Δ_S:

$$
\begin{aligned}
A_{tt} &= \frac{1}{4} \\
A_{ts} &= \begin{pmatrix} -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{16} & \frac{1}{16} & -\frac{1}{8} & \frac{1}{16} & \frac{1}{16} \end{pmatrix} \\
A_{st} &= \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}^T \\
A_{ss} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\Delta_S &= diag\begin{pmatrix} S_r & S_l & S_u & S_u & S_u & S_d & S_d & S_d \end{pmatrix}
\end{aligned} \tag{1.23}
$$

It is obvious that the connections between variables of surrounding subsystems and the central subsystem of the equations of (1.20) - the first, second, third and fifth equation - appear in matrix A_{st}, while the other *indirect* connections can be found in matrix A_{ss}.

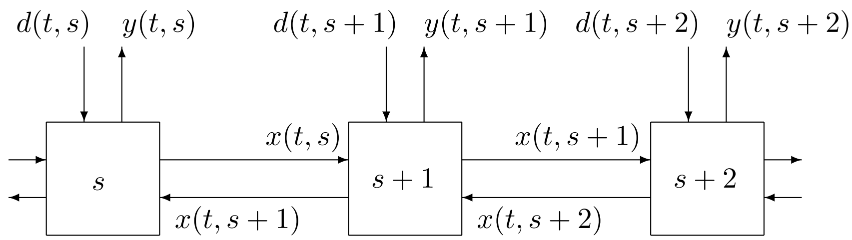

Figure 1.3: *One-dimensional system with single shift operators*

1.3 Multiple shift operators

In the previous sections only first order (or single) shift operators were used. Consider again the figure in section 1.1.2 in which a part of a one-dimensional interconnected system was depicted. In Figure 1.3, the left subsystem with index s is considered as the main building block of which we want to calculate the state.

Suppose the subsystem at the right with spatial coordinate $s+2$ contains a variable x which has influence on variable x of the state of the main subsystem with index s at the left. Then this variable needs to be part of the spatial vector, and has to be connected with our central subsystem. Two operations are necessary to make this connection, namely:

$$(S_l x)(t, s+2) = x(t, s+1) \tag{1.24}$$
$$(S_l x)(t, s+1) = x(t, s) \tag{1.25}$$

So we have to take two spatial steps to make the connection. As we have noticed in section 1.1.2, the operation in (1.24) will appear in A_{ss} and the one in (1.25) appears in A_{st} - unless that specific variable of the central subsystem is not a part of the temporal vector x_t. Now suppose we allow Δ_S to contain also higher order, or multiple, shift operators, like $S_r S_d$ or S_u^2, then it is possible to connect the variables of every subsystem with the main subsystem with just one spatial operation. If Δ_S contains higher order

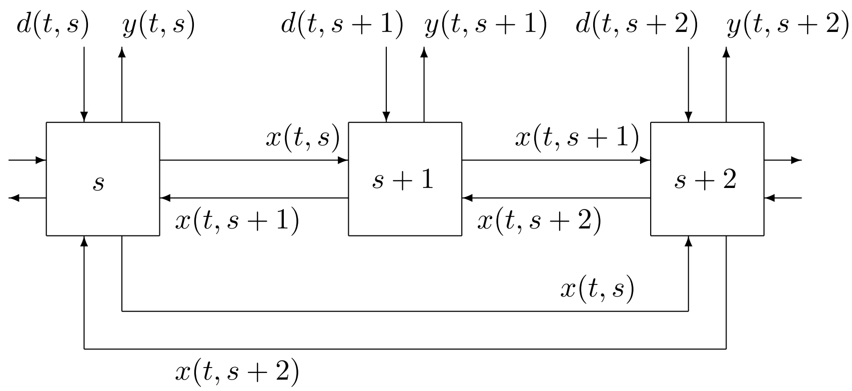

Figure 1.4: *One-dimensional system with multiple shift operators*

shift operators, we rename this operator as $\tilde{\Delta}_S$.

Consider Figure 1.4. We now not only have the shift operations in equation (1.24) and (1.25), but also the following operation:

$$(S_r^2 x)(t,s) = (S_r S_r x)(t,s) = (S_r x)(t,s+1) = x(t,s+2) \qquad (1.26)$$

When we use multiple shift operators, every variable that has influence on the state of the central subsystem can be connected with this building block in one spatial step. As a consequence, no connections will appear in A_{ss} anymore, so this matrix will become $\mathbf{0}$, all connections appear in A_{st}. If the temporal state contains only one variable, A_{st} becomes $\mathbf{1}$. As already mentioned in section 1.1.1, not every connection appears in A_{st}, if a variable of the central subsystem can not be a part of the state x_t. In that case, this variable appears in the spatial vector x_s and matrix A_{ss} will in general be unequal to zero, no matter if multiple shift operators are used.

Consider again the example in section 1.2. We had the following differential equation:

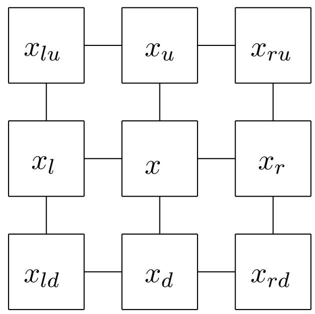

Figure 1.5: *The spatial notation of all variables is made according to their position with respect to the temporal state of the the main subsystem*

$$\dot{x} = \frac{1}{16}(S_r + S_l - 2)(S_u + S_d - 2)x \tag{1.27}$$

Again, all necessary variables of the surrounding subsystems are renamed to their spatial position with respect to the variable of the main subsystem, see Figure 1.5, we called this the spatial notation.

In case we allow multiple shift operators in Δ_S in this system, we can connect the variables of the four subsystems at the corners with the central subsystem by using only one spatial operation:

$$\begin{aligned}
(S_r S_u x_{ld}) &= (S_r S_u x)(t, s_1 - 1, s_2 - 1) &= x(t, s_1, s_2) = x \\
(S_l S_u x_{rd}) &= (S_l S_u x)(t, s_1 + 1, s_2 - 1) &= x(t, s_1, s_2) = x \\
(S_r S_d x_{lu}) &= (S_r S_d x)(t, s_1 - 1, s_2 + 1) &= x(t, s_1, s_2) = x \\
(S_l S_d x_{ru}) &= (S_l S_d x)(t, s_1 + 1, s_2 + 1) &= x(t, s_1, s_2) = x
\end{aligned} \tag{1.28}$$

The matrices and operator Δ_S of the spatial state then become:

$$
\begin{aligned}
A_{st} &= \mathbf{1}^T \\
A_{ss} &= \mathbf{0} \\
\tilde{\Delta}_S &= diag \begin{pmatrix} S_r & S_l & S_u & S_r S_u & S_l S_u & S_d & S_r S_d & S_l S_d \end{pmatrix} \quad (1.29)
\end{aligned}
$$

The variables of all surrounding subsystems are now directly connected with the one of the central subsystem, as a result A_{st} consists of only ones and A_{ss} is equal to a zero matrix. Note that the matrices A_{tt} and A_{ts} of the temporal state remain unchanged, simply because the differential equation in (1.27) does not change. Both sets of matrices in equation (1.23) and (1.29) present the same system, but the latter one is written in a more compact way. There is another difference between both ways of notation. Suppose we have the following equations of a system with one spatial dimension:

$$
\left\{ \begin{array}{lcl} \dot{x}_t(t,s) & = & x_s(t,s) \\ x_s(t,s) & = & \sum_{i=0}^{\infty} (\frac{1}{2})^i x_t(t,s+i) \end{array} \right. \quad (1.30)
$$

Then x_s can be rewritten as follows:

$$
\begin{aligned}
x_s(t,s) &= \sum_{i=0}^{\infty} (\frac{1}{2})^i x_t(t,s+i) \\
&= x_t(t,s) + \sum_{i=1}^{\infty} (\frac{1}{2})^i x_t(t,s+i) \\
&= x_t(t,s) + \frac{1}{2} \sum_{i=0}^{\infty} (\frac{1}{2})^i x_t(t,s+1+i) \\
&= x_t(t,s) + \frac{1}{2} x_s(t,s+1) \quad (1.31)
\end{aligned}
$$

Equation (1.31) contains an infinitely large sum, but it can - with use of a single shift operator - still be written in a very compact way:

$$
\begin{aligned}
x_s &= x_t + \frac{1}{2} S_r x_s \qquad \Rightarrow \\
S_r x_s &= -2x_t + 2x_s \quad (1.32)
\end{aligned}
$$

When we use multiple shift operators, we get an infinitely large equation:

$$x_s \quad = \quad x_t + \frac{1}{2}S_r x_t + \frac{1}{4}S_r^2 x_t + \frac{1}{8}S_r^3 x_t + \ldots \tag{1.33}$$

The spatial state of the interconnected system of equation (1.32) contains an operator Δ_S which consists of a single shift operator S_r and also a matrix A_{ss} which is equal to 2. It is not possible to define the spatial state when A_{ss} is set to zero, even if multiple shift operators are allowed, because an infinite number of shift operators would be necessary in operator Δ_S to catch all the surrounding subsystems. This property has to be taken into account when we are dealing with an interconnected system in which subsystems depend on other subsystems which are very far away, like in equation (1.31). So one can only set A_{ss} to zero when every subsystem depends on a finite number of neighbors. For physical reasons, the number of neighbors which have influence on a cell will always be finite, so in practice we can always set A_{ss} to zero. However, in some cases it can still be attractive to use A_{ss} in an approximate sense, because Δ_S can become very large when the number of subsystems which have influence on a central subsystem is very large. Besides, if a variable cannot be part of the temporal state because there is no differential equation defined, it moves to the spatial state and as a consequence A_{ss} will not be zero. This will become clear in the following chapters.

1.4 Existence and uniqueness of a solution

Consider the interconnected system in equation (1.1) which holds on a plane. A natural assumption is that for any variable $x(t, s)$ the values at any instant of time tend to some constant value when s tends to $+\infty$ or $-\infty$. If we subtract this constant value from all variables x in the interconnected system, the structure of the system remains the same, while any variable x now tends to zero for $s \to +\infty$ or $s \to -\infty$. Therefore, the l_2 space is a convenient vector space for the variables of this system. A vector x is an element of the l_2 space if its l_2 norm is finite:

$$\|x(t, s)\|_2 = \sqrt{\sum_i |x(t, s_i)|^2} < \infty \tag{1.34}$$

This means a variable is an element of the l_2 space if its elements are square summable. We will return to this subject in section 3.4. It is possible to eliminate the spatial vector x_s by making a substitution of the spatial state in the temporal state and the output. Note that we assume that the system is well-posed, in other words operator $(\Delta_S - A_{ss})$ is invertible from l_2 to l_2:

$$x_s(t, \mathbf{s}) \;=\; (\Delta_S - A_{ss})^{-1}(A_{st}x_t)(t, \mathbf{s}) + (\Delta_S - A_{ss})^{-1}(B_{sd}d)(t, \mathbf{s}) \tag{1.35}$$

The system can then be expressed as follows:

$$\begin{aligned} \dot{x}_t(t, \mathbf{s}) &= (\mathbf{A}x_t)(t, \mathbf{s}) + (\mathbf{B}d)(t, \mathbf{s}) \\ y(t, \mathbf{s}) &= (\mathbf{C}x_t)(t, \mathbf{s}) + (\mathbf{D}d)(t, \mathbf{s}) \end{aligned} \tag{1.36}$$

where

$$\begin{aligned} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \;:=\; & \begin{bmatrix} A_{tt} & B_{td} \\ C_{yt} & D_{yd} \end{bmatrix} + \begin{bmatrix} A_{ts} \\ C_{ys} \end{bmatrix} \\ & \times(\Delta_S - A_{ss})^{-1} \begin{bmatrix} A_{st} & B_{sd} \end{bmatrix} \end{aligned} \tag{1.37}$$

Note that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are bounded operators from l_2 to l_2, therefore the system is well-posed. Suppose we allow multiple shift operators in Δ_S, then A_{ss} can be set to zero. The remaining operator Δ_S, which consists of multiple shift operators on the diagonal, is always invertible on a plane; we just have to take the inverse shift operators on the diagonal. For example:

$$\Delta_S = \begin{pmatrix} S_r S_u & 0 \\ 0 & S_d^2 \end{pmatrix} \Rightarrow \Delta_S^{-1} = \begin{pmatrix} S_l S_d & 0 \\ 0 & S_u^2 \end{pmatrix} \tag{1.38}$$

We mention once again that it is not always possible to set A_{ss} to zero. If a system is well-posed, there exists a solution to the problem in (1.36). This solution is unique, if we have some initial condition $x_t(0, \mathbf{s}) = x_{t,0}(\mathbf{s}) \in l_2$. The solution looks like this:

$$x_t(t, \mathbf{s}) \;=\; e^{\mathbf{A}t}x_{t,0}(\mathbf{s}) + \int_0^t e^{\mathbf{A}(t-\tau)}(\mathbf{B}d)(\tau, \mathbf{s})d\tau \tag{1.39}$$

where $e^{\mathbf{A}t}$ is the continuous semigroup defined by

$$e^{\mathbf{A}t} \;:=\; \sum_{n=0}^{\infty} \frac{(\mathbf{A}t)^n}{n!} \tag{1.40}$$

Since $\mathbf{A}$ is bounded, $e^{\mathbf{A}t}$ is well-defined.

Chapter 2

Interconnected system on a half plane

In this chapter we consider the interconnected system introduced in the first chapter when it is applied to a half plane instead of a plane. This is necessary, because our problem is defined on a half plane as will become clear in chapter 3. Section 2.1 describes the properties of the two vertical shift operators in a half plane. These are much more complicated than when we work on a plane, because we have to cope with a boundary. This boundary corresponds with the surface of a wing of an airplane. The consequences of the properties of the vertical shift operators for the invertibility of operator Δ_S will be outlined in section 2.2. Finally, in section 2.3 we examine the solution of the state of the system in a half plane.

2.1 Vertical shift operators

The state-space equations of a building block of a spatially interconnected system were defined as follows in the previous chapter:

$$\begin{aligned} \begin{bmatrix} \dot{x}_t(t,\mathbf{s}) \\ \Delta_S x_s(t,\mathbf{s}) \\ y(t,\mathbf{s}) \end{bmatrix} &= \begin{bmatrix} A_{tt} & A_{ts} & B_{td} \\ A_{st} & A_{ss} & B_{sd} \\ C_{yt} & C_{ys} & D_{yd} \end{bmatrix} \begin{bmatrix} x_t(t,\mathbf{s}) \\ x_s(t,\mathbf{s}) \\ d(t,\mathbf{s}) \end{bmatrix} \\ x_t(0,\mathbf{s}) &= x_{t,0}(\mathbf{s}) \end{aligned} \quad (2.1)$$

These equations hold if they are applied to a plane, but our problem, of which a more detailed description can be found in chapter 3, is defined on a half plane, more specifically on the upper half plane. We have to deal with a boundary on which sensors measure data (through output vector y) and actuators produce control signals (through input vector c). Before we examine how the interconnected system and the solution for the state x_t look like in case of a half plane, we first have to investigate the boundary of the half plane and in particular how we have to deal with the vertical shift operators in this situation.

As said before, in case of a half plane, we have not only to cope with an initial condition, but also with one or more boundary conditions. The number of boundary conditions depends on the number of surrounding subsystems in vertical direction which have influence on the central subsystem. In chapter 3 we will see that the l_2 space imposes extra information and for this reason the number of boundary conditions will be reduced.

Furthermore, the question is how to define the shift operator at the boundary or just outside our half plane. For example, in a *plane* holds

$$(S_d x)(t, s_1, 0) \quad = \quad x(t, s_1, -1) \tag{2.2}$$

but in the upper half plane $x(t, s_1, -1)$ is not defined. For this reason, we have to make a distinction between the boundary ($s_2 = 0$) and the rest of the upper half plane, when we apply the downward shift operator to some variable x. We define S_d as follows:

$$(S_d x)(t, s_1, s_2) \quad = \quad \begin{cases} x(t, s_1, s_2 - 1), & s_2 > 0 \\ 0, & s_2 = 0 \end{cases} \tag{2.3}$$

Because we choose $(S_d x)(t, s_1, 0)$ to be zero for any variable x, operator S_d is still linear, since for any variables x and y and any constant a holds:

$$\begin{aligned} (S_d x)(t, s_1, s_2) + (S_d y)(t, s_1, s_2) &= \begin{cases} x(t, s_1, s_2 - 1) + y(t, s_1, s_2 - 1), & s_2 > 0 \\ 0, & s_2 = 0 \end{cases} \\ &= (S_d(x + y))(t, s_1, s_2) \\ a(S_d x)(t, s_1, s_2) &= \begin{cases} a x(t, s_1, s_2 - 1), & s_2 > 0 \\ 0, & s_2 = 0 \end{cases} \\ &= (S_d(ax))(t, s_1, s_2) \end{aligned} \tag{2.4}$$

This linearity is a useful property in our model. Operator S_d is not linear if we choose any other real number then zero for $(S_d x)(t, s_1, 0)$.

There is another property we have to take into account. This property is a consequence of our definition for S_d in (2.3). Suppose we have a variable which value at the boundary is unequal to zero, so $x(t, s_1, 0) \neq 0$. Then there is a difference between the operators $S_u S_d$ and $S_d S_u$ when we apply these to this variable x. We get the following equations for $S_u S_d$ when this operator is applied to some variable x:

$$(S_u S_d x)(t, s_1, s_2) = (S_u(S_d x))(t, s_1, s_2) = (S_d x)(t, s_1, s_2 + 1) = x(t, s_1, s_2) \tag{2.5}$$

So $S_u S_d$ corresponds with the identity operator I. However, when we apply $S_d S_u$ to a variable x with the same property at the boundary, we get the following:

$$\begin{aligned}(S_d S_u x)(t, s_1, s_2) = (S_d(S_u x))(t, s_1, s_2) &= \begin{cases} (S_u x)(t, s_1, s_2 - 1), & s_2 > 0 \\ 0, & s_2 = 0 \end{cases} \\ &= \begin{cases} x(t, s_1, s_2), & s_2 > 0 \\ 0, & s_2 = 0 \end{cases}\end{aligned} \tag{2.6}$$

So on the boundary $S_u S_d$ and $S_d S_u$ do not give the same result when these are applied to a variable of state x, in other words S_u and S_d are not each others inverse on a half plane. We have to take the difference between both operators into account when we further investigate the properties of the vertical shift operators in the following section.

2.2 Well-posedness

On a plane, the system was well-posed if the operator $(\Delta_S - A_{ss})$ was invertible on the l_2 space, because in this case the matrices of equation (1.37) were bounded. On a half plane, invertibility is not straightforward. It is no problem to invert the horizontal operators S_l and S_r, but we have to be careful with the vertical operators S_d and S_u because of the boundary. Suppose we have a two-dimensional system in which operator Δ_S contains an upward and a downward shift operator. We investigate how these operators behave at or close to the boundary.

We first consider the downward shift operator S_d. The definition of this operator, applied to a variable $x(t, s_1, s_2)$, was already given in the previous section:

$$(S_d x)(t, s_1, s_2) = \begin{cases} x(t, s_1, s_2 - 1), & s_2 > 0 \\ 0, & s_2 = 0 \end{cases} \tag{2.7}$$

Now suppose we have the equation $x_2 = S_d x_1$, then the following holds:

$$x_2(t, s_1, s_2) = \begin{cases} x_1(t, s_1, s_2 - 1), & s_2 > 0 \\ 0, & s_2 = 0 \end{cases} \tag{2.8}$$

The consequence of the restriction for $s_2 = 0$ is that the image of this shift function is equal to zero at the boundary. But not every function x_2 equals zero on the boundary. This means that these functions are not in the image of S_d and therefore operator S_d is *not surjective* on the upper half plane. For some applications, the restriction of this definition for $s_2 = 0$ might be useful, for example when we have to deal with a no-slip condition at the boundary. However, in general it will be a disadvantage, and we have to adapt the definition of the downward shift operator.

Equation 2.8 is not always solvable, because not every function x_2 is an element of the image of S_d. By adding the operator $S_d S_u$ to the right hand side of the equation, the image becomes zero for $s_2 = 0$ and therefore we avoid problems at the boundary. Operator $S_d S_u$ is in fact a projection on the image of S_d.

$$\begin{aligned} (S_d x_1)(t, s_1, s_2) &= (S_d S_u x_2)(t, s_1, s_2) \qquad \Rightarrow \\ \begin{cases} x_1(t, s_1, s_2 - 1), & s_2 > 0 \\ 0, & s_2 = 0 \end{cases} &= \begin{cases} (S_u x_2)(t, s_1, s_2 - 1), & s_2 > 0 \\ 0, & s_2 = 0 \end{cases} \qquad \Rightarrow \\ x_1(t, s_1, s_2) &= (S_u x_2)(t, s_1, s_2), \qquad s_2 \geq 0 \end{aligned} \tag{2.9}$$

Because of operator S_d, both the left and right hand side in the equation above become zero for $s_2 = 0$ following (2.7). Note that by adding operator $S_d S_u$ to the right hand side, the equation becomes equivalent with

$x_1 = S_u x_2$. In this form the equation is always solvable, because we assure that $S_d x_1$ has to be equal to some element of the image of S_d. Note that if we use the definition in equation (2.9), we have to accept that (vertical) shift operators do not only appear in Δ_S, but also in some matrices like A_{ss} and A_{st}, which therefore become operators.

Now we consider the upward shift operator S_u. This operator is defined as follows:

$$(S_u x)(t, s_1, s_2) = x(t, s_1, s_2 + 1) \tag{2.10}$$

Now suppose we have the equation $x_2 = S_u x_1$, then the following holds:

$$x_2(t, s_1, s_2) \quad = \quad x_1(t, s_1, s_2 + 1) \tag{2.11}$$

Shift operator S_u gives us a degree of freedom at the boundary, therefore the equation is not uniquely solvable. This degree of freedom might be useful for the implementation of the user defined input vector c in the state-space equations, we will investigate this in chapter 3. Because we can obtain the same image when we apply S_u to variables which have different values at the boundary, operator S_u is *not injective* on the upper half plane. We introduce an operator Λ to fill in the degree of freedom at the boundary. This operator is a function from $l_2(\mathbb{R}^3)$ to $l_2(\mathbb{R}^2)$ and is defined as follows:

$$(\Lambda x)(t, s_1, s_2) \quad = \quad \begin{cases} 0, & s_2 > 0 \\ x(t, s_1), & s_2 = 0 \end{cases} \tag{2.12}$$

This means that when we apply operator Λ to a variable, all values become zero, except for those at the boundary which can attain any magnitude. Consider the definition for $S_d S_u$ in equation (2.6). Note that operator Λ is more or less equivalent with operator $(I - S_d S_u)$, the only difference is that spatial coordinate s_2 is omitted at the right hand side of the equation. This coordinate is left out, since we are only interested in the values for $s_2 = 0$. We use Λ to define a unique solution when we have an equation with operator S_u. Suppose we have the equation $S_u x_1 = x_2$. Then there exists a variable c_f such that the following holds:

$$\begin{aligned} (S_u x_1)(t, s_1, s_2) \quad &= \quad x_2(t, s_1, s_2) \qquad \Rightarrow \\ x_1(t, s_1, s_2) \quad &= \quad (S_d x_2)(t, s_1, s_2) + (\Lambda c_f)(t, s_1, s_2) \end{aligned} \tag{2.13}$$

In equation (2.13), variable c_f fills in the degree of freedom and will be an element of the user defined input vector c at the boundary. More about c_f and its applications can be found in the following chapter.

The consequence of the restrictions for S_u (not injective) and S_d (not surjective) here above is that when the operator Δ_S contains an upward or downward shift operator, $(\Delta_S - A_{ss})$ is simply not invertible on a half plane. Since our problem certainly will contain these shift operators, we cannot use the ordinary inverse operator of $(\Delta_S - A_{ss})$ to define a solution. The alternative inverse operator we will use is the generalized inverse operator. This new definition contains a degree of freedom because of the property of S_u at the boundary. We need to fill in this degree of freedom, because we are interested in a unique solution for x_t. The new definition also contains a restriction because of the property of S_d at the boundary.

If we define $(\Delta_S - A_{ss})^\dagger$ as the generalized inverse operator of $(\Delta_S - A_{ss})$, the following properties have to hold:

$$\begin{aligned}
(\Delta_S - A_{ss})(\Delta_S - A_{ss})^\dagger(\Delta_S - A_{ss}) &= (\Delta_S - A_{ss}) \\
(\Delta_S - A_{ss})^\dagger(\Delta_S - A_{ss})(\Delta_S - A_{ss})^\dagger &= (\Delta_S - A_{ss})^\dagger
\end{aligned} \tag{2.14}$$

For the horizontal shift operators S_r and S_l there are no restrictions for the inverse operator, nor for the ordinary inverse, neither for the generalized case. S_r and S_l are simply each others generalized inverse operator. We have to investigate what the generalized inverse operators of the vertical shifts S_u and S_d are. It is clear that S_d and S_u should be each others generalized inverse on the half plane, so we want to prove that $S_d^\dagger = S_u$. Then the following equations must hold [6]:

$$S_d S_u S_d = S_d \tag{2.15}$$

$$S_u S_d S_u = S_u \tag{2.16}$$

We may immediately conclude that both (2.15) and (2.16) are true for all variables x, simply because the left hand side of both equations contains an operator $S_u S_d$ which is equal to I, and therefore can be omitted. However, in the interest of clarity we will apply both equations to a variable x:

First we consider equation (2.15). If we apply the operator at the left hand side to a variable $x(t, s_1, s_2)$, we get the following equation:

$$\begin{aligned} (S_d S_u S_d x)(t, s_1, s_2) &= (S_d(S_u S_d x))(t, s_1, s_2) \\ &= \begin{cases} (S_u(S_d x))(t, s_1, s_2 - 1), & s_2 > 0 \\ 0, & s_2 = 0 \end{cases} \\ &= \begin{cases} (S_d x)(t, s_1, s_2), & s_2 > 0 \\ 0, & s_2 = 0 \end{cases} \\ &= \begin{cases} x(t, s_1, s_2 - 1), & s_2 > 0 \\ 0, & s_2 = 0 \end{cases} \end{aligned} \tag{2.17}$$

This definition corresponds to the one of $S_d x$ in equation (2.7), so indeed we have that $S_d S_u S_d = S_d$. Now we apply operator $S_u S_d S_u$ of equation (2.16) to a variable x:

$$\begin{aligned} (S_u S_d S_u x)(t, s_1, s_2) &= (S_u(S_d S_u x))(t, s_1, s_2) \\ &= (S_d(S_u x))(t, s_1, s_2 + 1) \\ &= (S_u x)(t, s_1, s_2) \\ &= x(t, s_1, s_2 + 1) \end{aligned} \tag{2.18}$$

This definition is equivalent with the definition for $S_u x$ in equation (2.10), so we also have that $S_u S_d S_u = S_u$. So indeed we have that S_u and S_d are each others generalized inverse operator. As a consequence, the generalized inverse $\Delta_S^\dagger$ of operator Δ_S is now also defined.

The question remains what the generalized inverse operator of $(\Delta_S - A_{ss})$ is. We use multiple shift operators to find this operator. Assume that the evolution of all variables of our system depends on a finite number of variables of surrounding subsystems, and assume that all variables of the central subsystem can be found in the temporal state of the interconnected system. In section 1.3 we saw that if we connected all variables by just a single operation with the variable of the central subsystem, A_{ss} became $\mathbf{0}$ in this case. If $A_{ss} = \mathbf{0}$ in the multiple shift case, then for operator $\tilde{\Delta}_S$ holds:

$$\tilde{\Delta}_S = \Delta_S - A_{ss} \tag{2.19}$$

If we can find the generalized inverse operator for any multiple shift operator in $\tilde{\Delta}_S$, then the generalized inverse operator of $\tilde{\Delta}_S$ consists of all generalized inverse operators of its diagonal, and as a consequence we also have the generalized inverse operator of $(\Delta_S - A_{ss})$. Since the horizontal shift operators do not cause any problems, the question remains if the higher order vertical shift operators have a generalized inverse. We want to know if the following equation holds:

$$S_u^{\dagger k} = S_d^k, \qquad k \in \mathbb{N} \tag{2.20}$$

Then we have to prove the following equations:

$$S_d^k S_u^k S_d^k = S_d^k, \qquad k \in \mathbb{N} \tag{2.21}$$

$$S_u^k S_d^k S_u^k = S_u^k, \qquad k \in \mathbb{N} \tag{2.22}$$

If we use the property in equation (2.5), the proof becomes pretty straightforward, because if $S_u S_d = I$, then also $S_u^k S_d^k = I$ for all $k \in \mathbb{N}$, and we see directly that (2.21) and (2.22) hold.

However, we have to be careful with the assumptions we made in order to obtain the equation $\tilde{\Delta}_S = \Delta_S - A_{ss}$. We assumed that the evolution of all variables of the interconnected system depends on a finite number of variables of surrounding subsystems. For physical reasons, this is a realistic assumption, as we already concluded at the end of section 1.3. We also assumed that every variable of the central subsystem is a part of the temporal state of the system. This is in general *not* the case. If we do not have a differential equation for some variable, which will be the case for the pressure in our problem the next chapter, this variable appears in the spatial state, no matter if it is part of the central subsystem or one of the surrounding subsystems, and A_{ss} will be unequal to zero. The consequence of this property for our problem is that for most operators in $(\Delta_S - A_{ss})$ it is still more or less straightforward to find the generalized inverse operator, but the pressure variable in the spatial state causes an extra boundary value problem, which will be investigated in section 3.5.

Just like in the previous chapter, we want to give a definition for the spatial vector x_s. This will be pretty complicated for the upper half plane if we compare it with the case of a plane in the previous chapter. First we repeat the equation of the spatial state:

$$\begin{aligned} \Delta_S x_s &= A_{st}x_t + A_{ss}x_s + B_{sd}d \qquad \Rightarrow \\ (\Delta_S - A_{ss})x_s &= A_{st}x_t + B_{sd}d \end{aligned} \tag{2.23}$$

The right hand side of the latter equation, $A_{st}x_t + B_{sd}d$, must be an element of the image of $(\Delta_S - A_{ss})$, this requirement assures that $(\Delta_S - A_{ss})$ is surjective. The following equation must hold:

$$A_{st}x_t + B_{sd}d \;\in\; im\ (\Delta_S - A_{ss}) \tag{2.24}$$

Variable d corresponds with the disturbances and is therefore a free input vector. If $d = 0$, we have:

$$A_{st}x_t \;\in\; im\ (\Delta_S - A_{ss}) \tag{2.25}$$

Because of linearity, we may directly conclude that:

$$B_{sd}d \;\in\; im\ (\Delta_S - A_{ss}) \tag{2.26}$$

Since variable d is free to choose, it can not depend on d whether or not $B_{sd}d$ is an element of the image of $(\Delta_S - A_{ss})$. As a consequence of this property, we get the following:

$$im\ B_{sd} \;\in\; im\ (\Delta_S - A_{ss}) \tag{2.27}$$

Problems with surjectivity arise when the downward shift operator S_d is involved, as we have seen earlier in this section. Suppose we have the following equation:

$$(S_d x_1)(t, s_1, s_2) \;=\; x_2(t, s_1, s_2), \qquad s_2 \geq 0 \tag{2.28}$$

According to the definition of S_d in equation (2.7), this map is not surjective. By requiring that variable x_2 is zero for $s_2 = 0$, we can handle this problem. One way to formulate this requirement, is by using the definition of $S_d S_u$ in equation (2.6), then we get:

$$((I - S_d S_u)x_2)(t, s_1, s_2) = 0 \quad \Rightarrow \quad x_2(t, s_1, 0) = 0 \tag{2.29}$$

Next we investigate the problems which arise when a system equation contains an upward shift operator S_u. As we have seen already, this operator is not injective. Suppose we have the following equation:

$$(S_u x_1)(t, s_1, s_2) \quad = \quad x_2(t, s_1, s_2), \qquad s_2 \geq 0 \tag{2.30}$$

To deal with the injectivity at the boundary, we once again use operator Λ, this operator is used to fill in the degree of freedom we have at the boundary. If we have the equation in (2.30), then there exists an c_f such that:

$$x_1(t, s_1, s_2) \quad = \quad (S_d x_2)(t, s_1, s_2) + (\Lambda c_f)(t, s_1, s_2), \qquad s_2 \geq 0 \tag{2.31}$$

Note that our definition for Λ is probably not sufficient to define a solution for x_s, since Λ is only defined for an equation like $S_u x_1 = x_2$, while $(\Delta_S - A_{ss})$ can also contain equations like:

$$(S_u - 1)x_s \quad = \quad x_t \tag{2.32}$$

It is not clear how we have to fill in the degree of freedom in this equation. Another reason that we cannot stick to Λ is that it is not always possible to use its definition in our staggered grid, which we will introduce in the following chapter. Therefore, it is useful to extend the definition of Λ by introducing a new operator $\tilde{\Lambda}$ which describes this degree of freedom for all equations. This $\tilde{\Lambda}$ is an abstract operator with the following property:

$$ker\ (\Delta_S - A_{ss}) \quad = \quad im\ \tilde{\Lambda} \tag{2.33}$$

Then there exists an input vector c such that:

$$x_s = (\Delta_S - A_{ss})^{\dagger}[A_{st} x_t + B_{sd} d] + \tilde{\Lambda} c \tag{2.34}$$

2.3 Existence and uniqueness of a solution

With the definitions stated in the previous section, we can investigate the solution for the state x_t for our system in a half plane. First we repeat the equations of our interconnected system:

$$\begin{aligned} \begin{bmatrix} \dot{x}_t(t,\mathbf{s}) \\ (\Delta_S x_s(t))(\mathbf{s}) \\ y(t,\mathbf{s}) \end{bmatrix} &= \begin{bmatrix} A_{tt} & A_{ts} & B_{td} \\ A_{st} & A_{ss} & B_{sd} \\ C_{yt} & C_{ys} & D_{yd} \end{bmatrix} \begin{bmatrix} x_t(t,\mathbf{s}) \\ x_s(t,\mathbf{s}) \\ d(t,\mathbf{s}) \end{bmatrix} \\ x_t(0,\mathbf{s}) &= x_{t,0}(\mathbf{s}) \end{aligned} \tag{2.35}$$

Note that the boundary conditions are omitted in (2.35). These conditions will be investigated in chapter 3 and will depend on the staggered grid which we will introduce in that chapter. As we have seen in the previous section, the requirement

$$A_{st}x_t + B_{sd}d \quad \in \quad im\ (\Delta_S - A_{ss}) \tag{2.36}$$

handles problems with surjectivity caused by operator S_d and assures existence of a solution. The requirement

$$ker\ (\Delta_S - A_{ss}) \quad = \quad im\ \tilde{\Lambda} \tag{2.37}$$

deals with the injectivity caused by operator S_u and assures uniqueness of the solution. We can eliminate the spatial vector x_s by substituting of this vector in the temporal state and output; our interconnected system then becomes:

$$\begin{aligned} \dot{x}_t(t,\mathbf{s}) &= \mathbf{A}_x x_t(t,\mathbf{s}) + \mathbf{A}_a c(t,\mathbf{s}) + \mathbf{B}_d d(t,\mathbf{s}) \\ y(t,\mathbf{s}) &= \mathbf{C}_x x_t(t,\mathbf{s}) + \mathbf{C}_a c(t,\mathbf{s}) + \mathbf{D}_d d(t,\mathbf{s}) \end{aligned} \tag{2.38}$$

where

$$\begin{aligned} \begin{bmatrix} \mathbf{A}_x & \mathbf{A}_a & \mathbf{B}_d \\ \mathbf{C}_x & \mathbf{C}_a & \mathbf{D}_d \end{bmatrix} &:= \begin{bmatrix} A_{tt} & A_{ts}\tilde{\Lambda} & B_{td} \\ C_{yt} & C_{ys}\tilde{\Lambda} & D_{yd} \end{bmatrix} \\ &\quad + \begin{bmatrix} A_{ts} \\ C_{ys} \end{bmatrix} \times (\Delta_S - A_{ss})^{\dagger} \begin{bmatrix} A_{st} & 0 & B_{sd} \end{bmatrix} \end{aligned} \tag{2.39}$$

The requirement for injectivity causes an operator $\tilde{\Lambda}$ which appears explicitly in the equations for the solution, while we cannot directly see how we adapted the equations to handle with the surjectivity; but system matrices like A_{st} and A_{ss} may contain shift operators.

We mention once again that the user defined input vector at the boundary is called c, like in equation (2.34). Then the solution for x_t is:

$$x_t(t, \mathbf{s}) = e^{\mathbf{A}_x t} x_t(0, \mathbf{s}) + \int_0^t e^{\mathbf{A}_x (t-\tau)} (\mathbf{A}_a c(\tau, \mathbf{s}) + \mathbf{B}_d d(\tau, \mathbf{s})) d\tau \qquad (2.40)$$

Chapter 3

Navier-Stokes equations

In this chapter a flow over a flat solid wall is considered. A part of this wall, which corresponds with the surface of the wing of an airplane [10], is equipped with actuators and sensors. The actuators are in fact blowing and suction devices which are used to control the flow over the wall, these actuators enter the system as input vector c. This vector contains velocity and pressure components. The sensors measure data, and enter the system as output vector y. It is up to the user to compose this vector via the output equation of the interconnected system. The Navier-Stokes equations for our flow will be casted into the form of a spatially invariant interconnected system. In section 3.1 the linearized incompressible Navier-Stokes equations for the two-dimensional case are introduced and a discretization of these equations is made. In section 3.2 we will work out the discretization and cast the Navier-Stokes equations into the form of an interconnected system. We will make a distinction between two cases; one in which we use vertical shift operators in the equations and one in which we do not. In section 3.3 we use both the horizontal and vertical shift operators in the equations and take a look at the boundary conditions at the bottom wall. In subsection 3.3.1 is shortly outlined how the system looks like if we restrict ourselves to the horizontal shift operators. In section 3.4 we summarize the boundary conditions defined in the previous section and investigate which conditions are still necessary to guarantee existence and uniqueness of a solution to the problem. In section 3.5 we investigate once more the well-posedness of the system. In subsection 3.5.1 we look at the uniqueness of the solution, in 3.5.2 we consider existence. It will become clear that the continuity equation plays an important role.

3.1 Discretization of the Navier-Stokes equations

In the interest of clarity, we will start once again with the general form of the state-space representation of an interconnected system:

$$\begin{aligned}
\begin{bmatrix} \dot{x}_t(t,\mathbf{s}) \\ (\Delta_S x_s(t))(\mathbf{s}) \\ y(t,\mathbf{s}) \end{bmatrix} &= \begin{bmatrix} A_{tt} & A_{ts} & B_{td} \\ A_{st} & A_{ss} & B_{sd} \\ C_{yt} & C_{ys} & D_{yd} \end{bmatrix} \begin{bmatrix} x_t(t,\mathbf{s}) \\ x_s(t,\mathbf{s}) \\ d(t,\mathbf{s}) \end{bmatrix} \\
x_t(0,\mathbf{s}) &= x_{t,0}(\mathbf{s})
\end{aligned} \tag{3.1}$$

The linearized incompressible Navier-Stokes equations for the two-dimensional case are defined as follows:

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
\dot{u} + U\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial y} - \nu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + \frac{1}{\rho}\frac{\partial p}{\partial x} &= 0 \\
\dot{v} + U\frac{\partial v}{\partial x} + V\frac{\partial v}{\partial y} - \nu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) + \frac{1}{\rho}\frac{\partial p}{\partial y} &= 0
\end{aligned} \tag{3.2}$$

Some problems with Navier-Stokes equations can be found in [7]. To be able to solve the problem governed by these partial differential equations [5], we still need to define some boundary conditions. At a part of the bottom wall, these conditions are determined by the blowing and suction devices, this part is investigated in section 3.3. A natural way to define boundary conditions at the left, right and top is to cut off the half plane somewhere, at a (very) large distance from the devices, and take constant values for u and v at this 'virtual' boundary. This is necessary to be able to make computations in programs like Matlab. A more detailed explanation can be found in section 3.4. The first equation of (3.2) is the continuity equation, the second and third are called the momentum equations. In these equations, u is the velocity in horizontal (x) direction, v is the velocity in vertical (y) direction, ν is the kinematic viscosity, which is equal to the viscosity μ divided by the air density ρ, p is the pressure and U and V are the average velocities in horizontal and vertical direction, respectively. The equations are incompressible, which implies that the air density ρ is constant.

We make a discretization of the Navier-Stokes equations by putting them in a rectangular grid. When the central differences are used for the Navier-Stokes

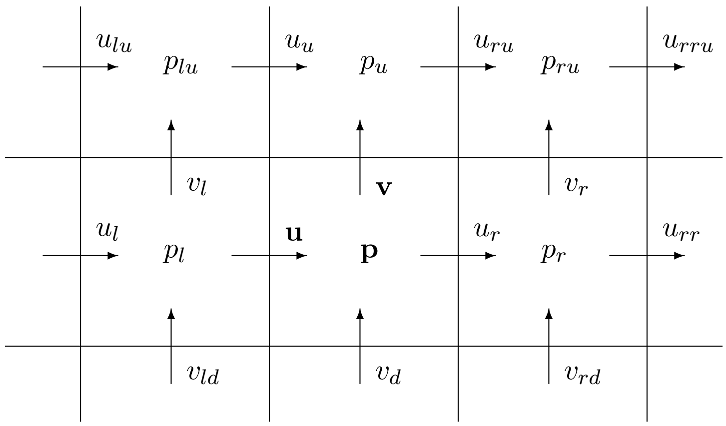

Figure 3.1: *Part of a staggered grid of an interconnected system in which both horizontal and vertical shifts are used*

equations, pressure oscillations can possibly occur if all three variables u, v and p are evaluated at the same grid points [1], [11]. To avoid these kind of problems, we will use a staggered grid. In a staggered grid, the pressure p is calculated at the center of a cell, the horizontal velocity u is evaluated in the midpoint of a vertical cell edge and the vertical velocity v is computed in the midpoint of a horizontal cell edge. In Figure 3.1 a part of a staggered grid is depicted. We used the spatial notation, the variables are renamed according to their spatial position with respect to the central subsystem. This is the subsystem which contains the bold variables u, v and p.

Before we define the spatial derivatives, we make an important assumption which we will use in the remaining of the report. We assume that the grid sizes in both horizontal and vertical direction, Δx and Δy respectively, are equal to 1. This assumption does not harm the structure of the problem, while it simplifies the equations.

We also assume that the (horizontal) grid size is equal to the distance between two adjacent devices. This is an important assumption, because it prevents unnecessarily complicated boundary conditions. The vertical velocity at the boundary is measured at the end of the devices.

We first give two definitions of the spatial partial derivative at position (x,y), which corresponds with the spatial coordinates (s_1,s_2). In the example mentioned below the derivative of the horizontal velocity u is taken in horizontal direction x:

$$(\frac{\partial u}{\partial x})(x,y) = \frac{u(x+1,y)-u(x-1,y)}{2} \tag{3.3}$$
$$(\frac{\partial u}{\partial x})(x,y) = u(x+1,y)-u(x,y) \tag{3.4}$$

The reason we need two definitions for the derivative is a consequence of the staggered grid. Consider again Figure 3.1 in which our staggered grid is depicted. Suppose we are interested in the point with variable **u**, and we need to calculate the derivative of the horizontal velocity u in horizontal direction in this point. We compute this derivative via variables u_l and u_r, and need the definition in (3.3) because the variables $u(x+1,y)$ and $u(x-1,y)$ differ two spatial steps, just like u_l and u_r do. If we would like to know the derivative of the pressure in horizontal direction in the same point, we calculate this derivative via variables p and p_l and need definition (3.4), because there is only one spatial step between p and p_l. For the second derivative, one definition is sufficient. Again, we look at the derivative of the horizontal velocity in horizontal direction:

$$(\frac{\partial^2 u}{\partial x^2})(x,y) = u(x+1,y)-2u(x,y)+u(x-1,y) \tag{3.5}$$

The derivatives for the vertical velocity and the pressure are defined in a similar way, we have to take the properties of the staggered grid into account.

3.2 State-space equations

For the axis hold $x = s_1\Delta x = s_1$ and $y = s_2\Delta y = s_2$, these equations connect the continuous coordinates x and y to the discrete coordinates s_1 and s_2, respectively. Discrete coordinate s_1 corresponds with horizontal direction x and s_2 presents the vertical direction y. We mention once again that we have to take our staggered grid into account when we compose the equations. Substitution of the derivatives in the Navier-Stokes equations of (3.2) results in:

$$
\begin{aligned}
0 &= u(t,s_1+1,s_2) - u(t,s_1,s_2) + v(t,s_1,s_2) - v(t,s_1,s_2-1) \\
\dot{u} &= -\frac{1}{\rho}(p(t,s_1,s_2) - p(t,s_1-1,s_2)) + R_u \\
\dot{v} &= -\frac{1}{\rho}(p(t,s_1,s_2+1) - p(t,s_1,s_2)) + R_v
\end{aligned} \tag{3.6}
$$

where

$$
\begin{aligned}
R_u &= -\frac{1}{2}U(u(t,s_1+1,s_2) - u(t,s_1-1,s_2)) - \frac{1}{2}V(u(t,s_1,s_2+1) - u(t,s_1,s_2-1)) \\
&+ \nu(u(t,s_1+1,s_2) - 2u(t,s_1,s_2) + u(t,s_1-1,s_2)) \\
&+ \nu(u(t,s_1,s_2+1) - 2u(t,s_1,s_2) + u(t,s_1,s_2-1))
\end{aligned}
$$

$$
\begin{aligned}
R_v &= -\frac{1}{2}U(v(t,s_1+1,s_2) - v(t,s_1-1,s_2)) - \frac{1}{2}V(v(t,s_1,s_2+1) - v(t,s_1,s_2-1)) \\
&+ \nu(v(t,s_1+1,s_2) - 2v(t,s_1,s_2) + v(t,s_1-1,s_2)) \\
&+ \nu(v(t,s_1,s_2+1) - 2v(t,s_1,s_2) + v(t,s_1,s_2-1))
\end{aligned} \tag{3.7}
$$

With the Navier-Stokes equations in (3.6) and the shift operators the state-space equations can be constructed. We will make a distinction between two cases; one in which we will *not* use the vertical shift operators S_u and S_d, and one in which we *will* use these operators together with the horizontal shift operators. In this report we are interested in a system in which both the horizontal and vertical shift operators are used, because in this case we take full advantage of the spatial structure of the problem and decrease the system's dimension significantly. That is why we will focus us on this situation. In 3.3.1 the case in which the vertical shifts are not used, like in [10], will shortly be outlined.

3.3 Vertical shift operator in state-space equations

Because we use both the horizontal and vertical shift operators in this section, we can use the spatial notation of the variables in Figure 3.1. However, we will start in the state-space equations with the shift operators S_r, S_l, S_u and S_d. Note that by using vertical shift operators in the equations, they generally do not hold at the boundary. Because of this, we have to be careful if we use the shift or the spatial notation. For example, u_d is not the same as $S_d u$, because $u_d(t, s_1, 0)$ is in general unequal to zero, while $(S_d u)(t, s_1, 0)$ *is* zero. First we consider the equations in the interior of the half plane, so S_d and S_u can be used without restrictions. Later in this section we will adapt the equations of the temporal and spatial state such that they also hold at the boundary. We will use the definitions of the previous chapter to obtain correct equations at the boundary. While doing this, the boundary conditions at the bottom wall will be defined.

The state-space equations (3.6) in spatial shift form are written as follows. In the interest of clarity, the temporal and spatial coordinates are left out:

$$
\begin{aligned}
0 &= S_r u - u + v - S_d v && (3.8)\\
\dot{u} &= (\nu - \frac{U}{2})S_r u + (\nu + \frac{U}{2})S_l u + (\nu - \frac{V}{2})S_u u + (\nu + \frac{V}{2})S_d u \\
&\quad -4\nu u - \frac{1}{\rho}p + \frac{1}{\rho}S_l p && (3.9)\\
\dot{v} &= (\nu - \frac{U}{2})S_r v + (\nu + \frac{U}{2})S_l v + (\nu - \frac{V}{2})S_u v + (\nu + \frac{V}{2})S_d v \\
&\quad -4\nu v - \frac{1}{\rho}S_u p + \frac{1}{\rho}p && (3.10)
\end{aligned}
$$

First we consider the continuity equation in (3.8). This equation is a consequence of the incompressible flow: the volume of every cell is constant, in other words the incoming flow is equal to the outgoing flow. Because of the blowing and suction devices at the boundary, it is not clear if the equation holds for $s_2 = 0$. Since we only consider the equations at the part of the half plane where we do not have restrictions for the vertical shift operators, that is for $s_2 > 0$, we can stick to the definition in equation (3.8) at this stage. Later we will investigate if this equation has to be adapted in order to satisfy the boundary conditions.

The momentum equations in (3.9) and (3.10) are part of the temporal state. Our interconnected system is defined in such a way, that shift operators do *not* appear in the temporal state. Therefore, we will omit the shift operators in the equations and use the spatial notation, like we did in the staggered grid in Figure 3.1. For example, $S_r u$ will be written as u_r, this will be done later in this section. As a consequence, the modifications made in the model in order to take the boundary into account, will be part of the spatial state of the interconnected system, since we definitely need shift operators to get a correct model.

Now we are going to implement the Navier-Stokes equations into the interconnected system. For the second and third Navier-Stokes equation, the momentum equations, this is more or less straightforward, because the velocities u and v together form the temporal vector x_t. We now have equations for $\dot{u}$ and $\dot{v}$, but lack one for the pressure p. We have to use the only equation left to construct an equation for p, which is the continuity equation. However, variable p itself is not a part of this equation. We can make p 'appear' by taking the derivative of the continuity equation and then substitute the equations for $\dot{u}$ and $\dot{v}$ in the new equation. The derivative of the continuity equation is:

$$S_r\dot{u} - \dot{u} + \dot{v} - S_d\dot{v} \quad = \quad 0 \tag{3.11}$$

Substitution of $\dot{u}$ and $\dot{v}$ gives:

$$\begin{aligned} 0 \quad = \quad & (\nu - \frac{U}{2})S_r^2 u + (\nu + \frac{U}{2})u + (\nu - \frac{V}{2})S_r S_u u \\ & +(\nu + \frac{V}{2})S_r S_d u - 4\nu S_r u - \frac{1}{\rho}S_r p + \frac{1}{\rho}p \\ & -(\nu - \frac{U}{2})S_r u - (\nu + \frac{U}{2})S_l u - (\nu - \frac{V}{2})S_u u \\ & -(\nu + \frac{V}{2})S_d u - 4\nu u + \frac{1}{\rho}p - \frac{1}{\rho}S_l p \\ & +(\nu - \frac{U}{2})S_r v + (\nu + \frac{U}{2})S_l v + (\nu - \frac{V}{2})S_u v \\ & +(\nu + \frac{V}{2})S_d v - 4\nu v - \frac{1}{\rho}S_u p + \frac{1}{\rho}p \\ & -(\nu - \frac{U}{2})S_r S_d v - (\nu + \frac{U}{2})S_l S_d v - (\nu - \frac{V}{2})v \\ & -(\nu + \frac{V}{2})S_d^2 v + 4\nu S_d v + \frac{1}{\rho}p - \frac{1}{\rho}S_d p \end{aligned} \tag{3.12}$$

This equation can be rewritten slightly easier:

$$
\begin{aligned}
0 \quad = \quad & (5\nu + \frac{U}{2})u + (-5\nu + \frac{U}{2})S_r u + (-\nu - \frac{U}{2})S_l u + (-\nu + \frac{V}{2})S_u u \\
& +(-\nu - \frac{V}{2})S_d u + (\nu - \frac{U}{2})S_r^2 u + (\nu - \frac{V}{2})S_r S_u u + (\nu + \frac{V}{2})S_r S_d u \\
& +(-5\nu + \frac{V}{2})v + (\nu - \frac{U}{2})S_r v + (\nu + \frac{U}{2})S_l v + (\nu - \frac{U}{2})S_u v \\
& +(5\nu + \frac{V}{2})S_d v + (-\nu + \frac{U}{2})S_r S_d v + (-\nu - \frac{U}{2})S_l S_d v + (\nu - \frac{V}{2})S_d^2 v + \\
& +\frac{4}{\rho}p - \frac{1}{\rho}S_r p - \frac{1}{\rho}S_l p - \frac{1}{\rho}S_u p - \frac{1}{\rho}S_d p
\end{aligned}
\tag{3.13}
$$

We mention once again that at this stage equation (3.13) is only considered on the part of the half plane where we do not have to deal with boundary conditions. In this case all shift operators in the equation are allowed and not only operators $S_l S_r$, but also $S_d S_u$ can be left out, because these operators are equal to the identity operator. However, eventually we have to use the spatial notation in (3.13) to satisfy the constraints of our model, which only accepts shift operators in Δ_S. We define some symbols for the constant values in (3.13):

$$
\begin{aligned}
\alpha &:= \nu + \tfrac{U}{2} & \theta &:= 5\nu + \frac{U}{2} \\
\beta &:= \nu - \tfrac{U}{2} & \lambda &:= -5\nu + \frac{U}{2} \\
\gamma &:= \nu + \tfrac{V}{2} & \tau &:= 5\nu + \frac{V}{2} \\
\delta &:= \nu - \tfrac{V}{2} & \varphi &:= -5\nu + \frac{V}{2}
\end{aligned}
\tag{3.14}
$$

Note that the unit of these values equals s^{-1}. The grid sizes Δx and Δy are not a component of the equations, because their magnitude was already fixed to 1 in section 3.1. With use of the symbols in equations (3.14) and the spatial notation the variables of u, v and p depicted in the staggered grid in Figure 3.1, the equation looks little less complicated:

$$
\begin{aligned}
0 &= \theta u + \lambda u_r - \alpha u_l - \delta u_u - \gamma u_d + \beta u_{rr} + \delta u_{ru} + \gamma u_{rd} \\
&+ \varphi v + \beta v_r + \alpha v_l + \beta v_u + \tau v_d - \beta v_{rd} - \alpha v_{ld} + \delta v_{dd} \\
&+ \frac{4}{\rho} p - \frac{1}{\rho} p_r - \frac{1}{\rho} p_l - \frac{1}{\rho} p_u - \frac{1}{\rho} p_d
\end{aligned}
\tag{3.15}
$$

By adding equation (3.15) to the interconnected system, the spatial vector x_s becomes large. Next to the pressure p and the variables of the adjacent cells of u, v and p, the vector now also contains three extra variables of horizontal velocities, u_{rr}, u_{ru} and u_{rd}, and three of vertical velocities, v_{rd}, v_{ld} and v_{dd}. Every variable has to be shared by the system and therefore needs its 'own' equation in the spatial state. We will choose to keep Δ_S first order, so the variables of indirect neighbors are connected with those of direct neighbors via A_{ss}. Equation (3.15) can be added to the system by writing it as a function for, for example, p_d. This variable will be written as $S_d p$ in the left-hand side of the equation. We mention once again that this equation will appear in the spatial state, since it is not a differential equation.

$$
\begin{aligned}
S_d p &= \rho \{\theta u + \lambda u_r - \alpha u_l - \delta u_u - \gamma u_d + \beta u_{rr} + \delta u_{ru} + \gamma u_{rd}\} \\
&+ \rho \{\varphi v + \beta v_r + \alpha v_l + \beta v_u + \tau v_d - \beta v_{rd} - \alpha v_{ld} + \delta v_{dd}\} \\
&+ 4p - p_u - p_r - p_l
\end{aligned}
\tag{3.16}
$$

Note that variable p_d itself is not a part of x_s. We give all the equations of the temporal and spatial state. These are the following for the temporal state if we use the symbols in (3.14) and the spatial notation of the staggered grid:

$$
\begin{aligned}
\dot{u} &= -4\nu u + \alpha u_l + \beta u_r + \gamma u_d + \delta u_u + \frac{1}{\rho} p_l - \frac{1}{\rho} p \\
\dot{v} &= -4\nu v + \alpha v_l + \beta v_r + \gamma v_d + \delta v_u + \frac{1}{\rho} p - \frac{1}{\rho} p_u
\end{aligned}
\tag{3.17}
$$

And for the spatial state:

$$
\begin{array}{lllllllll}
S_r u_l & = & u & S_r v_l & = & v & S_d p & = & (3.16) \\
S_l u_r & = & u & S_l v_r & = & v & S_r p_l & = & p \\
S_u u_d & = & u & S_u v_d & = & v & S_l p_r & = & p \\
S_d u_u & = & u & S_d v_u & = & v & S_d p_u & = & p \\
S_l u_{rr} & = & u_r & S_r v_{ld} & = & v_d & & & \\
S_u u_{rd} & = & u_r & S_l v_{rd} & = & v_d & & & \\
S_d u_{ru} & = & u_r & S_u v_{dd} & = & v_d & & &
\end{array}
\tag{3.18}
$$

The spatial vector x_s is ordered in the same way as the equations above:

$$
\begin{array}{lll}
x_s & = & \big(\; u_l \quad u_r \quad u_d \quad u_u \quad u_{rr} \quad u_{rd} \quad u_{ru} \quad v_l \quad v_r \quad v_d \quad v_u \quad v_{ld} \quad v_{rd} \quad v_{dd} \\
 & & \quad p \quad p_l \quad p_r \quad p_u \; \big)
\end{array}
\tag{3.19}
$$

We now investigate how the definitions of the previous chapter have influence on the equations of the spatial state in (3.18), in particular at or close to the boundary. It will be necessary to adapt some of the equations, this depends on the way we define the boundary conditions. In Figure 3.2 an interconnected system is depicted with vertical spatial coordinate $s_2 = 0$, so that we can see which variables need to be defined at the boundary. We mention once again that the measure points of the vertical velocity at the boundary lie at the end of the blowing and suction devices. We repeat the conditions of the previous chapter to assure surjectivity and injectivity, respectively:

$$
A_{st} x_t + B_{sd} d \;\; \in \;\; im \; (\Delta_S - A_{ss}) \tag{3.20}
$$

$$
ker \; (\Delta_S - A_{ss}) \;\; = \;\; im \; \tilde{\Lambda} \tag{3.21}
$$

We will start with the seven equations in which the horizontal velocity u is involved. There are no restrictions for the first two equations, because $u_l(t, s_1, s_2) = u(t, s_1 - 1, s_2)$ and $u_r(t, s_1, s_2) = u(t, s_1 + 1, s_2)$ for all t, s_1, s_2:

$$
\begin{array}{c}
S_r u_l = (S_r u_l)(t, s_1, s_2) = u_l(t, s_1 + 1, s_2) = u(t, s_1, s_2) = u \\
S_l u_r = (S_l u_r)(t, s_1, s_2) = u_r(t, s_1 - 1, s_2) = u(t, s_1, s_2) = u
\end{array}
\tag{3.22}
$$

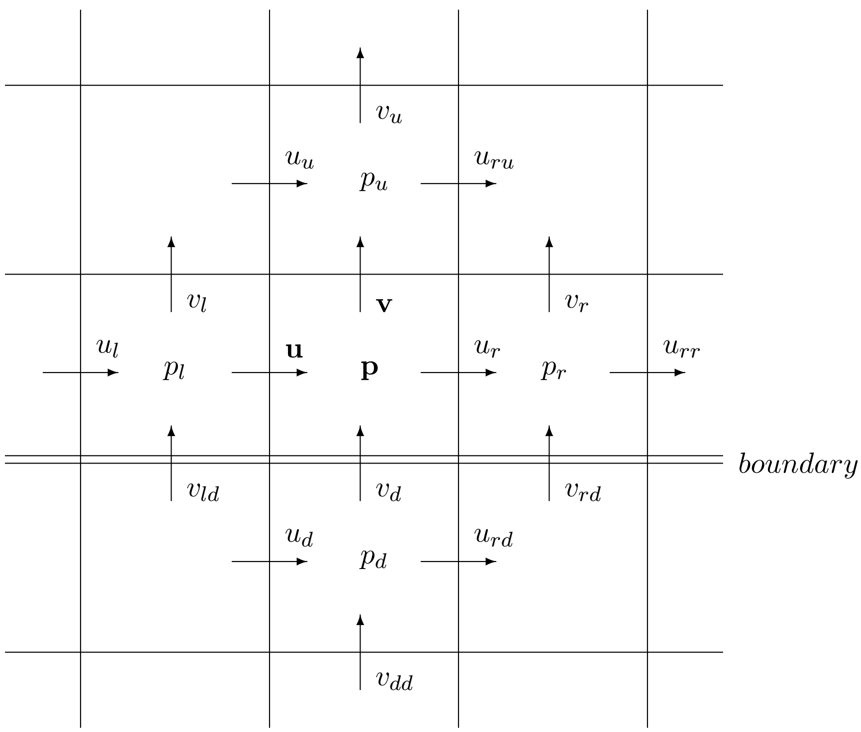

Figure 3.2: *All variables of a subsystem with $s_2 = 0$ of an interconnected system in a staggered grid, in which both horizontal and vertical shifts are used*

This means that the first two equations remain unchanged. Consider the third equation, $S_u u_d = u$. According to the definition for S_u in equation (2.13), we know that

$$\begin{aligned} (S_u u_d)(t, s_1, s_2) &= u(t, s_1, s_2) \qquad \Rightarrow \\ u_d(t, s_1, s_2) &= (S_d u)(t, s_1, s_2) + (\Lambda c_f)(t, s_1, s_2) \end{aligned} \tag{3.23}$$

for some variable c_f. This variable c_f corresponds in this equation with the value of $u_d(t, s_1, 0)$ and is determined by the boundary condition for the horizontal velocity. Note that we have to take the staggered grid into account for this definition. If the horizontal velocity at the boundary equals $c_u(t, s_1)$, then we have that:

$$u_d(t, s_1, 0) = 2c_u(t, s_1) - u(t, s_1, 0) \tag{3.24}$$

Hereby we assume that $c_u(t, s_1)$ is the average of $u_d(t, s_1, 0)$ and $u(t, s_1, 0)$. Note that the definition of Λ is not sufficient to fill in the degree of freedom for this equation, we also need the one for $(I - S_d S_u)$. This is a consequence of the way our staggered grid is composed; because $u_d(t, s_1, 0)$ is not measured at the boundary, we need variable $u(t, s_1, 0)$ to define this boundary condition, and we cannot use Λ to make this variable part of the definition. We get:

$$\begin{aligned} &(S_u u_d)(t, s_1, s_2) = u(t, s_1, s_2) \qquad \Rightarrow \\ &u_d(t, s_1, s_2) = (S_d u)(t, s_1, s_2) + 2(\Lambda c_u)(t, s_1, s_2) - ((I - S_d S_u)u)(t, s_1, s_2) \end{aligned} \tag{3.25}$$

We added s_2 to the coordinates of operation Λc_u, since this is necessary for operator Λ, see the definition in (2.12). Consider variable $c_u(t, s_1)$; the magnitude of this variable is closely related to the composition of our staggered grid. We already assumed that its value is the average of $u_d(t, s_1, 0)$ and $u(t, s_1, 0)$. If these velocities were measured just above and beneath the blowing and suction devices, like is done for the pressure in our staggered grid (see Figure 3.2), it would be plausible to set $c_u(t, s_1)$ to zero for all t, s_1. But because of the way we composed our staggered grid, the values of $c_u(t, s_1)$ are determined right between two adjacent devices, so we probably do *not* have a no-slip condition. Note that for this reason it is maybe not convenient to use input variable c for the horizontal velocity at the boundary, since it is not instantly determined by the devices like is done for the vertical velocity and the pressure, but we will leave it this way.

Now we take a look at the fourth equation, which is $S_d u_u = u$. According to equation (2.7), we get the following equation for u:

$$\begin{aligned} (S_d u_u)(t, s_1, s_2) &= u(t, s_1, s_2) \qquad \Rightarrow \\ u(t, s_1, s_2) &= \begin{cases} u_u(t, s_1, s_2 - 1), & s_2 > 0 \\ 0, & s_2 = 0 \end{cases} \end{aligned} \tag{3.26}$$

In this case operator S_d causes an implicit boundary condition, namely $u(t, s_1, 0) = 0$. If we have this condition, there is no horizontal flow just above the boundary. For physical reasons, it is very unlikely that we do have a horizontal flow *at* the boundary in this situation, so we may assume that we have a no-slip condition. This means that $c_u(t, s_1) = 0$ for all t, s_1, and as a result also that $u_d(t, s_1, 0) = 0$ for all t, s_1. Because $u(t, s_1, 0) = 0$ for all t, s_1, the same holds for the derivative of the horizontal velocity, so we have that $\dot{u}(t, s_1, 0) = 0$. By the spatial state, we know that the following equations hold:

$$\begin{aligned} (S_r u_l)(t, s_1, 0) &= u(t, s_1, 0) \\ (S_l u_r)(t, s_1, 0) &= u(t, s_1, 0) \end{aligned} \tag{3.27}$$

We may conclude that $u_l(t, s_1, 0) = 0$ and $u_r(t, s_1, 0) = 0$ for all t, s_1. Consider again the equation for $\dot{u}$ in the temporal state in (3.17):

$$\dot{u} = -4\nu u + \alpha u_l + \beta u_r + \gamma u_d + \delta u_u + \frac{1}{\rho} p_l - \frac{1}{\rho} p \tag{3.28}$$

We investigate the consequences for this equation for $s_2 = 0$, if $u(t, s_1, 0) = 0$. We already concluded that $\dot{u}(t, s_1, 0)$, $u_l(t, s_1, 0)$, $u_r(t, s_1, 0)$ and $u_d(t, s_1, 0)$ are equal to zero in this situation. If $u(t, s_1, 0) = 0$ for all t, s_1, there is no pressure difference for $s_2 = 0$, and so $p(t, s_1, 0) = p_l(t, s_1, 0)$. If we substitute these values in (3.28), we simply get that:

$$u_u(t, s_1, 0) = 0 \tag{3.29}$$

By definition, we know that $u_u(t, s_1, 0)$ corresponds with $u(t, s_1, 1)$. But then we have that $u(t, s_1, 1) = 0$ for all t, s_1. By induction, $u(t, s_1, s_2) = 0$ for all t, s_1, s_2. This implies that there is no horizontal flow on the half plane. Because this situation is far from realistic, we may conclude that our current model cannot handle a no-slip condition at the boundary. The main reason for this failure is that the spatial discretization of the Navier-Stokes equations causes a loss of information, which has a large influence on the behavior at the boundary. To avoid problems with no-slip, we add operator $S_d S_u$ to the right hand side of the equation $S_d u_u = u$:

$$\begin{aligned} & (S_d u_u)(t, s_1, s_2) = (S_d S_u u)(t, s_1, s_2) \\ \Rightarrow \quad & u_u(t, s_1, s_2 - 1) = (S_u u)(t, s_1, s_2 - 1), \qquad s_2 > 0 \\ \Rightarrow \quad & u_u(t, s_1, s_2) = (S_u u)(t, s_1, s_2), \qquad s_2 \geq 0 \end{aligned} \tag{3.30}$$

Note that equation (3.30) is now equivalent with $u_u = S_u u$. By adding $S_d S_u$ to the right hand side, we have to accept that matrix A_{st} now contains shift operators. With this operator in A_{st}, the requirement in (3.20) is satisfied for this specific equation.

The fifth equation, $S_l u_{rr} = u_r$ is more or less similar with the second equation, $S_l u_r = u$, so we can keep this equation unchanged. Note that the equation $S_l u_{rr} = u_r$ in our interconnected system corresponds with the equation $S_l u_r = u$ in the system with horizontal spatial coordinate $s_1 + 1$:

$$(S_l u_{rr})(t, s_1, s_2) = u_r(t, s_1, s_2) \quad \Leftrightarrow \quad (S_l u_r)(t, s_1 + 1, s_2) = u(t, s_1 + 1, s_2) \tag{3.31}$$

The sixth equation is $S_u u_{rd} = u_r$. At the same way we treated equation $S_u u_d = u$, we get:

$$\begin{aligned} (S_u u_{rd})(t, s_1, s_2) &= u_r(t, s_1, s_2) \qquad \Rightarrow \\ u_{rd}(t, s_1, s_2) &= (S_d u_r)(t, s_1, s_2) + 2(\Lambda c_u)(t, s_1 + 1, s_2) \\ & \quad -((I - S_d S_u)u)(t, s_1 + 1, s_2) \end{aligned} \tag{3.32}$$

The definition of u_{rd} in the equation above has to be equivalent with the definition of u_d for the interconnected system one spatial step to the right. This is the case, because the following holds:

$$
\begin{aligned}
u_{rd}(t,s_1,s_2) &= (S_d u_r)(t,s_1,s_2) + 2(\Lambda c_u)(t,s_1+1,s_2) \\
&\quad -((I - S_d S_u)u)(t,s_1+1,s_2) \qquad \Leftrightarrow \\
u_d(t,s_1+1,s_2) &= (S_d u)(t,s_1+1,s_2) + 2(\Lambda c_u)(t,s_1+1,s_2) \\
&\quad -((I - S_d S_u)u)(t,s_1+1,s_2)
\end{aligned} \tag{3.33}
$$

The last equation for the horizontal velocity in the spatial state is $S_d u_{ru} = u_r$. Just like in the fourth equation, we avoid possible problems at the boundary by adding operator $S_d S_u$ to the right hand side of the equation:

$$
\begin{aligned}
&(S_d u_{ru})(t,s_1,s_2) = (S_d S_u u_r)(t,s_1,s_2) \qquad \Rightarrow \\
&u_{ru}(t,s_1,s_2-1) = (S_u u_r)(t,s_1,s_2-1), \qquad s_2 > 0 \qquad \Rightarrow \\
&u_{ru}(t,s_1,s_2) = (S_u u_r)(t,s_1,s_2), \qquad s_2 \geq 0
\end{aligned} \tag{3.34}
$$

This equation now corresponds with $u_{ru} = S_u u_r$. The way we adapted the definition causes an operator $S_d S_u$ in matrix A_{ss}. Equation $S_d u_{ru} = u_r$ corresponds with equation $S_d u_u = u$ in the interconnected system one step at the right of ours.

Now we take a look at the seven equations for the vertical velocity v. These equations will have pretty much the same properties as those of the horizontal velocity. We immediately see that we do not have to investigate the equations $S_r v_l = v$ and $S_l v_r = v$, since we do not have any restrictions for the horizontal shift operators. Consider the third equation, $S_u v_d = v$. By definition (2.13), we know that there exists a variable c_f such that:

$$
\begin{aligned}
(S_u v_d)(t,s_1,s_2) &= v(t,s_1,s_2) \qquad \Rightarrow \\
v_d(t,s_1,s_2) &= (S_d v)(t,s_1,s_2) + (\Lambda c_f)(t,s_1,s_2)
\end{aligned} \tag{3.35}
$$

According to Figure 3.2, variable c_f has to be equivalent with the value of v_d at the boundary, which is $v_d(t,s_1,0)$. The magnitude of this variable is measured just at the devices, and is therefore determined by the vertical component of the input vector, c_v. Then we get the following equation:

$$
v_d(t,s_1,s_2) = (S_d v)(t,s_1,s_2) + (\Lambda c_v)(t,s_1,s_2) \tag{3.36}
$$

The fourth equation, $S_d v_u = v$, implies that v is zero for $s_2 = 0$. We do not want this property, simply because it is not true in general. Therefore we use the definition for S_d in equation (2.9). We add operator $S_d S_u$ to the right hand side, to prevent problems at the boundary. This results in the following equation:

$$\begin{array}{rcl} (S_d v_u)(t, s_1, s_2) & = & (S_d S_u v)(t, s_1, s_2) \qquad \Rightarrow \\ v_u(t, s_1, s_2 - 1) & = & (S_u v)(t, s_1, s_2 - 1), \qquad s_2 > 0 \qquad \Rightarrow \\ v_u(t, s_1, s_2) & = & (S_u v)(t, s_1, s_2), \qquad s_2 \geq 0 \end{array} \tag{3.37}$$

This equation is equivalent with $v_u = S_u v$. Equation (3.37) causes another shift operator $S_d S_u$ in matrix A_{st}.

Consider the fifth and sixth equation of the vertical velocity, $S_r v_{ld} = v_d$ and $S_l v_{rd} = v_d$. By defining $v_{ld}(t, s_1, 0) := c_v(t, s_1 - 1, 0)$ and $v_{rd}(t, s_1, 0) := c_v(t, s_1 + 1, 0)$, these equations hold for all t, s_1, s_2 and need not to be adapted. The last equation which gives information about the vertical velocity, is $S_u v_{dd} = v_d$. By definition, we know there exists some variable c_f such that:

$$S_u v_{dd} = v_d \quad \Rightarrow \quad v_{dd} = S_d v_d + \Lambda c_f \tag{3.38}$$

We already noticed that $v_d(t, s_1, 0)$ corresponds with the vertical velocity component of the user defined input at the boundary, $c_v(t, s_1, 0)$. It is not clear yet how we can determine the value of $v_{dd}(t, s_1, 0)$, but we will call this variable $c_{vv}(t, s_1, 0)$, we assume that it depends on the input vector. Then we get the following equation:

$$\begin{array}{rcl} v_{dd}(t, s_1, s_2) & = & (S_d v_d)(t, s_1, s_2) + \Lambda c_{vv}(t, s_1, s_2) \\ & = & \left\{ \begin{array}{ll} v_d(t, s_1, s_2 - 1), & s_2 > 1 \\ c_v(t, s_1, 0), & s_2 = 1 \\ c_{vv}(t, s_1, 0), & s_2 = 0 \end{array} \right. \end{array} \tag{3.39}$$

Finally, we consider the four pressure equations in (3.18). The first equation is:

$$
\begin{aligned}
S_d p &= \rho\left\{\theta u + \lambda u_r - \alpha u_l - \delta u_u - \gamma u_d + \beta u_{rr} + \delta u_{ru} + \gamma u_{rd}\right\} \\
&+ \rho\left\{\varphi v + \beta v_r + \alpha v_l + \beta v_u + \tau v_d - \beta v_{rd} - \alpha v_{ld} + \delta v_{dd}\right\} \\
&+ 4p - p_u - p_r - p_l \qquad (3.40)
\end{aligned}
$$

This equation plays an important role in the determination whether or not our problem has a unique solution, it is investigated in section 3.5. We add operator $S_d S_u$ to all elements of the right hand side of the equation to prevent problems with surjectivity. The second and third pressure equation, $S_r p_l = p$ and $S_l p_r = p$, do not need special attention, because these equation hold for all t, s_1, s_2. In the fourth and last equation, $S_d p_u = p$, we can add $S_d S_u$ to the right hand side to prevent possible problems at the boundary. Then we get:

$$
\begin{aligned}
& (S_d p_u)(t, s_1, s_2) = (S_d S_u p)(t, s_1, s_2), \qquad s_2 \geq 0 \\
\Rightarrow\ & p_u(t, s_1, s_2 - 1) = (S_u p)(t, s_1, s_2 - 1), \qquad s_2 > 0 \\
\Rightarrow\ & p_u(t, s_1, s_2) = (S_u p)(t, s_1, s_2), \qquad s_2 \geq 0
\end{aligned} \qquad (3.41)
$$

All equations of the spatial state have now been adapted to the bottom wall:

$$
\begin{array}{lll lll}
S_r u_l & = & u & S_u v_d & = & v \\
S_l u_r & = & u & S_d v_u & = & S_d S_u v \\
S_u u_d & = & u & S_r v_{ld} & = & v_d \\
S_d u_u & = & S_d S_u u & S_l v_{rd} & = & v_d \\
S_l u_{rr} & = & u_r & S_u v_{dd} & = & v_d \\
S_u u_{rd} & = & u_r & S_d p & = & S_d S_u (3.16) \\
S_d u_{ru} & = & S_d S_u u_r & S_r p_l & = & p \\
S_r v_l & = & v & S_l p_r & = & p \\
S_l v_r & = & v & S_d p_u & = & S_d S_u p
\end{array} \qquad (3.42)
$$

The following boundary conditions have to be taken into account:

$$
\begin{array}{lll}
u_d(t, s_1, 0) = & 2c_u(t, s_1) - u(t, s_1, 0), & \forall t, s_1 \\
v_d(t, s_1, 0) = & c_v(t, s_1), & \forall t, s_1 \\
p_d(t, s_1, 0) = & 2c_p(t, s_1) - p(t, s_1, 0), & \forall t, s_1 \\
v_{dd}(t, s_1, 0) = & c_{vv}(t, s_1), & \forall t, s_1
\end{array} \qquad (3.43)
$$

Now we are also able to construct the matrices and operators of the temporal and spatial state of the interconnected system:

$$
\begin{aligned}
A_{tt} &= \begin{pmatrix} -4\nu & 0 \\ 0 & -4\nu \end{pmatrix} \\
A_{ts} &= \begin{pmatrix} \alpha & \beta & \gamma & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{\rho} & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & \beta & \gamma & \delta & 0 & 0 & 0 & \frac{1}{\rho} & 0 & 0 & \frac{-1}{\rho} \end{pmatrix} \\
A_{st} &= \begin{pmatrix} 1 & 1 & 1 & ** & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & **\rho\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & ** & 0 & 0 & 0 & **\rho\varphi & 0 & 0 & 0 \end{pmatrix}^T \\
A_{ss} &= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & ** & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ** & 0 & 0 & 0
\end{pmatrix}
\end{aligned}
\tag{3.44}
$$

The elements with $**$ correspond with operator $S_d S_u$. The fifteenth row of A_{ss} is equal to the following vector:

$$
\begin{aligned}
A_{ss}(15,\cdot) = S_d S_u \big(& -\rho\alpha \quad \rho\lambda \quad -\rho\gamma \quad -\rho\delta \quad \rho\beta \quad \rho\gamma \quad \rho\delta \quad \rho\alpha \quad \rho\beta \\
& \rho\tau \quad \rho\beta \quad -\rho\alpha \quad -\rho\beta \quad \rho\delta \quad 4 \quad -1 \quad -1 \quad -1 \big)
\end{aligned}
\tag{3.45}
$$

Note that at this stage the Navier-Stokes equations have already been casted into the temporal and spatial state of the interconnected system in (2.35).

In section 3.4 we will summarize all boundary conditions at that part of the bottom wall where we have devices and investigate how we have to deal with the left, right and top of the half plane, but first we shortly consider our problem when the vertical shift operators are left out.

3.3.1 Restriction to horizontal shift operator

In this section only the horizontal shift operators S_l and S_r are used. As a consequence, we need equations for every single 'row' of the half plane. Before we can give a discretization of the Navier-Stokes equations, the staggered grid has to be redefined because indices $_u$ and $_d$ cannot be used. That is why the row number is part of the index of every variable, see Figure 3.3. In other words, we have separate equations for all s_2.

Because the vertical shifts are not used in this case, we cannot restrict ourselves to the definition of an interconnected system around just one subsystem like in section 3.3, but have to define system equations for every single row. To be able to make computations, we have to cut off the system somewhere at the top, say at row $i = N$. In the case with the vertical shifts operators, the temporal and spatial state together consisted of only 20 variables, while in this case we need approximately $10N$ variables. So if N equals for example 1000, our temporal and spatial vector consist of more than 10,000 variables.

An advantage of the case without vertical shifts is that it is easier to implement the boundary conditions in the equations. We have to define the Navier-Stokes equations for every single row, in the equations for the two bottom rows the boundary conditions occur. Like in the previous section, we first present the equations for the interior, afterwards we investigate how the equations look at and close to the boundary.

$$\begin{aligned}
0 &= S_r u_i - u_i + v_i - v_{i-1} \\
\dot{u}_i &= (\nu - \frac{U}{2})S_r u_i + (\nu + \frac{U}{2})S_l u_i + (\nu - \frac{V}{2})u_{i+1} + (\nu + \frac{V}{2})u_{i-1} \\
&\quad -4\nu u_i - \frac{1}{\rho}p_i + \frac{1}{\rho}S_l p_i \\
\dot{v}_i &= (\nu - \frac{U}{2})S_r v_i + (\nu + \frac{U}{2})S_l v_i + (\nu - \frac{V}{2})v_{i+1} + (\nu + \frac{V}{2})v_{i-1} \\
&\quad -4\nu v_i - \frac{1}{\rho}p_{i+1} + \frac{1}{\rho}p_i
\end{aligned} \tag{3.46}$$

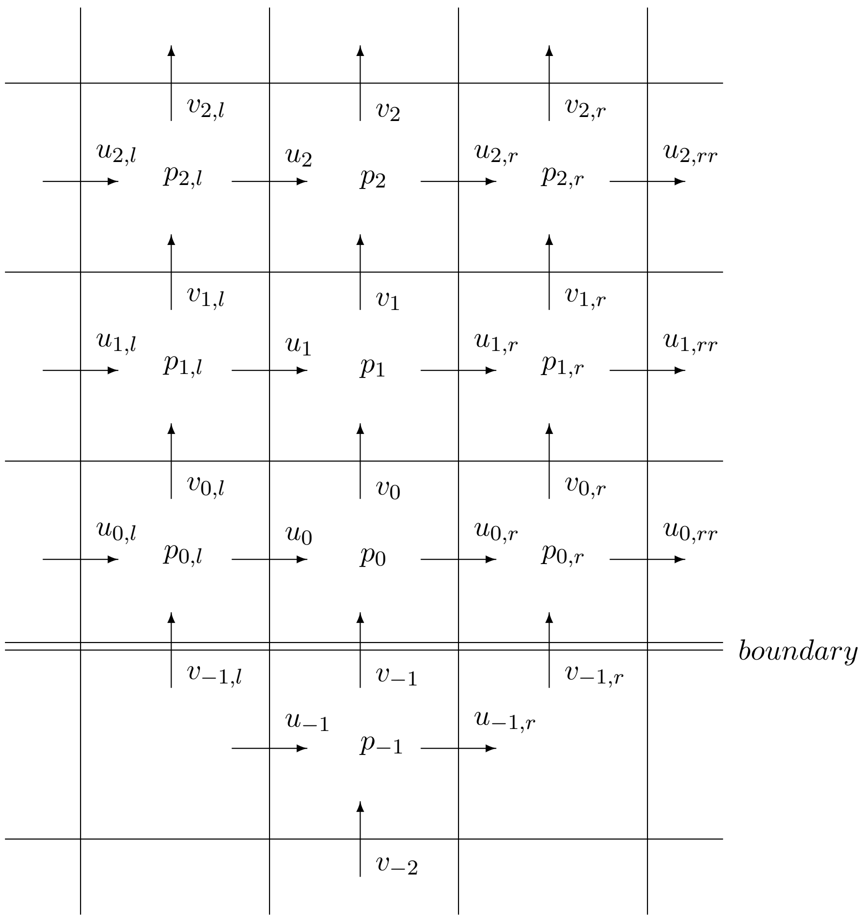

Figure 3.3: *Variables in the staggered grid of an interconnected system in which only horizontal shifts are used*

Just like in the 'vertical' case, the derivative of the continuity equation is used in order to get the pressure in the system. After substitution of $\dot{u}$ and $\dot{v}$ this equation becomes:

$$\begin{aligned}
0 \;=\; & (5\nu + \frac{U}{2})u_i + (-5\nu + \frac{U}{2})S_r u_i + (-\nu - \frac{U}{2})S_l u_i + (-\nu + \frac{V}{2})u_{i+1} \\
& +(-\nu - \frac{V}{2})u_{i-1} + (\nu - \frac{U}{2})S_r^2 u_i + (\nu - \frac{V}{2})S_r u_{i+1} + (\nu + \frac{V}{2})S_r u_{i-1} \\
& +(-5\nu + \frac{V}{2})v_i + (\nu - \frac{U}{2})S_r v_i + (\nu + \frac{U}{2})S_l v_i + (\nu - \frac{U}{2})v_{i+1} \\
& +(5\nu + \frac{V}{2})v_{i-1} + (-\nu + \frac{U}{2})S_r v_{i-1} + (-\nu - \frac{U}{2})S_l v_{i-1} + (\nu - \frac{V}{2})v_{i-2} + \\
& +\frac{4}{\rho}p_i - \frac{1}{\rho}S_r p_i - \frac{1}{\rho}S_l p_i - \frac{1}{\rho}p_{i+1} - \frac{1}{\rho}p_{i-1}
\end{aligned} \tag{3.47}$$

For the equations of the rows with $i = 0, 1$ we have to define some boundary conditions, which are equivalent with the case in which vertical shift operators were used. Note that we use indices -1 and -2 for the vertical spatial coordinate to fill in the boundary conditions, even if this might not be a convenient notation since we work in the upper half plane.

$$\begin{aligned}
u_{-1}(t, s_1) = &\; 2c_u(t, s_1) - u_0(t, s_1), && \forall t, s_1 \\
v_{-1}(t, s_1) = &\; c_v(t, s_1), && \forall t, s_1 \\
p_{-1}(t, s_1) = &\; 2c_p(t, s_1) - p_0(t, s_1), && \forall t, s_1 \\
v_{-2}(t, s_1) = &\; c_{vv}(t, s_1), && \forall t, s_1
\end{aligned} \tag{3.48}$$

Then the equations for $i = 0$ become:

$$
\begin{aligned}
\dot{u}_0 &= (\nu - \frac{U}{2})S_r u_0 + (\nu + \frac{U}{2})S_l u_0 + (\nu - \frac{V}{2})u_1 + (2\nu + V)c_u \\
&\quad +(-5\nu - \frac{V}{2})u_0 - \frac{1}{\rho}p_0 + \frac{1}{\rho}S_l p_0 \\
\dot{v}_0 &= (\nu - \frac{U}{2})S_r v_0 + (\nu + \frac{U}{2})S_l v_0 + (\nu - \frac{V}{2})v_1 + (\nu + \frac{V}{2})c_v \\
&\quad -4\nu v_0 - \frac{1}{\rho}p_1 + \frac{1}{\rho}p_0 \\
0 &= (6\nu + \frac{U}{2} + \frac{v}{2})u_0 + (-6\nu + \frac{U}{2} - \frac{V}{2})S_r u_0 + (-\nu - \frac{U}{2})S_l u_0 \\
&\quad +(-\nu + \frac{V}{2})u_1 + (\nu - \frac{U}{2})S_r^2 u_0 + (\nu - \frac{V}{2})S_r u_1 \\
&\quad +(-5\nu + \frac{V}{2})v_0 + (\nu - \frac{U}{2})S_r v_0 + (\nu + \frac{U}{2})S_l v_0 + (\nu - \frac{U}{2})v_1 \\
&\quad +(5\nu + \frac{V}{2})c_v + (-\nu + \frac{U}{2})S_r c_v + (-\nu - \frac{U}{2})S_l c_v + (\nu - \frac{V}{2})c_{vv} + \\
&\quad +\frac{5}{\rho}p_0 - \frac{1}{\rho}S_r p_0 - \frac{1}{\rho}S_l p_0 - \frac{1}{\rho}p_1 - \frac{2}{\rho}c_p
\end{aligned}
\tag{3.49}
$$

And finally the derivative of the continuity equation for $i = 1$ becomes:

$$
\begin{aligned}
0 &= (5\nu + \frac{U}{2})u_1 + (-5\nu + \frac{U}{2})S_r u_1 + (-\nu - \frac{U}{2})S_l u_1 + (-\nu + \frac{V}{2})u_2 \\
&\quad +(-\nu - \frac{V}{2})u_0 + (\nu - \frac{U}{2})S_r^2 u_1 + (\nu - \frac{V}{2})S_r u_2 + (\nu + \frac{V}{2})S_r u_0 \\
&\quad +(-5\nu + \frac{V}{2})v_1 + (\nu - \frac{U}{2})S_r v_1 + (\nu + \frac{U}{2})S_l v_1 + (\nu - \frac{U}{2})v_2 \\
&\quad +(5\nu + \frac{V}{2})v_0 + (-\nu + \frac{U}{2})S_r v_0 + (-\nu - \frac{U}{2})S_l v_0 + (\nu - \frac{V}{2})c_v + \\
&\quad +\frac{4}{\rho}p_1 - \frac{1}{\rho}S_r p_1 - \frac{1}{\rho}S_l p_1 - \frac{1}{\rho}p_2 - \frac{1}{\rho}p_0
\end{aligned}
\tag{3.50}
$$

When the half plane is cut off at the top for some $i = N$, far from the bottom wall, we obtain more boundary conditions. It is a natural assumption that the velocities tend to a constant value, but we will not investigate these conditions at this stage. With the symbols in (3.14) and the variables in the staggered grid in Figure 3.3, the equations can be rewritten somewhat easier. We only give the equations for the part of the plane on which the boundary does not have any (direct) influence, so the equations in (3.46) and (3.47):

$$
\begin{aligned}
\dot{u}_i &= \beta u_{i,r} + \alpha u_{i,l} + \delta u_{i+1} + \gamma u_{i-1} - 4\nu u_i - \frac{1}{\rho} p_i + \frac{1}{\rho} p_{i,l} \\
\dot{v}_i &= \beta v_{i,r} + \alpha v_{i,l} + \delta v_{i+1} + \gamma v_{i-1} - 4\nu v_i - \frac{1}{\rho} p_{i+1} + \frac{1}{\rho} p_i \\
0 &= \theta u_i + \lambda u_{i,r} + -\alpha u_{i,l} + -\delta u_{i+1} - \gamma u_{i-1} + \beta u_{i,rr} + \delta u_{i+1,r} + \gamma u_{i-1,r} \\
&\quad + \varphi v_i + \beta v_{i,r} + \alpha v_{i,l} + \beta v_{i+1} + \tau v_{i-1} + -\beta v_{i-1,r} - \alpha v_{i-1,l} + \delta v_{i-2} + \\
&\quad + \frac{4}{\rho} p_i - \frac{1}{\rho} p_{i,r} - \frac{1}{\rho} p_{i,l} - \frac{1}{\rho} p_{i+1} - \frac{1}{\rho} p_{i-1}
\end{aligned}
\tag{3.51}
$$

3.4 Boundary conditions

In Figure 3.2, all variables which are part of the temporal or spatial vector of a subsystem of our interconnected system were depicted. In order to see which boundary conditions had to be defined, we looked at a subsystem which vertical spatial coordinate s_2 was equal to zero. While we investigated the equations of the temporal and spatial state, we defined the necessary boundary conditions at the bottom wall. In this section, these conditions will first be summarized, next we consider how we have to define boundary conditions at the left, right and top of our half plane.

As we can see in Figure 3.2, we have to define four boundary conditions for every column of the grid, that is for all s_1. We need one boundary condition for the pressure, one for the horizontal velocity and two conditions for the vertical velocity. We defined these conditions already during the investigation of the equations of the temporal and spatial state in section 3.3. We assumed that the vertical velocity $v_d(t, s_1, 0)$ is equal to the vertical velocity component of the user defined input:

$$
v_d(t, s_1, 0) = c_v(t, s_1), \qquad \forall t, s_1 \tag{3.52}
$$

A natural way to obtain a boundary condition for the pressure $p_d(t, s_1, 0)$, was to assume that the pressure at the boundary, determined by the pressure component c_p of the input vector, is the average of the pressures p and p_d:

$$p_d(t, s_1, 0) \;=\; 2c_p(t, s_1) - p(t, s_1, 0) \tag{3.53}$$

At the same way we defined a boundary condition for the horizontal velocity $u_d(t, s_1, 0)$. However, it is not clear yet how to fill in $c_u(t, s_1)$.

$$u_d(t, s_1, 0) \;=\; 2c_u(t, s_1) - u(t, s_1, 0), \qquad \forall t, s_1 \tag{3.54}$$

Finally, the second boundary condition for the vertical velocity, $v_{dd}(t, s_1, 0)$, was defined as follows:

$$\begin{aligned} v_{dd}(t, s_1, s_2) &= (S_d v_d)(t, s_1, s_2) + (\Lambda c_{vv})(t, s_1, s_2) \\ &= \begin{cases} v_d(t, s_1, s_2 - 1), & s_2 > 1 \\ c_v(t, s_1), & s_2 = 1 \\ c_{vv}(t, s_1), & s_2 = 0 \end{cases} \end{aligned} \tag{3.55}$$

Consider Figure 3.2. For the variables v_{ld}, v_{rd} and u_{rd} hold:

$$\begin{aligned} v_{ld}(t, s_1, 0) &= v_d(t, s_1 - 1, 0) \\ v_{rd}(t, s_1, 0) &= v_d(t, s_1 + 1, 0) \\ u_{rd}(t, s_1, 0) &= u_d(t, s_1 + 1, 0) \end{aligned} \tag{3.56}$$

We do not need to define special boundary conditions for these three variables, because these conditions are already defined for the subsystems at the left $(s - 1)$ and right $(s + 1)$ of our subsystem of interest.

The equations of the temporal and spatial state of the system are known, and we have some boundary conditions at the bottom wall. To be able to calculate a (unique) velocity distribution, the half plane still needs to be cut off by boundaries at the left, right and top. We may assume that the horizontal and vertical velocities tend to a constant value (very) far away from the bottom wall:

$$
\begin{aligned}
\lim_{s_2\to\infty} u(s_1,s_2) &= U_0 \\
\lim_{s_1\to\pm\infty} u(s_1,s_2) &= U_0 \\
\lim_{s_2\to\infty} v(s_1,s_2) &= V_0 \\
\lim_{s_1\to\pm\infty} v(s_1,s_2) &= V_0
\end{aligned} \tag{3.57}
$$

So for some s_1 at the left and right and a certain s_2 at the top we assume that the horizontal and vertical velocity are equal to U_0 and V_0, respectively. When both the horizontal and vertical velocity are constant at a 'virtual' boundary of the half plane, the pressure is automatically also constant at that point. Of course, the boundary conditions at the bottom wall remain the same:

$$
\begin{aligned}
u_d(t,s_1,0) &= 2c_u(t,s_1) - u(t,s_1,0) \\
v_d(t,s_1,0) &= c_v(t,s_1) \\
p_d(t,s_1,0) &= 2c_p(t,s_1) - p(t,s_1,0) \\
v_{dd}(t,s_1,0) &= c_{vv}(t,s_1)
\end{aligned} \tag{3.58}
$$

Constants U_0 and V_0 are taken as the values at the boundaries at the left, right and top of the plane. We define $\tilde{u}$ and $\tilde{v}$ as follows:

$$
\begin{aligned}
\tilde{u} &= u - U_0 \\
\tilde{v} &= v - V_0
\end{aligned} \tag{3.59}
$$

Substitute these new $\tilde{u}$ and $\tilde{v}$ into the Navier-Stokes equations for u and v, respectively. Then the structure of the system remains the same, we only have to add (U_0, V_0) to our solution to get the real velocities. Moreover, the boundary conditions, except for those on the part of the bottom wall equipped with the devices, become equal to 0. If we do not make these substitutions, the values for u and v tend to U_0 and V_0, respectively, which are in general non-zero constants. But in that case it would not be possible to obtain a solution which satisfies the l_2 norm, so we have to use the substitutions in (3.59) in the Navier-Stokes equation. In the interest of clarity, we will rename $\tilde{u}$ and $\tilde{v}$ to u and v, respectively.

3.5 Well-posedness of the system

In this section we investigate the well-posedness of the system, which particularly depends on $(\Delta_S - A_{ss})$. We already know that this operator, which has the scape of an 18×18 matrix, is not invertible. We have to use the definitions of the previous sections to make sure we can obtain a unique solution for our problem.

As we have seen in the chapter 2, the system is well-posed if $(\Delta_S - A_{ss})$ has a well-defined generalized inverse operator. This operator is invertible if for any 18×1 vector q, there exists a unique 18×1 vector x_s such that $(\Delta_S - A_{ss})x_s = q$. Note that all components of both x_s and q are an element of l_2 and are defined on the upper half plane. We know that the following equation must hold:

$$A_{st}x_t + B_{sd}d \quad \in \quad im\ (\Delta_S - A_{ss}) \tag{3.60}$$

This means we have some restrictions for our vector q; some elements of this vector have to be zero for $s_2 = 0$ to satisfy the requirement above. We can obtain this situation by adding operator S_dS_u to these specific components of q. This will be done when we investigate the existence of a solution. To show that $(\Delta_S - A_{ss})$ is invertible in a generalized sense, we have to prove uniqueness and existence of a solution, we will start with the proof of uniqueness.

3.5.1 Uniqueness of the solution

Suppose there exist two solutions to the problem, say $x_{s,1}$ and $x_{s,2}$. Let $x_{s,diff}$ be the difference of both solutions:

$$x_{s,diff} := x_{s,1} - x_{s,2} \tag{3.61}$$

The problem has a unique solution if vector $x_{s,diff}$ is equivalent with zero. We obtain the following boundary value problem for $x_{s,diff}$:

$$\begin{cases} (\Delta_S - A_{ss})x_{s,diff} = 0 & \\ x_{s,diff} \in l_2(\mathbb{R} \times \mathbb{Z} \times \mathbb{Z}^+) & \\ u_d(t, s_1, 0) = -u(t, s_1, 0), & \forall t, s_1 \\ v_d(t, s_1, 0) = 0, & \forall t, s_1 \\ p_d(t, s_1, 0) = -p(t, s_1, 0), & \forall t, s_1 \\ v_{dd}(t, s_1, 0) = 0, & \forall t, s_1 \end{cases} \tag{3.62}$$

Due to the staggered grid we obtain somewhat different boundary conditions for the horizontal velocity and the pressure, these conditions correspond to the following:

$$
\begin{aligned}
u_d(t, s_1, 0) = -u(t, s_1, 0) &\quad\Leftrightarrow\quad c_u(t, s_1) = 0 \\
p_d(t, s_1, 0) = -p(t, s_1, 0) &\quad\Leftrightarrow\quad c_p(t, s_1) = 0
\end{aligned}
\tag{3.63}
$$

Operation $(\Delta_S - A_{ss})x_{s,diff} = 0$ gives us the following equations. In the interest of clarity, we omitted index $_{diff}$:

$$
\begin{array}{lllllllll}
S_r u_l & = & 0 & S_r v_l & = & 0 & S_d p & = & S_d S_u (3.65) \\
S_l u_r & = & 0 & S_l v_r & = & 0 & S_r p_l & = & p \\
S_u u_d & = & 0 & S_u v_d & = & 0 & S_l p_r & = & p \\
S_d u_u & = & 0 & S_d v_u & = & 0 & S_d p_u & = & S_d S_u p \\
S_l u_{rr} & = & u_r & S_r v_{ld} & = & v_d & & & \\
S_u u_{rd} & = & u_r & S_l v_{rd} & = & v_d & & & \\
S_d u_{ru} & = & S_d S_u u_r & S_u v_{dd} & = & v_d & & &
\end{array}
\tag{3.64}
$$

where (3.65) is equal to:

$$
\begin{aligned}
& \rho\left\{\lambda u_r - \alpha u_l - \delta u_u - \gamma u_d + \beta u_{rr} + \delta u_{ru} + \gamma u_{rd}\right\} \\
+ \;& \rho\left\{\beta v_r + \alpha v_l + \beta v_u + \tau v_d - \beta v_{rd} - \alpha v_{ld} + \delta v_{dd}\right\} \\
+ \;& 4p - p_u - p_r - p_l
\end{aligned}
\tag{3.65}
$$

Consider the equations in (3.64). We can immediately see that the variables u_l, u_r, v_l and v_r have to be zero for all t, s_1, s_2, because we have no restrictions for the horizontal shift operators. Because u_r is zero, we also have that u_{rr} is zero for all t, s_1, s_2. We simply have that $u(t, s_1, s_2)$ and $v(t, s_1, s_2)$ are equal to zero for all t, s_1, s_2. We *do* have some restrictions at the boundary for the vertical shift operators. Consider equation $S_u u_d = 0$, for this equation holds:

$$
\begin{aligned}
(S_u u_d)(t,s_1,s_2) &= 0 \qquad \Rightarrow \\
u_d(t,s_1,s_2) &= 0 + 2(\Lambda c_u)(t,s_1,s_2) + ((I - S_d S_u)u)(t,s_1,s_2) \\
&= \begin{cases} 0, & s_2 > 0 \\ 2c_u(t,s_1) - u(t,s_1,0), & s_2 = 0 \end{cases} \\
&= \begin{cases} 0, & s_2 > 0 \\ -u(t,s_1,0), & s_2 = 0 \end{cases}
\end{aligned} \tag{3.66}
$$

Because $u(t, s_1, 0)$ equals zero, we also have that $u_d(t, s_1, 0) = 0$. Since this variable u_d is also zero for $s_2 > 0$, we may conclude that u_d is zero for all t, s_1, s_2. Next we take a look at the following equation, which is $S_d u_u = 0$. By definition, we have:

$$
(S_d u_u)(t,s_1,s_2) = u_u(t,s_1,s_2-1) = 0, \qquad s_2 > 0 \tag{3.67}
$$

We may directly conclude that u_u is zero for all t, s_1, s_2. Then also the other variables of the horizontal and vertical velocity u and v have to be equal to zero for all t, s_1, s_2, this follows directly from both results for u_d and u_u and the remaining boundary conditions in (3.64). This means that we still have to prove that the pressure variables p, p_l, p_r and p_u are equal to zero. The main reason that uniqueness of the solution depends on the pressure, is that the pressure variable p is part of the spatial state instead of the temporal state, like the horizontal and vertical velocity. We have four pressure equations remaining. The equation in (3.65) is reduced to the following because all velocity variables are zero:

$$
\begin{aligned}
S_d p &= S_d S_u (4p - p_u - p_r - p_l) \qquad \Rightarrow \\
p &= S_u (4p - p_u - p_r - p_l)
\end{aligned} \tag{3.68}
$$

For the other pressure equations the following holds:

$$
\begin{aligned}
S_r p_l = p &\quad \Rightarrow \quad p_l = S_l p \\
S_l p_r = p &\quad \Rightarrow \quad p_r = S_r p \\
S_d p_u = S_d S_u p &\quad \Rightarrow \quad p_u = S_u p
\end{aligned} \tag{3.69}
$$

Substitution of the equations of (3.69) in equation (3.68) gives:

$$p \quad = \quad 4S_u p - S_u S_l p - S_u S_r p - S_u^2 p \qquad \Rightarrow \tag{3.70}$$

This equation can be rewritten as follows:

$$S_u(S_l + S_r + S_d + S_u - 4)p = 0 \tag{3.71}$$

Equation (3.71) corresponds with the discrete Laplace equation. Some problems in which the Laplace equation is involved can be found in [4]. The discrete Laplace equation can be written as:

$$\Delta p \quad = \quad 0 \tag{3.72}$$

The only difference is that operator S_u is added to the equation, this is done in order to avoid problems at the boundary. In other words, the Laplace equation holds for $s_2 > 0$, for $s_2 = 0$ we have to use the boundary condition for the pressure:

$$p_d(t, s_1, 0) \quad = \quad -p(t, s_1, 0) \tag{3.73}$$

Note that if we can prove that p is equal to zero for all t, s_1, s_2, then also p_l, p_r and p_u are zero along with all the other elements of x_s, and the problem has a unique solution. That means that we have to prove that $p = 0$ in the following boundary value problem:

$$\begin{cases} p(t, s_1, s_2) & \in \quad l_2(\mathbb{R} \times \mathbb{Z} \times \mathbb{Z}^+) \\ S_u \Delta p(t, s_1, s_2) & = \quad 0 \\ p_d(t, s_1, 0) & = \quad -p(t, s_1, 0) \end{cases} \tag{3.74}$$

Because $p \in l_2$, the function attains a maximum, say M, at some point in V. Suppose this maximum value value of p is attained at the boundary. Since $c_p(t, s_1) = 0$, we see immediately that $p(t, s_1, s_2)$ has to be zero for all t, s_1, s_2.

Now suppose the maximum is attained at an interior point of V, say p_{int}. The discrete Laplace equation in (3.74) states that the value in any point is equal to the average of its four neighbors. As a consequence, every neighbor of p_{int} must have the same value as p_{int}, simply because else the maximum would not be attained in p_{int}. By induction, p must have the same value in every point of V, except for the boundary. This means that p is a constant function, equivalent with the maximum M. Since p has to be an element of the l_2 space, the only function which satisfies (3.74) is equivalent to the zero function. We may conclude that $p(t, s_1, s_2) = 0$ for all t, s_1, s_2, and as a result all elements of x_s in equation (3.64) are equal to zero for all t, s_1, s_2. In other words, the solution is unique, if it exists.

Note that we cannot use the maximum principle [9] to prove uniqueness, since V is unbounded.

3.5.2 Existence of a solution

Now the existence of a solution is investigated. The problem has a solution if for any q, there exists a x_s such that $(\Delta_S - A_{ss})x_s = q$. We have the following equations:

$$\begin{array}{rclrcl} S_r u_l &=& q_1 & S_u v_d &=& q_{10} \\ S_l u_r &=& q_2 & S_d v_u &=& q_{11} \\ S_u u_d &=& q_3 & S_r v_{ld} - v_d &=& q_{12} \\ S_d u_u &=& q_4 & S_l v_{rd} - v_d &=& q_{13} \\ S_l u_{rr} - u_r &=& q_5 & S_u v_{dd} - v_d &=& q_{14} \\ S_u u_{rd} - u_r &=& q_6 & S_d p - S_d S_u (3.65) &=& q_{15} \\ S_d u_{ru} - S_d S_u u_r &=& q_7 & S_r p_l - p &=& q_{16} \\ S_r v_l &=& q_8 & S_l p_r - p &=& q_{17} \\ S_l v_r &=& q_9 & S_d p_u - S_d S_u p &=& q_{18} \end{array} \qquad (3.75)$$

Where (3.65) is equivalent with:

$$\begin{aligned} & \rho\left\{\lambda u_r - \alpha u_l - \delta u_u - \gamma u_d + \beta u_{rr} + \delta u_{ru} + \gamma u_{rd}\right\} \\ + \; & \rho\left\{\beta v_r + \alpha v_l + \beta v_u + \tau v_d - \beta v_{rd} - \alpha v_{ld} + \delta v_{dd}\right\} \\ + \; & 4p - p_u - p_r - p_l \end{aligned} \qquad (3.76)$$

Like we mentioned before, one requirement is that q is an element of the image of $(\Delta_S - A_{ss})$. This leads to some restrictions for our vector q, caused by the downward shift operator in some of the equations in (3.75). We have to assure that $q_i(t, s_1, 0) = 0$ for $i = 4, 7, 11, 15, 18$. To obtain this situation, we add operator S_dS_u to these components of q. Then the equations of $(\Delta_S - A_{ss})x_s = q$ are written as follows:

$$\begin{array}{rclrcl}
S_r u_l &=& q_1 & S_u v_d &=& q_{10} \\
S_l u_r &=& q_2 & S_d v_u &=& S_d S_u q_{11} \\
S_u u_d &=& q_3 & S_r v_{ld} - v_d &=& q_{12} \\
S_d u_u &=& S_d S_u q_4 & S_l v_{rd} - v_d &=& q_{13} \\
S_l u_{rr} - u_r &=& q_5 & S_u v_{dd} - v_d &=& q_{14} \\
S_u u_{rd} - u_r &=& q_6 & S_d p - S_d S_u (3.65) &=& S_d S_u q_{15} \\
S_d u_{ru} - S_d S_u u_r &=& S_d S_u q_7 & S_r p_l - p &=& q_{16} \\
S_r v_l &=& q_8 & S_l p_r - p &=& q_{17} \\
S_l v_r &=& q_9 & S_d p_u - S_d S_u p &=& S_d S_u q_{18}
\end{array} \tag{3.77}$$

For some elements of x_s we can immediately give the solution:

$$\begin{array}{lcl}
(S_r u_l)(t,s_1,s_2) = q_1(t,s_1,s_2) & \Rightarrow & u_l(t,s_1,s_2) = (S_l q_1)(t,s_1,s_2) \\
(S_l u_r)(t,s_1,s_2) = q_2(t,s_1,s_2) & \Rightarrow & u_r(t,s_1,s_2) = (S_r q_2)(t,s_1,s_2) \\
(S_r v_l)(t,s_1,s_2) = q_8(t,s_1,s_2) & \Rightarrow & v_l(t,s_1,s_2) = (S_l q_8)(t,s_1,s_2) \\
(S_l v_r)(t,s_1,s_2) = q_9(t,s_1,s_2) & \Rightarrow & v_r(t,s_1,s_2) = (S_r q_9)(t,s_1,s_2)
\end{array} \tag{3.78}$$

Because we have a solution for u_r, we can also give the solution for u_{rr}:

$$\begin{array}{rcl}
(S_l u_{rr})(t,s_1,s_2) &=& u_r(t,s_1,s_2) + q_5(t,s_1,s_2) \qquad \Rightarrow \\
u_{rr}(t,s_1,s_2) &=& (S_r u_r)(t,s_1,s_2) + (S_r q_5)(t,s_1,s_2) \\
&=& (S_r^2 q_2)(t,s_1,s_2) + (S_r q_5)(t,s_1,s_2)
\end{array} \tag{3.79}$$

If we have a vertical shift operator in an equation, we use the definitions for the boundary conditions given in section 3.3. Consider equation $S_u u_d = q_3$, we know that there exists a variable c_u such that:

$$\begin{array}{rcl}
(S_u u_d)(t,s_1,s_2) &=& q_3(t,s_1,s_2) \qquad \Rightarrow \\
u_d(t,s_1,s_2) &=& (S_d q_3)(t,s_1,s_2) + 2(\Lambda c_u)(t,s_1,s_2) - ((I - S_d S_u)u)(t,s_1,s_2)
\end{array} \tag{3.80}$$

And for equation $S_d u_u = S_d S_u q_4$ holds:

$$
\begin{aligned}
(S_d u_u)(t,s_1,s_2) &= (S_d S_u q_4)(t,s_1,s_2) \qquad \Rightarrow \\
u_u(t,s_1,s_2) &= (S_u q_4)(t,s_1,s_2)
\end{aligned}
\tag{3.81}
$$

For the other elements of u and v we can find the solution in a similar way:

$$
\begin{aligned}
u_{rd}(t,s_1,s_2) &= (S_d u_r)(t,s_1,s_2) + (S_d q_6)(t,s_1,s_2) \\
&\quad +2(\Lambda c_u)(t,s_1+1,s_2) - ((I - S_d S_u)u)(t,s_1+1,s_2) \\
&= (S_d S_r q_2)(t,s_1,s_2) + (S_d q_6)(t,s_1,s_2) \\
&\quad +2(\Lambda c_u)(t,s_1+1,s_2) - ((I - S_d S_u)u)(t,s_1+1,s_2) \\
u_{ru}(t,s_1,s_2) &= (S_u u_r)(t,s_1,s_2) + (S_u q_7)(t,s_1,s_2) \\
&= (S_u S_r q_2)(t,s_1,s_2) + (S_u q_7)(t,s_1,s_2) \\
v_d(t,s_1,s_2) &= (S_d q_{10})(t,s_1,s_2) + (\Lambda c_v)(t,s_1,s_2) \\
v_u(t,s_1,s_2) &= (S_u q_{11})(t,s_1,s_2) \\
v_{ld}(t,s_1,s_2) &= (S_l v_d)(t,s_1,s_2) + (S_l q_{12})(t,s_1,s_2) \\
&= (S_l S_d q_{10})(t,s_1,s_2) + (\Lambda c_v)(t,s_1-1,s_2) + (S_l q_{12})(t,s_1,s_2) \\
v_{rd}(t,s_1,s_2) &= (S_r v_d)(t,s_1,s_2) + (S_r q_{13})(t,s_1,s_2) \\
&= (S_r S_d q_{10})(t,s_1,s_2) + (\Lambda c_v)(t,s_1+1,s_2) + (S_r q_{13})(t,s_1,s_2) \\
v_{dd}(t,s_1,s_2) &= (S_d v_d)(t,s_1,s_2) + (S_d q_{14})(t,s_1,s_2) + (\Lambda c_{vv})(t,s_1,s_2) \\
&= (S_d^2 q_{10})(t,s_1,s_2) + (S_d q_{14})(t,s_1,s_2) + (\Lambda c_{vv})(t,s_1,s_2)
\end{aligned}
\tag{3.82}
$$

This means that we still have to prove that there also exists a solution for the pressure variables p, p_l, p_r and p_u. We have four pressure equations. For the last three equations the solution is straightforward, however, we still have to determine p:

$$
\begin{aligned}
S_r p_l = p + q_{16} &\quad\Rightarrow\quad p_l = S_l p + S_l q_{16} \\
S_l p_r = p + q_{17} &\quad\Rightarrow\quad p_r = S_r p + S_r q_{17} \\
S_d p_u = S_d S_u p + S_d S_u q_{18} &\quad\Rightarrow\quad p_u = S_u p + S_u q_{18}
\end{aligned}
\tag{3.83}
$$

The first pressure equation is related to the continuity equation:

$$
\begin{aligned}
S_d p &= S_d S_u \rho \{\lambda u_r - \alpha u_l - \delta u_u - \gamma u_d + \beta u_{rr} + \delta u_{ru} + \gamma u_{rd}\} \\
&+ S_d S_u \rho \{\beta v_r + \alpha v_l + \beta v_u + \tau v_d - \beta v_{rd} - \alpha v_{ld} + \delta v_{dd}\} \\
&+ S_d S_u \{4p - p_u - p_r - p_l\} + S_d S_u q_{15} \qquad (3.84)
\end{aligned}
$$

When we substitute the solutions for the elements of u, v and p in (3.84), we get the following very large equation which contains all components of vector q:

$$
\begin{aligned}
S_d p = \; & S_d S_u \rho \{\lambda S_r q_2 - \alpha S_l q_1 - \delta S_u q_4 - \gamma S_d q_3 + 2\gamma \Lambda c_u \\
& -\gamma (I - S_d S_u) u + \beta S_r^2 q_2 + \beta S_r q_5 + \delta S_u S_r q_2 + \delta S_u q_7 \\
& +\gamma S_d S_r q_2 + \gamma S_d q_6 + \beta S_r q_9 + \alpha S_l q_8 + \beta S_u q_{11} \\
& +\tau S_d q_{10} + \tau \Lambda c_v - \beta S_r S_d q_{10} - \beta S_r \Lambda c_v - \beta S_r q_{13} \\
& -\alpha S_l S_d q_{10} - \alpha S_l \Lambda c_v - \alpha S_l q_{12} + \delta S_d^2 q_{10} - \delta S_d \lambda c_v \\
& +\delta S_d q_{14} + \delta \Lambda c_{vv}\} \\
& +S_d S_u \{(4 - S_u - S_l - S_r) p + q_{15} + q_{16} + q_{17} + q_{18}\}
\end{aligned} \qquad (3.85)
$$

The boundary conditions cause also some variables which have influence on the boundary via Λ or $(I - S_d S_u)$, however, since we apply $S_d S_u$ to all elements, they have no influence and can therefore be omitted:

$$
\begin{aligned}
(S_d S_u \Lambda c_u)(t, s_1, s_2) &= \begin{cases} (S_u \Lambda c_u)(t, s_1, s_2 - 1), & s_2 > 0 \\ 0, & s_2 = 0 \end{cases} \\
&= \begin{cases} (\Lambda c_u)(t, s_1, s_2), & s_2 > 0 \\ 0, & s_2 = 0 \end{cases} \\
&= \begin{cases} 0, & s_2 > 0 \\ 0, & s_2 = 0 \end{cases} \\
(S_d S_u (I - S_d S_u) u)(t, s_1, s_2) &= ((S_d S_u - S_d S_u S_d S_u) u)(t, s_1, s_2) \\
&= ((S_d S_u - S_d S_u) u)(t, s_1, s_2) \\
&= 0
\end{aligned} \qquad (3.86)
$$

If we let f_q be the sum of all elements of q in (3.85), this equation can be rewritten as follows:

$$
\begin{aligned}
S_d p &= S_d S_u(4p - S_l p - S_r p - S_u p) + S_d S_u f_q \qquad \Rightarrow \\
S_u f_q &= S_u(S_l + S_r + S_d + S_u - 4)p \qquad (3.87)
\end{aligned}
$$

Except for the boundary, this equation corresponds with the inhomogenous Laplace equation, or the Poisson equation:

$$
\Delta p = f_q \qquad (3.88)
$$

If p exists for any function f_q, then also any other component of x_s exists and existence of a solution is proved. We use discrete Fourier transformation in order to prove existence of a solution. We have to solve the following boundary value problem. In the interest of clarity, f_q is renamed as q. Boundary function g replaces c_p. Note that we use a simplified boundary function for g, that is, we do not take our staggered grid into account.

$$
\begin{cases}
V = \{(t, s_1, s_2) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}^+\} \\
p, q : V \to \mathbb{R} \\
p(t, s_1, s_2) \in l_2(V) \\
q(t, s_1, s_2) \in l_2(V) \\
\Delta p(t, s_1, s_2) = q(t, s_1, s_2) \\
g(t, s_1) \in l_2(\mathbb{Z}) \\
p(t, s_1, 0) = g(t, s_1)
\end{cases} \qquad (3.89)
$$

We suppose that q is an element of l_2 and search for a solution $p \in l_2$. However, it is not straightforward that p is an element of l_2. Suppose for instance that $q(t, s_1, s_2) = 0$ and $g(t, s_1) = 0$. Then the function $p(t, s_1, s_2) = s_1 \cdot s_2$ is a solution of the Poisson equation, while p is definitely not an element of l_2.

The vector space we are working with is the l_2 space, but for Fourier transformations the l_1 space is more convenient. Therefore, we assume at this stage that $q(t, s_1, s_2)$ and $g(t, s_1)$ are elements of l_1 and we search for a $p(t, s_1, s_2) \in l_1$. The Fourier transformations of $p(t, s_1, s_2)$ and $q(t, s_1, s_2)$ with respect to variable s_1 are:

$$\begin{aligned} p(t,s_1,s_2) &\rightarrow \sum_{k=-\infty}^{\infty} e^{-ik\omega} p(t,k,s_2) = \hat{p}(t,\omega,s_2) \\ q(t,s_1,s_2) &\rightarrow \sum_{k=-\infty}^{\infty} e^{-ik\omega} q(t,k,s_2) = \hat{q}(t,\omega,s_2) \end{aligned} \tag{3.90}$$

If $p, q \in l_1(V)$, then $\hat{p}, \hat{q} \in L_1(-\pi, \pi)$ [12]. We obtain the following Fourier transformations for $p(t, s_1 + 1, s_2)$ and $p(t, s_1 - 1, s_2)$:

$$\begin{aligned} p(t,s_1+1,s_2) &\rightarrow \sum_{k=-\infty}^{\infty} e^{-ik\omega} p(t,k+1,s_2) = \sum_{k=-\infty}^{\infty} e^{-i(k-1)\omega} p(t,k,s_2) \\ &= e^{i\omega} \sum_{k=-\infty}^{\infty} e^{-ik\omega} p(t,k,s_2) = e^{i\omega} \hat{p}(t,\omega,s_2) \\ p(t,s_1-1,s_2) &\rightarrow \sum_{k=-\infty}^{\infty} e^{-ik\omega} p(t,k-1,s_2) = \sum_{k=-\infty}^{\infty} e^{-i(k+1)\omega} p(t,k,s_2) \\ &= e^{-i\omega} \sum_{k=-\infty}^{\infty} e^{-ik\omega} p(t,k,s_2) = e^{-i\omega} \hat{p}(t,\omega,s_2) \end{aligned} \tag{3.91}$$

Then the Poisson equation can be written in Fourier form:

$$\begin{aligned} q(t,s_1,s_2) &= 4p(t,s_1,s_2) - p(t,s_1+1,s_2) - p(t,s_1-1,s_2) \\ &\quad -p(t,s_1,s_2+1) - p(t,s_1,s_2-1) \qquad \Rightarrow \\ \hat{q}(t,\omega,s_2) &= 4\hat{p}(t,\omega,s_2) - e^{i\omega}\hat{p}(t,\omega,s_2) - e^{-i\omega}\hat{p}(t,\omega,s_2) \\ &\quad -\hat{p}(t,\omega,s_2+1) - \hat{p}(t,\omega,s_2-1) \qquad \Rightarrow \\ \hat{p}(t,\omega,s_2+1) &= (4 - e^{i\omega} - e^{-i\omega})\hat{p}(t,\omega,s_2) - \hat{p}(t,\omega,s_2-1) - \hat{q}(t,\omega,s_2) \end{aligned} \tag{3.92}$$

Next, we introduce an operator σ, which is defined as follows:

$$\sigma\hat{p}(t,\omega,s_2) = \hat{p}(t,\omega,s_2+1) \tag{3.93}$$

Note that operator σ is closely related to the vertical shift operator S_u. We also make a substitution $s_2 := s_2 + 1$. Then we get the following equation:

$$[\sigma^2 - (4 - e^{i\omega} - e^{-i\omega})\sigma + 1]\hat{p}(t, \omega, s_2) = \hat{q}(t, \omega, s_2 + 1) \tag{3.94}$$

Because $e^{i\omega} + e^{-i\omega} = 2\cos(\omega)$, we can rewrite the equation as follows:

$$[\sigma^2 - (4 - 2\cos(\omega))\sigma + 1]\hat{p}(t, \omega, s_2) = \hat{q}(t, \omega, s_2 + 1) \tag{3.95}$$

The coefficient of $\sigma\hat{p}(t, \omega, s_2)$ plays an important role in this equation, we will call this coefficient D:

$$D \quad := \quad -(4 - 2\cos(\omega)) \tag{3.96}$$

We make a distinction between two cases, in the first place we consider the equation if ω is equal to $2n\pi$, for $n \in \mathbb{Z}$. After that, we look at the equation for any other value of ω. First we consider the case that $\omega = 2n\pi$. The value of coefficient D is then equal to -2, and equation (3.95) becomes:

$$\begin{aligned} (\sigma^2 - 2\sigma + 1)\hat{p}(t, \omega, s_2) &= \hat{q}(t, \omega, s_2 + 1) \qquad \Rightarrow \\ (\sigma - 1)^2\hat{p}(t, \omega, s_2) &= \hat{q}(t, \omega, s_2 + 1) \end{aligned} \tag{3.97}$$

Define $\hat{r_1}$ as $\hat{r_1}(t, \omega, s_2) := (\sigma - 1)\hat{p}(t, \omega, s_2)$. For the L_1 norm of $\hat{r_1}$ holds:

$$\begin{aligned} \|\hat{r_1}(t, \omega, \cdot)\|_1 &= \sum_{k=0}^{\infty} |\hat{r_1}(t, \omega, k)| = \sum_{k=0}^{\infty} |(\sigma - 1)\hat{p}(t, \omega, k)| \\ &\leq \sum_{k=0}^{\infty} |\hat{p}(t, \omega, k + 1)| + \sum_{k=0}^{\infty} |\hat{p}(t, \omega, k)| \\ &\leq 2\|\hat{p}\|_1 \end{aligned} \tag{3.98}$$

We obtain the following equation for $(\sigma - 1)\hat{r_1}$:

$$\begin{aligned} (\sigma - 1)\hat{r_1}(t, \omega, s_2) &= \hat{q}(t, \omega, s_2 + 1) \qquad \Rightarrow \\ \hat{r_1}(t, \omega, s_2 + 1) &= \hat{r_1}(t, \omega, s_2) + \hat{q}(t, \omega, s_2 + 1) \qquad \Rightarrow \\ \hat{r_1}(t, \omega, n) &= \hat{r_1}(t, \omega, 0) + \sum_{k=1}^{n} \hat{q}(t, \omega, k) \end{aligned} \tag{3.99}$$

If $\hat{r_1}$ is an element of the L_1 space, the following must hold:

$$\hat{r_1}(t, \omega, n) \to 0, \qquad \text{for } n \to \infty \tag{3.100}$$

This implies that:

$$\hat{r_1}(t, \omega, 0) = -\sum_{k=1}^{\infty} \hat{q}(t, \omega, k) \tag{3.101}$$

This sum is well-defined since $\hat{q} \in L_1$. For $\hat{r_1}(t, \omega, n)$ now holds:

$$\begin{aligned} \hat{r_1}(t, \omega, n) &= -\sum_{k=1}^{\infty} \hat{q}(t, \omega, k) + \sum_{k=1}^{n} \hat{q}(t, \omega, k) \\ &= -\sum_{k=n+1}^{\infty} \hat{q}(t, \omega, k) \end{aligned} \tag{3.102}$$

Then we get for the L_1 norm of $\hat{r_1}$:

$$\begin{aligned} \|\hat{r_1}(t, \omega, \cdot)\|_1 &= \sum_{k=0}^{\infty} |\hat{r_1}(t, \omega, k)| \\ &= \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} |\hat{q}(t, \omega, k)| \end{aligned} \tag{3.103}$$

We return to the definition of $\hat{r_1}(t, \omega, s_2)$ to find an expression for $\hat{p}(t, \omega, n)$. Hereby, we use that $\hat{r_1}(t, \omega, 0) = -\sum_{k=1}^{\infty} \hat{q}(t, \omega, k)$. Then the following holds:

$$\begin{aligned} (\sigma - 1)\hat{p}(t, \omega, s_2) &= \hat{r_1}(t, \omega, s_2) \qquad \Rightarrow \\ \hat{p}(t, \omega, s_2 + 1) &= \hat{p}(t, \omega, s_2) + \hat{r_1}(t, \omega, s_2) \qquad \Rightarrow \\ \hat{p}(t, \omega, n) &= \hat{p}(t, \omega, 0) - \sum_{k=1}^{\infty} \hat{q}(t, \omega, k) + \sum_{k=1}^{n-1} \hat{r_1}(t, \omega, k) \end{aligned} \tag{3.104}$$

This leads to the following definition for the L_1 norm of $\hat{p}$:

$$\begin{aligned}\|\hat{p}(t,\omega,\cdot)\|_1 &= \sum_{k=0}^{\infty} |\hat{p}(t,\omega,k)| \\ &= \sum_{n=1}^{\infty} |\hat{p}(t,\omega,0) - \sum_{k=1}^{\infty} \hat{q}(t,\omega,k) + \sum_{k=1}^{n-1} \hat{r_1}(t,\omega,k)|\end{aligned} \tag{3.105}$$

With equation (3.102), we can obtain the following expression:

$$\sum_{l=1}^{n-1} \hat{r_1}(t,\omega,l) = -\sum_{l=2}^{n} \sum_{k=l}^{\infty} \hat{q}(t,\omega,k) \tag{3.106}$$

Substitution in (3.105) leads to the following equation:

$$\begin{aligned}\|\hat{p}(t,\omega,\cdot)\|_1 &= \sum_{n=1}^{\infty} |\hat{p}(t,\omega,0) - \sum_{k=1}^{\infty} \hat{q}(t,\omega,k) - \sum_{l=2}^{n} \sum_{k=l}^{\infty} \hat{q}(t,\omega,k)| \\ &= \sum_{n=1}^{\infty} |\hat{p}(t,\omega,0) - \sum_{l=1}^{n} \sum_{k=l}^{\infty} \hat{q}(t,\omega,k)|\end{aligned} \tag{3.107}$$

Every element of $\hat{q}$ appears infinitely many times in the equation above. If $\hat{q} \neq 0$, the L_1 norm of p is definitely not finite. For the same reason, $\hat{p}(t,\omega,0) = 0$. As a consequence, the only $\hat{p} \in L_1$ that satisfies the equation, is $\hat{p} = 0$.

Now suppose that $\omega \neq 2n\pi$. In this case we can rewrite equation (3.95) as follows:

$$(\sigma - \alpha_1(\omega))(\sigma - \alpha_2(\omega))\hat{p}(t,\omega,s_2) = \hat{q}(t,\omega,s_2+1) \tag{3.108}$$

where $\alpha_1(\omega) + \alpha_2(\omega) = 4 - 2\cos(\omega)$ and $\alpha_1(\omega)\alpha_2(\omega) = 1$. The last equation implies that $0 < \alpha_1(\omega) < 1 < \alpha_2(\omega)$. Define $\hat{r_2}$ as follows:

$$\begin{aligned}\hat{r_2}(t,\omega,s_2) &:= (\sigma - \alpha_2(\omega))\hat{p}(t,\omega,s_2) \\ &= \hat{p}(t,\omega,s_2+1) - \alpha_2(\omega)\hat{p}(t,\omega,s_2)\end{aligned} \tag{3.109}$$

For the L_1 norm of $\hat{r_2}$ holds:

$$\begin{aligned}
\|\hat{r_2}(t,\omega,\cdot)\|_1 &= \sum_{k=0}^{\infty} |\hat{r_2}(t,\omega,k)| = \sum_{k=0}^{\infty} |(\sigma - \alpha_2(\omega))\hat{p}(t,\omega,k)| \\
&\leq \sum_{k=0}^{\infty} |\hat{p}(t,\omega,k+1)| + \alpha_2(\omega)\sum_{k=0}^{\infty} |\hat{p}(t,\omega,k)| \\
&\leq (1+\alpha_2(\omega))\|\hat{p}(t,\omega,\cdot)\|_1
\end{aligned} \tag{3.110}$$

For $\hat{r_2}(t,\omega,n)$ we can find the following expression:

$$\begin{aligned}
(\sigma - \alpha_1(\omega))\hat{r_2}(t,\omega,s_2) &= \hat{q}(t,\omega,s_2+1) \qquad \Rightarrow \\
\hat{r_2}(t,\omega,s_2+1) &= \alpha_1(\omega)\hat{r_2}(t,\omega,s_2) + \hat{q}(t,\omega,s_2+1) \qquad \Rightarrow \\
\hat{r_2}(t,\omega,n) &= \alpha_1(\omega)^n \hat{r_2}(t,\omega,0) + \sum_{k=1}^{n} \alpha_1(\omega)^{n-k}\hat{q}(t,\omega,k)
\end{aligned} \tag{3.111}$$

And for the L_1 norm of $\hat{r_2}$ holds:

$$\begin{aligned}
\|\hat{r_2}(t,\omega,\cdot)\|_1 &= \sum_{k=0}^{\infty} |\hat{r_2}(t,\omega,k)| \\
&= \sum_{k=0}^{\infty} \alpha_1(\omega)^k |\hat{r_2}(t,\omega,0)| + \sum_{n=1}^{\infty}\sum_{k=1}^{n} \alpha_1(\omega)^{n-k}|\hat{q}(t,\omega,k)| \\
&\leq \sum_{k=0}^{\infty} \alpha_1(\omega)^k |\hat{r_2}(t,\omega,0)| + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} \alpha_1(\omega)^{n}|\hat{q}(t,\omega,k)| \\
&= \frac{1}{1-\alpha_1(\omega)}|\hat{r_2}(t,\omega,0)| + \frac{1}{1-\alpha_1(\omega)}\sum_{k=1}^{\infty}|\hat{q}(t,\omega,k)| \\
&\leq \frac{1}{1-\alpha_1(\omega)}|\hat{r_2}(t,\omega,0)| + \frac{1}{1-\alpha_1(\omega)}\|\hat{q}(t,\omega,\cdot)\|_1
\end{aligned} \tag{3.112}$$

For $|\hat{r_2}(t,\omega,0)|$, the following holds:

$$|\hat{r_2}(t,\omega,0)| = |\hat{p}(t,\omega,1) - \alpha_2(\omega)\hat{p}(t,\omega,0)| \tag{3.113}$$

We may conclude that the L_1 norm of $\hat{r_2}$ is finite:

$$\begin{aligned} \|\hat{r_2}(t,\omega,\cdot)\|_1 &\leq \frac{1}{1-\alpha_1(\omega)}|\hat{p}(t,\omega,1) - \alpha_2(\omega)\hat{p}(t,\omega,0)| \\ &+ \frac{1}{1-\alpha_1(\omega)}\|\hat{q}(t,\omega,\cdot)\|_1 < \infty \end{aligned} \tag{3.114}$$

The question remains if this implies that also the L_1 norm of $\hat{p}$ is finite. If this is the case, there exists a non-trivial solution for $\hat{p}$ in L_1 to the problem. If $\hat{p}, \hat{q} \in L_1$, then also $\hat{p}, \hat{q} \in L_2$, since L_1 is a subspace of L_2. As a consequence, $p \in l_2$ [12].

Chapter 4

Conclusions and recommendations

4.1 Conclusions

4.1.1 Refinement of the grid

In section 3.1 we made a discretization of the Navier-Stokes equations by putting them in a rectangular grid. This simplification of these equations leads to loss of information, in particular at the boundary. Our grid is chosen in such a way that the grid size in horizontal direction is equal to the distance between two adjacent devices at the bottom wall. Moreover, we introduced a staggered grid to prevent possible problems with pressure oscillations. Because of these restrictions, the magnitude of the variables at or close to the boundary heavily depend on the position they are measured. The vertical velocities at the bottom wall are continually measured at the end of the devices, one can imagine that if the measure points would lie right between the devices, this would result in totally different values for the vertical velocity close at the boundary. The same holds for the horizontal velocity and the pressure, the magnitude of these variables near the boundary is closely related to their position of measurement.

An increase of the number of measure points might lead to better results. Therefore, the horizontal grid size has to be decreased. It is recommended to keep the horizontal and vertical grid sizes equal, else we have to adapt the discretization of the Navier-Stokes equations because we cannot simply omit Δx and Δy like is done in chapter 3. We depicted this situation in

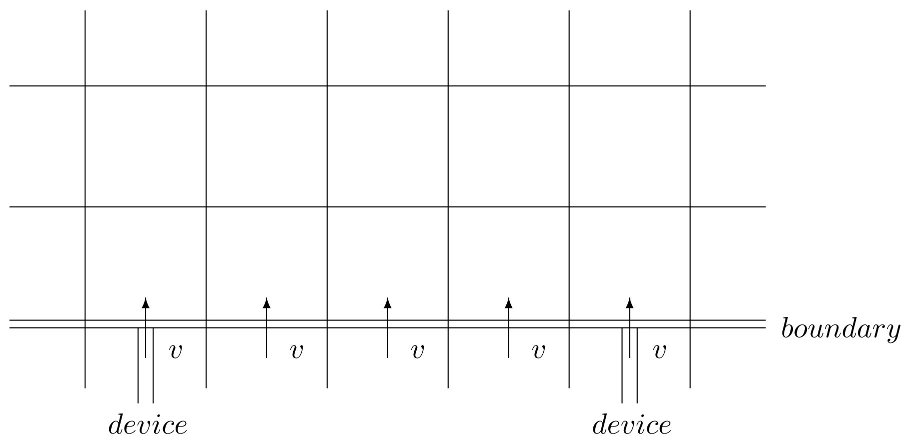

Figure 4.1: *Measure points of the vertical velocity in a 4 times denser grid than the original*

Figure 4.1. A denser grid might lead to periodic boundary conditions, where the period depends on the proportion between the horizontal grid size and the distance between adjacent devices.

4.1.2 Properties of the airplane

By filling in the boundary conditions, we have to take some properties of the airplane into account. Suppose the wings of the plane have a very rough surface, then it is imaginable that we have (approximately) a no-slip condition at the boundary, since the horizontal velocity at the bottom wall may become very small. Another property we have to deal with is the width of a blowing and suction device. Without mentioning it, we assumed in the previous chapter that such a device is one-dimensional. This means that only the magnitude of variables with measure points right at the top of the devices are directly influenced by the user defined input, like the vertical velocity at the boundary $v_d(t, s_1, 0)$ in our staggered grid. Suppose we enlarge the width of the devices, then it is recommended to use a denser grid, like we introduced in the previous section. This situation is depicted in Figure 4.2. A denser grid is especially helpful if the velocity at a device depends

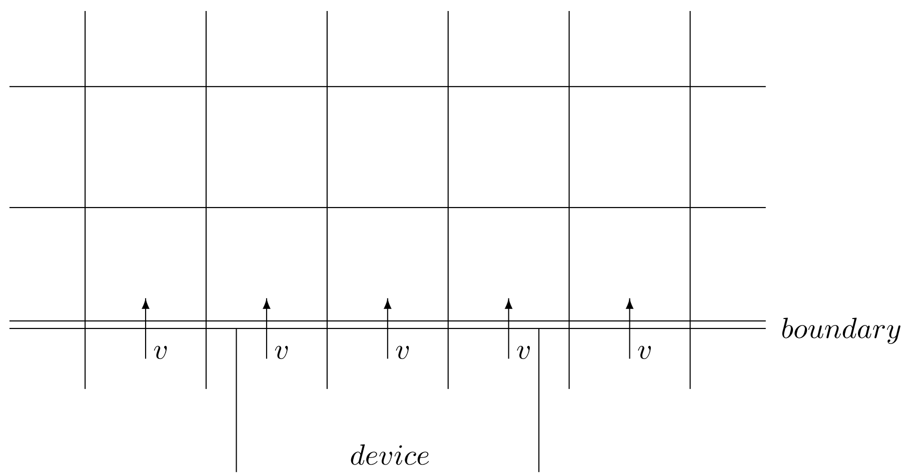

Figure 4.2: *Measure points of the vertical velocity in a grid with a very wide device*

on its position, for instance, the vertical velocity at the center of a device might be larger than the velocity close to the boundary of the device.

One more property that can affect the boundary conditions is whether or not the blowing and suction devices are fully integrated in the surface of the wing. Suppose the devices extend a little bit above the surface. Then it is plausible to assume that the horizontal velocity between two devices does not have influence on any variable at a device and these magnitudes are fully determined by the user defined input. Moreover, we might have a no-slip condition between the devices, but this also depends on how far the devices extend above the surface and on their width.

As we can see, there are many properties we have to take into account, especially at the boundary. Moreover, it might be desirable to take a denser grid. However, it is in general no problem to implement these properties in our model.

4.2 Recommendations

4.2.1 Extension to three-dimensional case

In this report we considered a flow problem on a two-dimensional grid. To obtain a realistic model for this flow problem, a logical step is to extend the Navier-Stokes equations to the three-dimensional case. We get a third velocity component, say w, to describe the velocity in forward and backward (z) direction. As a consequence, we obtain an extra momentum equation:

$$\begin{aligned} \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} &= 0 \\ \dot{u}+U\frac{\partial u}{\partial x}+V\frac{\partial u}{\partial y}+W\frac{\partial u}{\partial z}-\nu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})+\frac{1}{\rho}\frac{\partial p}{\partial x} &= 0 \\ \dot{v}+U\frac{\partial v}{\partial x}+V\frac{\partial v}{\partial y}+W\frac{\partial v}{\partial z}-\nu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2})+\frac{1}{\rho}\frac{\partial p}{\partial y} &= 0 \\ \dot{w}+U\frac{\partial w}{\partial x}+V\frac{\partial w}{\partial y}+W\frac{\partial w}{\partial z}-\nu(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2})+\frac{1}{\rho}\frac{\partial p}{\partial w} &= 0 \end{aligned} \tag{4.1}$$

It will definitely be useful to introduce two extra shift operators for the third dimension we have in this case, we can call these operators for instance the forward shift operator S_f and the backward shift operator S_b. We would obtain six shift operations, which will be defined as follows:

$$\begin{aligned} (S_r x)(t,s_1,s_2,s_3) &= x(t,s_1+1,s_2,s_3) \\ (S_l x)(t,s_1,s_2,s_3) &= x(t,s_1-1,s_2,s_3) \\ (S_f x)(t,s_1,s_2,s_3) &= x(t,s_1,s_2+1,s_3) \\ (S_b x)(t,s_1,s_2,s_3) &= x(t,s_1,s_2-1,s_3) \\ (S_u x)(t,s_1,s_2,s_3) &= x(t,s_1,s_2,s_3+1) \\ (S_d x)(t,s_1,s_2,s_3) &= x(t,s_1,s_2,s_3-1) \end{aligned} \tag{4.2}$$

Our problem would be defined on the space $V := \{\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^+\}$. The boundary is in this case a plane with blowing and suction devices. Note that the properties of the forward and backward shift operators are more or less equivalent with those of the horizontal shift operators, for instance we do not have problems with surjectivity or injectivity with S_f or S_b. That is why the forward and backward operators do not lead to a more complicated model. The boundary conditions would look pretty similar to the two-dimensional case:

$$
\begin{aligned}
u_d(t, s_1, s_2, 0) &= 2c_u(t, s_1, s_2) - u(t, s_1, s_2, 0) \\
v_d(t, s_1, s_2, 0) &= c_v(t, s_1, s_2) \\
p_d(t, s_1, s_2, 0) &= 2c_p(t, s_1, s_2) - p(t, s_1, s_2, 0) \\
v_{dd}(t, s_1, s_2, 0) &= c_{vv}(t, s_1, s_2)
\end{aligned} \tag{4.3}
$$

4.2.2 Replacing the Navier-Stokes equations

For our flow problem, we linearized the incompressible Navier-Stokes equations and made a discretization onto a rectangular, uniform grid. We used the Navier-Stokes equations, since these were also used in [10] and our report is a continuation of [10]. However, it might be more realistic to other equations in our model. The Euler equations for instance might better fit in our model, since the air density ρ will in reality not be equal to a constant value.

4.2.3 Grid sizes in discretization

In order to simplify the equations, the grid sizes Δx and Δy of the rectangular grid were set to a constant value 1. The results probably improve if we take a denser grid in vertical direction nearby the horizontal wall. We have to keep in mind that we cannot use the vertical shift operators S_u and S_d if we have a non-constant grid in vertical direction.

However, since our model has a relatively low dimension thanks to the vertical shift operators, we might permit ourselves to take a (very) dense grid over the whole half plane, in which the grid sizes Δx and Δy are constant and equal to each other. Then the results may become better, while we are still able to use the vertical shift operators. It is recommended to take the distance between the devices into account when a denser grid is constructed, this might prevent unnecessary complicated boundary conditions.

4.2.4 Control design

We described a flow process in our model, so now we are able to design a controller. This controller is used to control the velocity distribution at a certain distance from the boundary. The user defined controller determines the velocity of a blowing and suction device, based on the measurements of the actuators, and has to be implemented in the model through the output equation of the spatially interconnected system.

Appendix A

Implementation of the pressure

With respect to the report of Tilma [10], the approach in this report is different in a few ways. One difference is the use of the vertical shift operators S_d and S_u in this report. In the report of Tilma only horizontal shift operators S_l and S_r were used in order to avoid problems at the boundary. In this report also the vertical shifts are used to take full advantage of the spatial structure of the interconnected system. In chapter 2 we investigated how to deal with the boundary when vertical shift operator were used.

Another major difference is how the pressure is treated in both systems. In Appendix D.6 in [10], the way the values of u, v and p in the flow are calculated is outlined. First initial values for the velocities u and v, and pressure p are chosen for the whole area, and the system matrices for the temporal and spatial state, A_{tt}, A_{ts}, A_{st} and A_{ss}, are composed. To investigate the evolution of this flow a loop is started to calculate the variables for a certain number of time steps: In the first step of the loop, the boundary values for u and v are determined according to some boundary conditions. The following step is significantly different with respect to our method. Recall the continuity equation of the Navier-Stokes equations:

$$\nabla \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{A.1}$$

To assure that the problem satisfies equation (A.1), $\nabla \mathbf{u}$ is calculated over the whole area and added up to get a resulting 'error' e. This error is corrected by adapting the values of the horizontal velocity u of the boundary at the right-hand side in order to satisfy the continuity equation. In contrast to this method, in our report the (derivative of the) continuity equation is part of the interconnected system, in order to have an equation for the pressure.

For the next step of the loop, consider the momentum equations of the Navier-Stokes equations:

$$\begin{aligned} \dot{u} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + R_u \\ \dot{v} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + R_v \end{aligned} \tag{A.2}$$

with

$$R_u = -\tfrac{1}{2}U(u_{(s_1+1,s_2)} - u_{(s_1-1,s_2)}) - \tfrac{1}{2}V(u_{(s_1,s_2+1)} - u_{(s_1,s_2-1)}) + \\ \nu(u_{(s_1+1,s_2)} - 2u_{(s_1,s_2)} + u_{(s_1-1,s_2)} + u_{(s_1,s_2+1)} - 2u_{(s_1,s_2)} + u_{(s_1,s_2-1)})$$

$$R_v = -\tfrac{1}{2}U(v_{(s_1+1,s_2)} - v_{(s_1-1,s_2)}) - \tfrac{1}{2}V(v_{(s_1,s_2+1)} - v_{(s_1,s_2-1)}) + \\ \nu(v_{(s_1+1,s_2)} - 2v_{(s_1,s_2)} + v_{(s_1-1,s_2)} + v_{(s_1,s_2+1)} - 2v_{(s_1,s_2)} + v_{(s_1,s_2-1)}) \tag{A.3}$$

In [10], $\dot{u}$, $\dot{v}$, R_u and R_v are determined via the initial values for u and v and the associated system matrices A_{tt}, A_{ts}, A_{st} and A_{ss}. According to these calculations, the derivatives of the pressure $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ are determined with the equations in (A.2). After this a new velocity distribution for u and v is calculated by integration of the temporal state of the interconnected system.

This method is pretty doubtful, because the pressure is in fact considered as an exogenous input; this is confirmed on pages 4 and 45 of [10]. Suppose we eliminate the spatial vector x_s, the associating temporal state with this method looks like this:

$$\dot{x} = Ax + Bc + Ep \tag{A.4}$$

while in this report p is part of the state x, simply because the pressure is *not* an exogenous input. The pressure depends on the other variables of x so $p = p(x)$, This is an important difference when we investigate properties of the system, like stability. The method in [10] might lead to wrong conclusions.

Another consequence of the treatment of the continuity equation is that our method causes some extra boundary conditions. These include a condition for the pressure and another condition for the vertical velocity. They appear in the system when we take the derivative of the continuity equation and then substitute the differential equations for $\dot{u}$ and $\dot{v}$ in this derivative, see section 3.3. In [10], the continuity equation is not a part of the interconnected system, that is why only single conditions for u and v have to be defined for every boundary. Just like in this report, this is done via the no-slip condition and the user defined input, respectively.

Appendix B

Pressure-correction method

In our model, the continuity equation has been used to get the pressure variable in the equations. In this appendix we present another method to deal with the pressure. By applying the *pressure-correction method* to the Navier-Stokes equations, we eliminate the pressure out of the equations. First we repeat the two-dimensional linearized incompressible Navier-Stokes equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{B.1}$$

$$\dot{u} + U\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial y} - \nu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{B.2}$$

$$\dot{v} + U\frac{\partial v}{\partial x} + V\frac{\partial v}{\partial y} - \nu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0 \tag{B.3}$$

Let $\mathbf{u} = (u, v)$. Define the function f as follows:

$$f(\mathbf{u}) := (U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y})\mathbf{u} - \nu(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\mathbf{u} \tag{B.4}$$

Then equation (B.2) and (B.3) become:

$$\dot{u} + f(\mathbf{u}) = -\frac{1}{\rho}\nabla p \Leftrightarrow \rho(\mathbf{u} + f(\mathbf{u})) = -\nabla p \tag{B.5}$$

A spatial discretization gives:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = \begin{cases} g(c), & s_2 = 0 \\ 0, & s_2 \neq 0 \end{cases} \tag{B.6}$$

$$\rho(\dot{U} + F(\mathbf{U})) = -GP \tag{B.7}$$

Equation (B.1) is replaced by (B.6) and equations (B.2) and (B.3) are replaced by (B.7). Function $g(c)$ is a function of the input c at the boundary, G stands for the discretization of the grad-operator and F is a spatial discretization of function f. $\mathbf{U}$ and P are numerical approximations of $\mathbf{u}$ and p.

Time integration gives:

$$\frac{\partial U^{n+1}}{\partial x} + \frac{\partial V^{n+1}}{\partial y} = \begin{cases} g(c), & s_2 = 0 \\ 0, & s_2 \neq 0 \end{cases}$$

$$\rho(\frac{1}{\Delta t}(\mathbf{U}^{n+1} - \mathbf{U}^n) + F(\mathbf{U}^{n+1})) = -GP^{n+1} \tag{B.8}$$

The question is how to make a connection between $\mathbf{U}^{n+1}$ and P^{n+1}. We define a predictor $\mathbf{U}^*$ as follows:

$$\rho(\frac{1}{\Delta t}(\mathbf{U}^* - \mathbf{U}^n) + F(\mathbf{U}^*)) = -GP^n \tag{B.9}$$

Then we define a corrector $\mathbf{U}^{n+1}$ as follows:

$$\rho(\frac{1}{\Delta t}(\mathbf{U}^{n+1} - \mathbf{U}^n) + F(\mathbf{U}^*)) = -GP^{n+1} \tag{B.10}$$

Subtracting equation (B.9) from equation (B.10) gives:

$$\rho\frac{1}{\Delta t}(\mathbf{U}^{n+1} - \mathbf{U}^*) = -GQ^n, \qquad Q^n := P^{n+1} - P^n \tag{B.11}$$

Taking the div-operator on both sides gives:

$$\rho\frac{1}{\Delta t}(D\mathbf{U}^{n+1} - D\mathbf{U}^*) = -DGQ^n, \qquad Q^n := P^{n+1} - P^n \tag{B.12}$$

The operator DG is the Laplace operator L. We get:

$$\frac{\partial^2 Q^n}{\partial x^2} + \frac{\partial^2 Q^n}{\partial y^2} = \frac{\rho}{\Delta t}(D\mathbf{U}^* - g(c)) \Rightarrow Q^n \Rightarrow P^{n+1} \tag{B.13}$$

So we finally get:

$$\mathbf{U}^{n+1} = \mathbf{U}^* - \frac{\Delta t}{\rho} G Q^n \tag{B.14}$$

To find a solution to the problem, we still need initial values U^0 and P^0. With these values, predictor U^* can be found. Equation (B.13) gives the value of Q^0, and then U^1 and P^1 can be calculated, etcetera. When we take proper values for U^0 and P^0, this method should converge to a solution for U and P.

Note that we cannot use the pressure-correction method for our problem. Recall the time integration step in B.8:

$$\begin{aligned} \frac{\partial U^{n+1}}{\partial x} + \frac{\partial V^{n+1}}{\partial y} &= \begin{cases} g(c), & s_2 = 0 \\ 0, & s_2 \neq 0 \end{cases} \\ \rho\left(\frac{1}{\Delta t}(\mathbf{U}^{n+1} - \mathbf{U}^n) + F(\mathbf{U}^{n+1})\right) &= -GP^{n+1} \end{aligned} \tag{B.15}$$

By this iteration, we want to calculate the magnitude of the pressure. The iteration uses some boundary function $g(c)$. In our model however, this function $g(c)$ will depend on the pressure. That is why it is not possible to use this method in our model.

Index

Bibliography

[1] J.D. Anderson: Computational Fluid Dynamics, McGraw-Hill (1995)

[2] A. Daghighi: The Dirichlet problem for certain discrete structures, Department of Mathematics, Uppsala University (2005)

[3] R. D'Andrea and G.E. Dullerud: Distributed Control Design for Spatially Interconnected Systems, IEEE Transactions on Automatic Control, Vol. 48, No. 9, pages 1478-1495 (2003)

[4] E. DiBenedetto: Partial Differential Equations, Birkhäuser (1995)

[5] Yu.V. Egorov and M.A. Shubin (Eds.): Partial Differential Equations I, Encyclopaedia of Mathematical Sciences, Vol. 30, Springer-Verlag (1991)

[6] T.N.E. Greville: Note on the Generalized Inverse of a Matrix Product, SIAM Review, Vol. 8, No. 4 (1966)

[7] H.O. Kreiss and J. Lorenz: Initial-Boundary Value Problems and the Navier-Stokes Equations, Applied Mathematics, California Institute of Technology (1989)

[8] C. Langbort and R. D'Andrea: Distributed Control of Spatially Reversible Interconnected Systems with Boundary Conditions, SIAM J. Control Optim., Vol. 44, No. 1, pages 1-28 (2005)

[9] M.H. Protter and H.F. Weinberger: Maximum Principles in Differential Equations, Prentice-Hall (1967)

[10] B.P. Tilma: Contributions to Active Flow Control - A Spatially Interconnected System Based Approach, M.Sc. Thesis Report, Faculty of Aerospace Engineering, Delft University of Technology (2002)

[11] M. Xiao, Y. Lin, J.H. Myatt, R.C. Camphouse and S.S. Banda: Rotation Implementation of a Circular Cylinder in Incompressible Flow via Staggered Grid Approach, J. Appl. Math. & Computing Vol. 22, No. 1-2, pages 67-82 (2006)

[12] Lineaire Algebra en Lineaire Analyse 3, Dictaat faculteit Wiskunde en Informatica, Technische Universiteit Eindhoven (2000)